# Category Theory

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## 1 Preface

These notes offer an elementary introduction to category theory. Why bother writing a new text when so many exist already? Two main features distinguish this text from all others known to the author:

- The fraction of the text occupied by examples is considerably larger.
- A much larger area of mathematics is covered by examples. In particular, areas such as measure theory, functional analysis, smooth manifolds, and partial differential equations are emphasized.

## 2 Notation

Bold letters N, Z, Q, R, C denote the (semi)rings of natural, integer, rational, real, and complex numbers. Sans-serif letters like Set denote categories. Euler calligraphic letters like Mor denote functors. Roman letters like sup denote ordinary mathematical operators.

## 3 Introduction

Category theory is omnipresent in such branches of mathematics as algebraic geometry, algebraic topology, number theory, complex geometry, logic, commutative algebra, K-theory. More recently, categories made their way into a variety of applied areas such as condensed matter physics, signal processing, statistics, etc.

Very roughly, categories fit into the following chain of abstractions:

- Antiquity and middle ages: numbers and figures as mathematical objects. Abstraction: some numbers and figures need not be present in nature. Operations: addition, multiplication, division of numbers; compass and straightedge constructions with geometric objects.
- 18th and 19th century: functions as mathematical objects axiomatizing sequences of operations mentioned in the previous item: polynomials, analytic functions, smooth functions, continuous functions. Abstraction: some functions might not be specified by an explicit formula. Operations on functions: addition, multiplication, limit, infinite sums, etc.
- Early 20th century: abstract mathematical structures axiomatizing the above operations on functions: sets, groups, rings, fields, vector spaces, topological spaces, Banach spaces, C\*-algebras, measurable spaces, Lie groups. Abstraction: some mathematical structures might not have functions as their elements. Operations on structures: direct sum, product, direct and inverse limits, etc.
- Middle of 20th century: categories (abstract collections of mathematical structures axiomatizing the above operations): categories, abelian categories, toposes, regular categories, sites and Grothendieck topologies, etc. Abstraction: some categories need not arise as categories of mathematical structures. Operations on categories: coproducts and products, functor categories, etc.
- 21st century: higher categories (abstract collections of gadgets mentioned in the previous item): 2categories, model categories,  $\infty$ -categories,  $(\infty, n)$ -categories, etc. Abstraction: some higher categories need not arise from specific classes of categories. Operations: same as above (roughly, higher categories themselves form a higher category and higher category theory can process itself).

When trying to characterize the structure of category theory and its role in mathematics, it is useful to compare the notion of a category to that of a complex number. Both are omnipresent in mathematics: it is hard to name an area of mathematics untouched by category theory or complex numbers. Another unifying property of both notions is that there are relatively few deep theorems about categories or complex numbers per se, i.e., not belonging to some other field of mathematics.<sup>†</sup>

For instance, there are many theorems in other fields of mathematics for which the notion of a complex number is essential:

- the field of complex numbers is algebraically closed (the fundamental theorem of algebra);
- bounded entire functions on the complex plane are constant (Liouville's theorem in complex analysis);
- a compact Kähler manifold with vanishing first Chern class has a Kähler metric with vanishing Ricci curvature (differential geometry).

However, one cannot say that these results form a "theory of complex numbers" in the sense one normally uses the word "theory" in mathematics.

In the same way, categories are essential components of many theorems throughout mathematics:

- Pushforward along proper morphisms of locally Noetherian schemes preserves coherent sheaves (algebraic geometry);
- Čech cohomology, de Rham cohomology, and singular cohomology of a smooth manifold are isomorphic (topology);
- On a Stein manifold, the first Cousin problem is always solvable, whereas the second Cousin problem is always solvable if and only if the second integer cohomology vanishes (complex analysis).

It is important to point out, though, that these theorems are only a tiny sample of the enormous variety of results making use of categories, in the same way as with the three theorems using complex numbers above.

<sup>&</sup>lt;sup>†</sup> It is not entirely true that "pure" category theory is devoid of deep theorems. Among nontrivial results in category theory proper one can cite the Barr–Beck monadicity theorem, the Giraud theorem, and the Freyd–Mitchell embedding theorem.

The analogy with complex numbers breaks down when we consider the learning aspects. Complex numbers can be introduced and their basic properties proven in less than an hour. In contrast, category theory requires at least two orders of magnitude more time to get acquainted with. Acquiring a working understanding of category theory resembles climbing the Tibetan Plateau: one first has to expend a substantial amount of effort simply to climb 5 kilometers (3 miles) to the top of the plateau (i.e., learn and understand the relevant notions such as categories, functors, adjunctions, Kan extensions, etc.). After this, one still has to spend a considerable amount of time acclimatizing to the high altitude of the plateau (i.e., the high level of abstraction associated with the categorical language). The first few days one is guaranteed to have altitude sickness (i.e., difficulty managing the high level of abstraction and using the associated notions and tools), which eventually disappears once one spends a sufficient amount of time on the plateau.

## Categories, functors, and natural transformations

#### 4 Categories

Definition 4.1. A category C is a collection of the following data:

- a class<sup>†</sup> Ob(C) of *objects* (we write  $X \in C$  instead of  $X \in Ob(C)$ ;
- for any  $X \in \mathsf{C}$  and  $Y \in \mathsf{C}$  we have a set<sup>‡</sup>  $\mathsf{Mor}_{\mathsf{C}}(X, Y)$  of *morphisms* from X to Y (alias *maps*, *arrows*), but instead of  $f \in \mathsf{Mor}_{\mathsf{C}}(X, Y)$  we write  $f: X \to Y$  or  $X \xrightarrow{f} Y$ ;
- for any  $X \in \mathsf{C}$  an *identity morphism* on X:  $\mathrm{id}_X : X \to X$ ;
- for any  $X, Y, Z \in C$  the composition of morphisms  $\circ: Mor_{C}(Y, Z) \times Mor_{C}(X, Y) \to Mor_{C}(X, Z)$ , but instead of  $\circ(g, f)$  we write  $g \circ f$  or gf.
- This data must satisfy the following properties:
- unitality: for any morphism  $f: X \to Y$  we have  $id_Y f = f id_X = f$ ;
- associativity: for any morphisms  $f: W \to X, g: X \to Y, h: Y \to Z$  we have (hg)f = h(gf).

Used in 2.0\*, 4.1, 4.1\*, 4.2, 4.3, 4.26, 4.28, 5.1, 5.2, 6.1, 6.29, 6.30, 6.31, 6.32, 6.33, 6.34, 6.38, 7.1, 7.2, 7.4, 9.0\*, 9.1, 9.2, 9.5\*, 9.16, 9.36, 10.0\*, 10.2, 10.5, 10.34\*, 11.5, 11.19, 13.3, 13.4, 13.8, 23.6, 25.2, 26.1, 26.5\*, 26.6, 26.6\*, 26.7\*, 27.3, 27.6, 27.7\*, 30.1, 30.11\*, 31.1, 31.2, 32.2, 33.2, 33.4, 33.5.

We often write Mor instead of Mor<sub>C</sub> when no ambiguity can arise. If  $f: X \to Y$  is a morphism in C, we say that X is the *domain* (alias *source*) of f and Y is the *codomain* (alias *target*) of f. We also write X = dom f and Y = codom f.

The primordial example of a category is the *category of sets*:

**Example 4.2.** The category Set has sets as objects and Mor(X, Y) is the set of functions from X to Y. Composition is given by the composition of functions and  $id_X$  is the identity function  $X \to X$ . Used in 2.0\*, 4.5, 4.15, 4.41, 5.3, 6.3, 6.5, 6.9, 6.28, 6.30, 6.31, 6.32, 6.33, 6.36, 6.38, 6.39, 7.10, 7.12, 7.13, 7.34, 8.4, 8.6, 9.17, 10.14, 10.19, 10.22, 10.31, 10.32, 10.33, 10.34\*, 11.2, 11.14, 11.17, 11.19, 12.3, 13.3, 13.6, 13.8, 14.4, 15.3, 15.5, 15.6, 16.2, 16.4, 18.3, 19.2, 20.2, 20.5, 21.2, 21.3, 21.4, 22.2, 23.7, 25.1, 25.2, 25.3, 25.4, 26.0\*, 26.1, 26.2, 26.3, 26.4, 26.5, 26.5\*, 26.6, 26.7, 26.7\*, 27.0\*, 27.1, 27.2, 27.4, 27.7, 28.1, 28.3, 29.3, 29.5, 30.6, 30.7, 30.11, 30.11\*, 30.12, 31.1, 31.3, 33.3.

**Example 4.3.** The category Group has groups as objects and Mor(X, Y) is the set of group homomorphisms  $X \to Y$ . Composition is given by the composition of group homomorphisms (which is again a group homomorphism) and  $id_X$  is the identity group homomorphism on a group X. Used in 4.6, 6.5, 6.9, 6.10, 6.14, 6.16, 6.36, 8.6, 8.7, 8.8, 8.9, 8.14, 11.14, 12.6, 13.5, 14.7, 15.5, 19.3, 20.3, 21.3, 22.2, 28.4, 29.5, 31.4.

Before we continue with more examples, we introduce an important construction on categories.

**Definition 4.4.** The *full subcategory* of a category C on a class of objects  $D \subset C$  is a category that has D as its class of objects, whereas the sets of morphisms as well as identities and composition are inherited from C.

**Example 4.5.** The category FinSet of finite sets is the full subcategory of Set on the class of finite sets. Used in 4.15, 5.5.

<sup>&</sup>lt;sup>†</sup> A class is like a set, except that it can be much bigger. For instance, there is a class of all sets, but there is no set of all sets (by Russell's paradox).

<sup>&</sup>lt;sup>‡</sup> Some mathematicians also allow a class here, in which case our variant is referred to as a "locally small category".

**Example 4.6.** The category Ab of abelian groups is the full subcategory of Group on the class of abelian groups. Used in 5.3, 5.5, 6.3, 6.5, 8.6, 8.8, 8.14, 12.5, 13.6, 13.9, 20.3.

We now give more examples of categories from various areas of mathematics.

#### 4.7. Algebra

**Example 4.8.** The category Ring of rings has (associative) rings as objects and homomorphisms of rings as morphisms. (We require associative rings to have a unit and their homomorphisms to preserve units.) It has a full subcategory CRing of commutative rings. The latter has a full subcategory Field of fields. Used in 4.11, 5.3, 6.5, 6.10, 6.11, 6.36, 7.49, 8.6, 8.9, 12.7, 12.8, 13.5, 19.4, 20.3, 21.3, 21.6, 26.4, 26.5, 31.4.

**Example 4.9.** Given a ring R, the category  $Mod_R$  has right R-modules as objects and R-linear homomorphisms of modules as morphisms. If k = R is a field, we denote this category by  $Vect_k$  (vector spaces over a field k). Likewise, we have the categories  $Alg_k$  (associative unital algebras over k) and  $LieAlg_k$  (Lie algebras over k). The category  $Alg_k$  has a full subcategory  $CAlg_k$  of commutative algebras. Used in 4.9, 4.32, 5.3, 5.5, 6.5, 6.13, 6.14, 6.21, 6.24, 6.25, 6.36, 7.10, 7.11, 7.16, 7.47, 8.6, 8.8, 8.11, 9.34, 10.5, 12.5, 13.5, 13.6, 13.9, 20.3, 21.3, 25.3, 31.4, 31.5, 32.5, 33.4.

**Example 4.10.** Given a group G, we define the category **GSet** of sets with a *G*-action. Its objects are pairs  $(S, \rho)$ , where S is a set and  $\rho: G \to \Sigma_S$  is a homomorphism of groups, i.e., every element of G acts via a permutation on S. We denote  $g \cdot s = \rho(g)(s)$ . Morphisms  $(S, \rho) \to (S', \rho')$  are functions  $f: S \to S'$  such that  $f(\rho(g)(s)) = \rho'(g)(f(s))$ , i.e., f commutes with the action of  $G: g \cdot f(s) = f(g \cdot s)$ . Used in 10.34\*, 10.36\*.

**Example 4.11.** The category BoolAlg is the full subcategory of Ring on *Boolean algebras*: rings in which all elements are idempotent, i.e.,  $x^2 = x$ . (Such rings are automatically commutative.) We will also make use of the (nonfull) subcategory ComplBoolAlg of *complete* Boolean algebras and continuous homomorphisms (a Boolean algebra A is complete if any subset of A has a supremum with respect to the order  $x \le y \equiv x = xy$  and a homomorphism of Boolean algebras is continuous if it preserves these suprema). Finally the category ComplAtomBoolAlg is the full subcategory of ComplBoolAlg consisting of complete *atomic* Boolean algebras (a Boolean algebra is *atomic* if for any nonzero  $z \in A$  there is an atom  $a \in A$  such that  $a \le z$ , where a is an *atom* if  $a \ne 0$  and for any  $b \in A$  such that  $0 \le b \le a$  either b = 0 or b = a). Used in 4.11, 6.39, 7.12, 7.13, 7.35, 8.16.

**Example 4.12.** Other algebraic structures, far too numerous to be named here, also form categories. Morphisms are maps of underlying sets that preserve all algebraic operations. Examples include monoids, magmas, loops, heaps, rigs, G-actions for a fixed group G, division rings, algebras over a ring R, Lie algebras over a field k, k-vector spaces with an inner product, etc. Order-theoretic notions, such as posets, linearly ordered sets, ordered groups, ordered fields, etc., also form categories.

#### 4.13. Combinatorics

**Example 4.14.** The category Graph of (directed graphs) has graphs (i.e., pairs of functions  $s, t: E \to V$ ) as objects and homomorphisms of graphs (i.e., functions  $v: V \to V'$  and  $e: E \to E'$  such that vs = s'e and vt = t'e) as morphisms.

**Example 4.15.** The category of *species* plays an important role in combinatorics. We will define it later as a *category of functors* from  $FinSet^{\times}$  to Set.

#### 4.16. General topology

**Example 4.17.** The category Top has topological spaces as objects and continuous maps as morphisms. (The composition of continuous maps is again continuous.) The category Top<sub>\*</sub> has pointed topological spaces as objects and continuous maps that preserve the basepoint as morphisms. We also have the following full subcategories of Top: Haus (Hausdorff spaces) and CompHaus (compact Hausdorff spaces).  $U_{sed in 4.17, 4.39, 5.3, 6.5, 6.6, 6.16, 6.36, 7.15, 7.30, 7.31, 7.32, 7.36, 7.45, 8.13, 8.14, 9.10, 9.12, 9.15, 10.7, 10.8*, 10.11*, 10.16, 10.20, 10.21, 10.27, 10.31, 10.32, 10.33, 11.17, 12.4, 13.7, 14.5, 14.6, 15.6, 16.4, 21.4, 25.4, 30.12.$ 

**Example 4.18.** The category **TopGroup** has topological groups as objects and continuous homomorphisms of topological groups as morphisms. The categories of topological rings, topological fields, topological modules,

and topological vector spaces over a topological field k (denoted TopVect<sub>k</sub> or simply TopVect if no ambiguity can arise) can be defined in an analogous fashion. We will also need the full subcategory HausGroup (Hausdorff topological groups), LocCompHausGroup (locally compact Hausdorff groups), LocCompHausAb (locally compact Hausdorff abelian groups), CompHausGroup (compact Hausdorff groups). Finally, the category LocCompHausGroup<sub>Open</sub> has locally compact Hausdorff topological groups as objects and continuous *open* homomorphisms as morphisms. Used in 4.18, 4.22, 6.20, 6.21, 6.36, 7.16, 7.18, 7.23, 8.14, 9.32, 16.5.

**Example 4.19.** There are two different categories of metric spaces. The category  $Met_1$  of metric spaces and contractive maps has metric spaces as objects and contractive maps as morphisms. (A map  $f: X \to Y$  is contractive if  $d(f(x), f(x')) \leq d(x, x')$  for any points  $x, x' \in X$ .) The category Met of metric spaces and continuous maps has metric spaces as objects and continuous maps as morphisms. (Every contractive map is continuous, but not vice versa.) These two categories have different properties and illustrate the fact that in category theory morphisms are as important as objects. Used in 4.21, 6.5, 6.7, 6.36.

#### 4.20. Functional analysis

The spaces below can be either real or complex, but we omit this data in the notation.

**Example 4.21.** Continuing the examples with metric spaces (Met<sub>1</sub> and Met), one can define two different categories of Banach spaces:  $Ban_1$  has Banach spaces as objects and contractive linear maps as morphisms, whereas Ban has Banach spaces as objects and continuous linear maps as morphisms. One also has the categories Hilb<sub>1</sub> and Hilb for Hilbert spaces. Used in 4.22, 5.3, 6.5, 6.7, 6.18, 6.36, 6.37, 7.15, 7.22, 7.23, 7.24, 7.26, 7.27, 8.11, 9.9, 9.10, 9.21, 10.7, 10.11\*, 11.19, 14.8, 21.5, 33.5.

**Remark 4.22.** As we will see later, the categories Ban and Hilb can be identified (in the appropriate sense) with certain full subcategories of TopVect, the category of topological vector spaces. This is *not* true for  $Ban_1$  and  $Hilb_1$ : a topological vector space contains no information about norms or inner products.

**Example 4.23.** The theory of operator algebras delivers many examples of categories. The category BanAlg has Banach algebras as objects and continuous homomorphisms of algebras as morphisms. The category  $C^*$  has C\*-algebras as objects and \*-homomorphisms as morphisms. The category  $W^*$  has von Neumann algebras (alias W\*-algebras) as objects and *ultraweakly continuous* \*-homomorphisms as morphisms. The full subcategories  $CC^*$  and  $CW^*$  of commutative algebras are also important. Used in 4.26, 6.5, 6.36, 7.29, 7.30, 7.31, 7.32, 7.44, 7.45, 9.12, 9.13.

**Remark 4.24.** The examples given so far may create an impression that objects in a category are sets with structures, whereas morphisms are functions that preserve these structures. (Such an approach is explained in Chapter IV of Bourbaki's Set Theory.) However, this is not always the case and below we define the categories Meas, HoTop, and  $\Psi DO_M^{\infty}$ , none of which can be interpreted as "sets with structures". This situation is analogous to the one with groups. Groups were originally defined as sets of permutations of a fixed set S closed under composition and inverses.

#### 4.25. Measure theory

**Example 4.26.** A naive approach to defining an appropriate category for measure theory would take pairs (X, M) as objects, where X is a set and M is a  $\sigma$ -algebra of measurable subset of X. Morphisms  $(X, M) \to (X', M')$  would be functions  $f: X \to X'$  such that for any  $m \in M'$  we have  $f^{-1}(m) \in M$ , i.e., preimages of measurable sets are measurable.

The problem with this approach is that there is not enough data to formulate any nontrivial theorem of measure theory using this category: the notion of a *negligible set* (alias set of measure 0) features prominently in all main results of measure theory. Furthermore, measure theory *identifies* different maps that differ on a set of measure 0, which is not reflected in the above category.

We modify our definition accordingly and define a category Meas whose objects are triples (X, M, N), where X and M are as above and  $N \subset M$  is a  $\sigma$ -ideal of *negligible* sets. (A  $\sigma$ -ideal is a  $\sigma$ -algebra that is additionally closed under passage to subsets, which reflects the fact that subsets of sets of measure 0 again have measure 0.) We remark that the data of N encodes exactly the same data as a *measure class*, i.e., an equivalence class of measures on (X, M) with respect to the following equivalence relation:  $\mu \sim \nu$  if  $\mu \ll \nu$ and  $\nu \ll \mu$ . Morphisms  $(X, M, N) \to (X', M', N')$  are equivalence classes of functions  $f: X \to X'$  such that  $f^{-1}$ sends elements of M' to M and elements of N' to N (the latter condition is motivated below). The equivalence relation says that  $f \sim g$  if  $\{x \in X \mid f(x) \neq g(x)\} \in N$  (two functions are identified if they differ on a negligible set). In fact, to get a satisfactory theory, one must also allow functions  $f: X_0 \to X'$ , where  $X_0 \subset X$  is a measurable subset of X with negligible complement, i.e.,  $X \setminus X_0 \in N$ . The equivalence relation is defined in a similar fashion,  $f \sim g$  if  $X \setminus \{x \in X \mid x \in \text{dom } f \cap \text{dom } g \land f(x) = g(x)\} \in N$ .

The operation of composition descends to equivalences classes: if  $f \sim g$  for some

$$f, g: (X_1, M_1, N_1) \to (X_2, M_2, N_2),$$

then  $fe \sim ge$  for any  $e: (X_0, M_0, N_0) \to (X_1, M_1, N_1)$  and  $hf \sim hg$  for any  $h: (X_2, M_2, N_2) \to (X_3, M_3, N_3)$ . Indeed,

$$\{x_0 \in X_0 \mid f(e(x_0)) \neq g(e(x_0))\} = e^{-1}\{x_1 \in X_1 \mid f(x_1) \neq g(x_1)\},\$$

and we have

$$A = \{x_1 \mid f(x_1) \neq g(x_1)\} \in N_1,$$

so  $e^{-1}(A) \in N_0$  because  $e^{-1}$  sends elements of  $N_1$  to  $N_0$ . Likewise,

$$B = \{x_1 \in X_1 \mid h(f(x_1)) \neq h(g(x_1))\} \subset \{x_1 \in X_1 \mid f(x_1) \neq g(x_1)\} \in N_1,\$$

so  $B \in N_1$  because  $N_1$  is closed under passage to subsets. (See Definition 4.28 for an abstract formulation of this construction.)

As a vindication of this definition, we will see later that a subcategory of Meas consisting of *localizable* measurable spaces can be identified with  $CW^*$ , the category of commutative von Neumann algebras. Used in 4.24, 4.26, 4.27, 4.37, 5.3, 5.5, 6.20, 6.39, 7.33\*, 7.34, 7.35, 7.36, 7.38\*, 7.39, 7.42, 7.43, 7.45, 8.16, 9.17, 10.7, 11.20\*.

**Remark 4.27.** All previous categories have the following pattern: objects are sets equipped with additional structure, morphisms are functions that preserve this structure. The category Meas is *not* of this type because we identified functions that differ on a set of measure 0. In particular, one can *prove* that given a morphism in Meas, there is no way to choose a representative function in such a way that these choices respect composition (i.e., the composition of two representatives is again a representative). In other words, there *no* reasonable notion of an "underlying set" in Meas.

The above quotient construction occurs often enough to deserve a precise formalization.

**Definition 4.28.** A congruence R on a category C is an equivalence relation  $R_{X,Y}$  on Mor(X,Y) for any pair of objects  $X, Y \in C$  that satisfies the following compatibility condition: for any objects  $X, Y, Z \in C$ , morphisms  $f, f': X \to Y, g, g': Y \to Z$ , if  $f \sim f'$  and  $g \sim g'$ , then also  $gf \sim g'f'$ . The quotient category C/R has the same objects as C, whereas morphisms from X to Y are elements of the quotient set  $Mor(X,Y)/R_{X,Y}$ . Used in 4.26, 4.37.

## 4.29. Differential geometry

**Example 4.30.** The category Man has smooth manifolds as objects and smooth maps as morphisms. (Smooth means infinitely differentiable. A smooth manifold can be defined as a subset of  $\mathbf{R}^n$  that is locally diffeomorphic to some coordinate inclusion  $\mathbf{R}^k \to \mathbf{R}^n$ .) Analysts like to work with a full subcategory of this category consisting of open subsets of  $\mathbf{R}^n$  for all  $n \ge 0$ . Used in 4.33, 5.3, 6.23, 6.24, 6.25, 6.36.

**Example 4.31.** The category LieGroup has Lie groups as objects and smooth homomorphisms of groups as morphisms. Used in 6.25, 6.36, 9.34.

**Example 4.32.** Given a smooth manifold M, the category  $\mathsf{VBun}_M$  has vector bundles over M as objects and smooth linear maps of vector bundles as morphisms. As we will see later, this category can be identified with a certain subcategory of  $\mathsf{Mod}_{\mathbb{C}^{\infty}(M)}$ , where  $\mathbb{C}^{\infty}(M)$  denotes the algebra of smooth functions on M. Analysts like to work with a full subcategory of this category consisting of *trivial vector bundles*, whose objects are  $\mathbb{R}^n$  and morphisms  $\mathbb{R}^n \to \mathbb{R}^{n'}$  are smooth functions  $M \to \mathsf{Hom}(\mathbb{R}^n, \mathbb{R}^{n'})$ . Used in 4.32, 4.33, 4.35, 6.23, 6.24, 6.25, 6.36, 16.5.

**Example 4.33.** The category VBun (here we do not fix a manifold) has pairs (M, V)  $(M \in Man)$  and  $V \in VBun_M$ ) as objects and pairs  $(f,g): (M, V) \to (M', V')$   $(f: M \to M')$  and  $g: M \to f^*M'$  as morphisms. Composition is defined as  $(f',g') \circ (f,g) = (f' \circ f, f^*g' \circ g)$ .

## 4.34. Partial differential equations

**Example 4.35.** Given a smooth manifold M, the category  $\mathsf{DO}_M$  has vector bundles over M as objects and differential operators as morphisms. Specifically, a morphism  $V \to V'$  is a linear map of real vector spaces  $T: \mathbb{C}^{\infty}(V) \to \mathbb{C}^{\infty}(V')$  that preserves support: for any  $f \in \mathbb{C}^{\infty}(V)$  we have  $\operatorname{supp} T(f) \subset \operatorname{supp} f$ , where  $\operatorname{supp} g$  is the closure of the set  $\{m \in M \mid g(m) \neq 0\}$ . Equivalently, we can say that the Schwartz kernel of T is supported on the diagonal of  $M \times M$ . By Peetre's theorem this definition is equivalent to the coordinate definition that defines differential operators using local coordinate expressions of the form  $\sum_k a_k \partial^k f$ , where the sum is finite, k is a multi-index,  $a_k$  is a smooth function on M, and  $\partial^k f$  denotes the partial derivative of f corresponding to the multi-index k. Used in 4.35, 5.3.

**Example 4.36.** Given a smooth manifold M, the category  $\Psi DO_M^{ps}$  has vector bundles over M as objects and properly supported *pseudodifferential operators* as morphisms. Specifically, a morphism  $V \to V'$  is a linear map of real vector spaces  $T: C_{cs}^{\infty}(V) \to C_{cs}^{\infty}(V')$  ( $C_{cs}^{\infty}$  denotes smooth compactly supported sections) whose Schwartz kernel is properly supported and is a conormal distribution on  $M \times M$  with respect to its diagonal. Used in 4.36, 16.5.

**Example 4.37.** Another important category  $\Psi DO_M^{\infty}$  is obtained from vector bundles on M and pseudodifferential operators between them using the quotient construction of Definition 4.28 that we already used to define Meas: we declare two pseudodifferential operators equivalent if their difference is a smoothing operator (i.e., its Schwartz kernel is a smooth function on  $M \times M$ ). One can verify that composition of such equivalence classes form a sheaf on M (i.e., satisfy a gluing property) and their composition can be defined without the proper support condition. (Smoothing operators composed on either side with a pseudodifferential operator give a smoothing operator.) Thus we indeed get a category. This category is important in the *calculus of pseudodifferential operators*. For instance, elliptic differential operators become *isomorphisms* (defined below) in this category, and their inverse is known as a *parametrix*. Used in 4.24, 5.3.

#### 4.38. Homotopy theory

**Example 4.39.** In homotopy theory and algebraic topology a key role is played by the homotopy category of topological spaces, sometime denoted HoTop. It is formed from Top by identifying homotopic continuous maps: two continuous maps  $f, g: X \to Y$  of topological spaces are homotopic if there is a homotopy between them, i.e., a continuous map  $h: X \times [0,1] \to Y$  whose restrictions to  $X \times \{0\}$  and  $X \times \{1\}$  are f and g respectively. (Strictly speaking, the category that is actually used in homotopy theory is the full subcategory of HoTop consisting of *CW-complexes*, but we ignore such details for now.) Used in 4.24, 4.39, 5.3, 6.6, 6.40.

#### 4.40. Sheaf theory

**Example 4.41.** In set theory, categories of presheaves and sheaves (to be defined below), allow one to prove the independence of the continuum hypothesis and the axiom of choice from the Zermelo–Fraenkel axioms. Roughly speaking, some of these categories behave like the category Set, except that the continuum hypothesis or the axiom of choice fails in them.

**Example 4.42.** Presheaves and sheaves also play a very important role in complex analysis and algebraic geometry, for instance, they are used to define *sheaf cohomology*, which is one of the most important invariants of complex manifolds and algebraic varieties.

## 4.43. Algebraic geometry

**Example 4.44.** Fix an algebraically closed field k. The category  $AffVar_k$  has affine algebraic varieties over k (i.e., subsets of  $k^n$  defined by polynomial equations with coefficients in k) as objects and regular maps (restrictions of polynomial maps  $k^m \to k^n$ ) as morphisms. Used in 7.47, 7.48, 7.48\*, 32.5.

#### 5 Isomorphisms

Any set with one element can be turned into a group in the obvious fashion. Different sets with one element (e.g.,  $\{\emptyset\}, \{\{\emptyset\}\}, \{\{\emptyset, \{\emptyset\}\}\}, \text{etc.}$ ) give rise to groups (e.g.,  $G_1, G_2, G_3$  for the above sets) that from a formal viewpoint are different groups:  $G_1 \neq G_2, G_2 \neq G_3$ , etc. However, in group theory we perceive these groups as the "same" group: even though they are not *equal* groups, they are *isomorphic* groups. The situation in other categories is entirely analagous, and the notion of an isomorphism in a category generalizes the notion of an isomorphism in a group.

**Definition 5.1.** A morphism  $f: X \to Y$  in some category C is an *isomorphism* if there is a morphism  $g: Y \to X$  (typically denoted  $f^{-1}$ ) such that  $gf = id_X$  and  $fg = id_Y$ .

**Remark 5.2.** The morphism  $f^{-1} = g$  defined above is unique. Indeed, if some g' has the same property, then  $g' = \operatorname{id}_X g' = (gf)g' = gfg' = g(fg') = g\operatorname{id}_Y = g$ .

Examples 5.3. We list several categories and describe isomorphisms in them.

- Set: bijections (alias one-to-one and onto functions);
- Ab, Ring, Mod<sub>R</sub>, and other categories algebraic of algebraic structures: isomorphisms;
- Top: homemorphisms;
- HoTop: homotopy equivalences of topological spaces;
- Man: diffeomorphisms;
- Ban<sub>1</sub>: isometric isomorphisms of Banach spaces;
- Ban: linear homeomorphisms of Banach spaces (not necessarily norm-preserving);
- $DO_M$ : zeroth order differential operators given by multiplication by a smooth nonvanishing function;
- $\Psi DO_M^{\infty}$ : a large class that contains all elliptic differential (and pseudodifferential) operators;
- Meas: an isomorphisms of the underlying sets with  $\sigma$ -algebras and  $\sigma$ -ideals, after possibly removing negligible sets from source and target (e.g., **R** and **R** \ {0,1,2} are isomorphic, if we use Lebesgue structures).

**Definition 5.4.** An *endomorphism* is a morphism whose source and target are the same. An *automorphism* is an endomorphism that is also an isomorphism. Given an object X in a category C, the *group of automorphisms* of X is the set of all automorphisms of X equipped with the operation of composition and is denoted by  $Aut_{C}(X)$  or simply Aut(X) if C is clear from the context. Used in 5.4, 5.5, 5.9, 5.10, 5.12, 9.7.

#### Examples 5.5.

- $X \in \mathsf{FinSet}$  (finite sets):  $\mathsf{Aut}(X)$  is the symmetric group on X;
- $X = \mathbf{Z}^n \in \mathsf{Ab}$  (lattices):  $\operatorname{Aut}(X) = \mathbf{Z}/2 \times \operatorname{SL}_n(\mathbf{Z})$  is (up to the factor  $\mathbf{Z}/2$ ) the unimodular group of degree n;
- $X = k^n \in \text{Vect}_k$ :  $\text{Aut}(X) = \text{GL}(k^n) = \text{GL}(n,k)$  is the general linear group of degree n over a field k;
- $X \in Meas: Aut(X)$ , the group of measurable automorphisms, plays an important role in ergodic theory.

#### 5.6. Groupoids

**Definition 5.7.** A groupoid is a category in which all morphisms are isomorphisms.

**Example 5.8.** The fundamental groupoid  $\pi_{\leq 1}(X)$  of a topological space X is defined as follows. Objects are points of X. Morphisms  $x \to y$  are equivalences classes of continuous maps  $f:[0,1] \to X$  such that f(0) = x and f(1) = y modulo the equivalence relation of relative homotopy:  $f \sim g$  if there is a continuous map  $h:[0,1] \times [0,1] \to X$  such that  $h|_{0 \times [0,1]} = f$ ,  $h|_{1 \times [0,1]} = g$ ,  $h|_{[0,1] \times 0} = \hat{x}$ , and  $h|_{[0,1] \times 1} = \hat{y}$ . Here  $\hat{x}$  and  $\hat{y}$  denote the constant maps  $[0,1] \to X$  with values x and y respectively. Composition is defined by composing the underlying representative functions  $f:[0,1] \to X$  and  $g:[0,1] \to X$  as follows:

$$gf = \begin{cases} f(2t) & 0 \le t \le 1/2\\ g(2t-1) & 1/2 \le t \le 1 \end{cases}$$

One checks that this operation respects the above equivalence relation, which gives a well-defined composition. The identity morphism  $x \to x$  is the equivalence class of the constant map  $[0,1] \to X$  with value x.

The category that we just defined is a groupoid: the inverse can be defined on a representative function  $f:[0,1] \to X$  as  $g:[0,1] \to X$ , g(t) = f(1-t). Used in 5.9, 10.31, 10.32, 10.33, 10.33\*, 10.34\*.

**Example 5.9.** Consider a topological space X with a point  $x \in X$ . The group  $\operatorname{Aut}_{\pi_{\leq 1}(X)}(x)$  is denoted  $\pi_1(X, x)$  and is referred to as the *fundamental group* of the pointed space (X, x). It is an important invariant in topology. A different choice x' of a basepoint yields a *noncanonically isomorphic* fundamental group  $\pi_1(X, x')$ . More precisely, a morphism  $f: x \to x'$  (i.e., an homotopy class of paths) in  $\pi_{\leq 1}(X)$  gives rise to an isomorphism  $\pi_1(X, x) \to \pi_1(X, x')$  (namely,  $p \mapsto fpf^{-1}$ , where the right side uses composition in  $\pi_{\leq 1}(X)$ ), and different morphisms  $x \to x'$  can give different morphisms of groups. Used in 5.9, 6.16, 10.34\*.

**Example 5.10.** The (absolute) Galois groupoid  $\operatorname{Gal}(k)$  of a field k has as objects algebraically closed (or separably closed if char  $k \neq 0$ ) extensions L/k, whereas morphisms are isomorphisms  $L \to L'$  of extensions over k (i.e., the action on k is identity). The automorphism group  $\operatorname{Aut}_{\operatorname{Gal}(k)}(L/k)$  of some algebraic closure L of k is known as the (absolute) Galois group of k. A different choice of L/k yields a noncanonically isomorphic Galois group. More precisely, isomorphisms  $L \to L'$  produce isomorphisms  $\operatorname{Gal}(L/k) \to \operatorname{Gal}(L'/k)$ , and different isomorphisms can produce different isomorphisms of groups. Used in 5.10.

**Remark 5.11.** The above examples of fundamental groupoids and Galois groupoids seem to be analogous. Indeed, both groupoids can be defined using the same construction: the *fundamental groupoid of a topos*. For topological spaces one takes the topos of sheaves, whereas for fields one takes the etale topos.

**Remark 5.12.** The above arguments can be generalized to show that in any groupoid G an isomorphism  $X \to X'$  induces a homomorphism of groups  $\operatorname{Aut}_{\mathsf{G}}(X) \to \operatorname{Aut}_{\mathsf{G}}(X')$ .

## 6 Functors

In the previous section we gave many examples of categories in different areas of mathematics. One glaring omission from this list is category theory itself. Morphisms between categories are known as *functors*.

**Definition 6.1.** A *functor*  $F: C \to D$  from a category C to a category D is given by the following data:

- a function  $Ob(F): Ob(C) \rightarrow Ob(D)$  that sends objects of C to objects of D;
- for any object  $X, Y \in C$  we have a function  $Mor_{\mathsf{F}}(X, Y): Mor_{\mathsf{C}}(X, Y) \to Mor_{\mathsf{D}}(\mathsf{F}(X), \mathsf{F}(Y))$ , i.e.,  $\mathsf{F}$  maps a morphism  $f: X \to Y$  to a morphism  $\mathsf{F}(f): \mathsf{F}(X) \to \mathsf{F}(Y)$ . This data must satisfy the following properties:
- for any morphisms  $f: X \to Y$  and  $g: Y \to Z$  in the category  $\mathsf{C}$  we have  $\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$ , i.e.,  $\mathsf{F}$  preserves composition;
- for any object  $X \in \mathsf{C}$  we have  $\mathsf{F}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{F}(X)}$ , i.e.,  $\mathsf{F}$  preserves identity morphisms.

**Remark 6.2.** Thus, to define a functor  $F: C \to D$  one must specify an object  $F(X) \in D$  for any object  $X \in C$ , a morphism  $F(f): F(X) \to F(Y)$  of D for any morphism  $f: X \to Y$  in C such that the operation of composition and identity morphisms are preserved.

**Example 6.3.** Forgetful functors are one of the easiest functors to define. For instance, the forgetful functor  $Ab \rightarrow Set$  is defined as follows. We send an abelian group A to its underlying set with the algebraic operations discarded. A homomorphism of abelian groups  $X \rightarrow Y$  is a function between the underlying sets, hence already a morphism of sets. The forgetful functor preserves compositions because morphisms of abelian groups are composed by composing their underlying functions. The identity morphism is preserved for the same reason.

**Remark 6.4.** Although one can give rigorous definitions of the adjective "forgetful", typically this term is used in an informal manner, with somewhat imprecise corner cases.

**Example 6.5.** In an entirely analogous fashion we have forgetful functors  $\text{Group} \rightarrow \text{Set}$ ,  $\text{Ring} \rightarrow \text{Set}$ ,  $\text{Top} \rightarrow \text{Set}$ ,  $\text{Ban} \rightarrow \text{Set}$ ,  $\text{Ban}_1 \rightarrow \text{Set}$ , etc. Somewhat less obviously one also has forgetful functors  $\text{Vect}_k \rightarrow \text{Ab}$ ,  $\text{Ring} \rightarrow \text{Ab}$ ,  $\text{Ban} \rightarrow \text{Met}$ ,  $\text{Ban}_1 \rightarrow \text{Met}_1$  (but *not*  $\text{Ban} \rightarrow \text{Met}_1$ ),  $W^* \rightarrow C^*$ ,  $CW^* \rightarrow CC^*$ . For instance, any vector space has an underlying abelian group, and linear maps of vector spaces are maps of abelian

groups with additional properties (namely, preservation of multiplication), hence we have a forgetful functor  $\mathsf{Vect}_k \to \mathsf{Ab}$ .

**Example 6.6.** By definition of HoTop we have a functor  $\text{Top} \rightarrow \text{HoTop}$ . Applying this functor can be seen as discarding the nontopological information.

**Example 6.7.** We have the obvious *inclusion functors* (another informal term)  $Met_1 \rightarrow Met_1 \rightarrow Ban_1 \rightarrow Ban$ .

## 6.8. Algebra

**Example 6.9.** The free group functor  $\operatorname{Free}_{\operatorname{Group}}$ : Set  $\to$  Group sends a set S to the free group  $\operatorname{Free}_{\operatorname{Group}}(S)$  on the generating set S. A function  $f: X \to Y$  is sent to the (unique) homomorphism of free groups  $\operatorname{Free}_{\operatorname{Group}}(f)$ :  $\operatorname{Free}_{\operatorname{Group}}(X) \to \operatorname{Free}_{\operatorname{Group}}(Y)$  that sends elements of  $X \subset \operatorname{Free}_{\operatorname{Group}}(X)$  to their images in  $Y \subset \operatorname{Free}_{\operatorname{Group}}(Y)$  via f. (Here we used the universal property of free groups to extend the above map to a homomorphism of groups.) Used in 6.9.

**Example 6.10.** The group of units functor  $-\times$ : Ring  $\rightarrow$  Group sends a ring R to its group  $R^{\times}$  of invertible elements, i.e., elements  $x \in R$  for which there is  $y \in R$  such that 1 = xy = yx. Any homomorphism of rings  $R \rightarrow S$  preserves invertible elements and therefore induces a homomorphism of groups  $U(R) \rightarrow U(S)$ .

**Example 6.11.** The polynomial ring functor -[x]: Ring  $\rightarrow$  Ring sends a ring R to the ring R[x] of polynomials in a single variable x with coefficients in R. A homomorphism of rings  $R \rightarrow S$  is sent to the homomorphism of rings  $R[x] \rightarrow S[x]$  given by applying it to each coefficient.

**Nonexample 6.12.** The group center construction sends a group G to its center Z(G) defined as  $\{g \in G \mid \forall x \in G : gx = xg\}$ . A homomorphism of groups  $G \to H$  does not restrict to a homomorphism of groups  $Z(G) \to Z(H)$ . For instance, take  $G = \mathbb{Z}/2$ ,  $H = \Sigma_3$ , and  $G \to H$  sends the nontrivial element of  $\mathbb{Z}/2$  to a permutation in  $\Sigma_3$  that permutes two of the elements and leaves the third one untouched. We have  $Z(\mathbb{Z}/2) = \mathbb{Z}/2$ , but  $Z(\Sigma_3) = \{1\}$ .

**Example 6.13.** The exterior algebra functor  $\Lambda: \operatorname{Vect}_k \to \operatorname{Alg}_k$  sends a k-vector space V to its exterior algebra  $\Lambda V$  and a linear map  $V \to W$  to the induced homomorphism of algebras  $\Lambda V \to \Lambda W$ .

**Example 6.14.** Fix a field k. The group algebra functor  $k[-]: \operatorname{Group} \to \operatorname{Alg}_k$  sends a group G to its group algebra k[G] and a homomorphism of groups  $G \to H$  to the induced homomorphism of algebras  $k[G] \to k[H]$ . The group of units functor  $-^{\times}: \operatorname{Alg}_k \to \operatorname{Group}$  sends a k-algebra A to the group of its units (invertible elements)  $A^{\times}$ , with the induced multiplication.

## 6.15. Topology

**Example 6.16.** The fundamental group functor  $\mathsf{Top}_* \to \mathsf{Group}$  was defined in Example 5.9.

#### 6.17. Measure theory

**Example 6.18.** The functor  $L^1$ : LocMeas  $\rightarrow Ban_1$  sends a localizable measurable space (X, M, N) to the Banach space of finite measures on (X, M, N), defined as the Banach space of countably additive maps  $\mu: M \rightarrow \mathbf{C}$  whose restriction on N vanishes. (The norm of  $\mu$  is  $|\mu|(X) = \sup_{|f| \leq 1} \int f d\mu$ .) A morphism  $f: (X, M, N) \rightarrow (X', M', N')$  maps a finite measure  $\mu$  on (X, M, N) to the finite measure  $f_*\mu$  on (X', M', N') that sends  $m' \in M'$  to  $\mu(f^{-1}(m'))$ . Used in 6.19, 7.44.

**Remark 6.19.** More traditionally, one could define  $L^1(X, M, \mu)$ , where X is a set, M is a  $\sigma$ -algebra of subset of X, and  $\mu$  is a finite measure on (X, M), i.e., a countably additive map  $M \to \mathbb{C}$ , as the Banach space of equivalence classes of measurable functions f such that  $\int |f| d\mu$  exists and is finite. Equivalences classes are taken with respect to the  $\sigma$ -ideal  $N = \{m \in M \mid \mu(m) = 0\}$ . Assume that (X, M, N) is localizable. We have an isomorphism  $L^1(X, M, \mu) \to L^1(X, M, N)$  that sends  $f \mapsto f \cdot \mu$ , where  $(f \cdot \mu)(m) = \int_m f d\mu$ . Thus, as long as two different measures  $\mu$  and  $\mu'$  have the same  $\sigma$ -ideal N of sets of measure 0, the Banach spaces  $L^1(X, M, \mu)$  and  $L^1(X, M, \mu')$  are isometrically isomorphic. (This also follows directly from the Radon– Nikodym theorem, which holds for localizable measurable spaces.)

**Example 6.20.** The *Haar measurable space functor* HaarMeas: LocCompHausGroup<sub>Open</sub>  $\rightarrow$  Meas sends a locally compact Hausdorff topological group G to a measurable space (X, M, N), where X is the underlying

set of G, N is the  $\sigma$ -ideal of sets of measure 0 with respect to some (hence all) left (or right) Haar measure on G, and M is the  $\sigma$ -algebra generated by N and open sets. (A left Haar measure is a left-invariant Radon measure on G, or, equivalently, a left-invariant continuous functional on the space of compactly supported continuous functions on G equipped with the topology of uniform convergence on compact subsets.) An open continuous homomorphism of locally compact groups is sent to the equivalence class of its underlying function. (Negligible sets are preserved under preimages of open maps.)

**Example 6.21.** The functor MeasConv: LocCompHausGroup  $\rightarrow$  Alg<sub>R</sub> sends a locally compact Hausdorff topological group G to the real algebra of bounded measures on G, with the product of  $\mu$  and  $\nu$  given by the convolution  $\mu * \nu$  of measures. A continuous homomorphism of groups  $f: G \rightarrow H$  is mapped to the homomorphism of real algebras MeasConv $(G) \rightarrow$  MeasConv(H) that sends a bounded measure  $\mu$  on G to its pushforward  $f_*\mu$  along f, defined as  $(f_*\mu)(E) = \mu(f^{-1}(E))$  for any open set E in H. Used in 6.21, 7.16.

# 6.22. Smooth manifolds and Lie groups

**Example 6.23.** The functor  $T: \mathsf{Man} \to \mathsf{VBun}$  sends a smooth manifold M to its *tangent bundle* TM and a smooth map  $f: X \to Y$  to the induced tangent map  $TM \to TN$ .

**Example 6.24.** Given a manifold  $M \in Man$  with a basepoint  $* \in M$ , we define a functor fiber:  $VBun_M \rightarrow Vect_k$  by sending a vector bundle over M to its fiber over  $* \in M$  and a morphism of vector bundles to the induced morphism of fibers.

**Example 6.25.** The functor LieGroup  $\rightarrow$  Vect<sub>R</sub> is defined as the composition LieGroup  $\rightarrow$  Man  $\rightarrow$  VBun  $\rightarrow$  Vect<sub>k</sub>, where the first functor is the forgetful functor, the second functor is the tangent functor T, and the third functor is the fiber functor with respect to the identity element of the Lie group. As shown in any book on Lie groups, this functor factors as the composition LieGroup  $\rightarrow$  LieAlg<sub>R</sub>  $\rightarrow$  Vect<sub>R</sub>, where the second functor is the forgetful functor.

#### 6.26. Category theory

Recall that not every class is a set. For instance, by Russell's paradox, the class of all sets is not a set.

**Definition 6.27.** A category C is *small* if the class of its objects is a set.

**Example 6.28.** The category Set is not a small category. The full subcategory of Set on objects that are subsets of some fixed set X is a small category.

**Definition 6.29.** The category Cat of small categories has small categories as objects and functors as morphisms. Composition of morphisms  $G: D \to E$  and  $F: C \to D$  is given by the *composition of functors*: the functor  $G \circ F$  sends an object  $X \in C$  to the object  $G(F(X)) \in E$  and a morphism  $f: X \to Y$  in C to the morphism  $G(F(f)): G(F(X)) \to G(F(Y))$  in E. The identity morphism on C is the identity functor  $id_C: C \to C$ such that  $id_C(X) = X$  and  $id_C(f) = f$ . Used in 6.31, 6.32, 7.3, 9.0\*, 9.36.

**Remark 6.30.** The above definition restricts to *small* categories because functors between small categories form a set (as opposed to a mere class) and we require a *set* (not a class) of morphisms between any pair of objects. There is no category of categories because functors (say)  $\mathsf{Set} \to \mathsf{Set}$  form a class that is not a set: this class contains *constant* functors, i.e., functors  $\mathsf{Set} \to \mathsf{Set}$  that send any object  $X \in \mathsf{Set}$  to some fixed set A and any morphism f in  $\mathsf{Set}$  to  $\mathsf{id}_A$ . Thus there are as many constant functors as there are sets, so in particular the class of functors  $\mathsf{Set} \to \mathsf{Set}$  contains a subclass isomorphic to the class of all sets, and therefore cannot be a set by Russell's paradox. This problem is easily circumvented by introducing *conglomerates*, which are collections that can contain classes (and not just sets) as elements. This yields a "huge" category  $\mathsf{CAT}$  of categories. Used in 7.4, 7.5, 9.0\*.

**Definition 6.31.** The functor  $Ob: Cat \rightarrow Set$  sends a category C to its set of objects and a functor to its underlying function on objects.

**Definition 6.32.** The functor  $\pi_0: Cat \to Set$  sends a category C to the set  $Ob(C)/\sim$ , where  $\sim$  denotes the equivalence relation of isomorphism of objects. Used in 9.0\*, 9.1, 9.4, 9.5\*, 9.7, 10.36, 10.36\*.

**Definition 6.33.** Given a category C, the functor  $Mor_C: C^{op} \times C \to Set$  (see Definition 7.1 for a definition of  $C^{op}$ ) sends a pair of objects (X, Y) in C to the set  $Mor_C(X, Y)$ . Morphisms  $(X, Y) \to (X', Y')$  are pairs of

functions  $f: X' \to X$  and  $g: Y \to Y'$ , which are sent to the induced function  $Mor_{\mathsf{C}}(X, Y) \to Mor_{\mathsf{C}}(X', Y')$ . (In the above  $\times$  denotes the product of categories  $\mathsf{C}$  and  $\mathsf{C}^{\mathsf{op}}$ , a construction that will be explained below.)

Many examples of categories given above have "sets with structures" as objects and "functions that preserve the structure" as morphisms, e.g., groups and group homomorphisms, topological spaces and continuous maps, smooth manifolds and smooth maps, etc. We do not give a definition of a "structure" here, but see Chapter IV of Bourbaki's *Theory of Sets* for one possible definition. We can, however, rather easily formalize such types of categories as *concrete categories*.

**Definition 6.34.** A functor  $F: C \to D$  is *faithful* if for any pair of object  $X, Y \in C$  the induced function  $Mor_{C}(X, Y) \to Mor_{D}(F(X), F(Y))$  is injective.

In other words, F is faithful if F(f) = F(g) implies f = g for any pair of morphisms  $f, g: X \to Y$ .

**Definition 6.35.** A concrete category is a pair (C, U), where C is a category and  $U: C \to Set$  is a faithful functor. A category C is concretizable if there is U such that (C, U) is a concrete category.

**Examples 6.36.** The following categories are concrete for the obvious choice of the functor U, the underlying set functor:

- Set;
- Group, Ring, Vect<sub>k</sub>, any other category of algebraic objects;
- Met, Met<sub>1</sub>;
- Top, TopGroup, TopVect<sub>k</sub>;
- Ban, Ban<sub>1</sub>, C<sup>\*</sup>, W<sup>\*</sup>;
- Man, LieGroup, VBun.

**Example 6.37.** A given category C can admit many different functors U that make it concrete. For instance, for Ban<sub>1</sub> apart from the underlying set functor we can take the functor  $U(X) = \{x \in X \mid ||x|| \le 1\}$ .

**Example 6.38.** We show that  $\operatorname{Set}^{\operatorname{op}}$  (see Definition 7.1 below) is concrete by defining a faithful functor  $U:\operatorname{Set}^{\operatorname{op}} \to \operatorname{Set}$ . Set  $U(X) = 2^X$ , the set of all subsets of X. For a morphism  $q: X \to Y$  in  $\operatorname{Set}^{\operatorname{op}}$  (i.e., a function  $f: Y \to X$ ), we have to define a function  $U(q): U(X) \to U(Y)$ , i.e., a function  $2^X \to 2^Y$ . We take  $f^{-1}$ , the function that sends a subset  $A \subset X$  to its preimage  $f^{-1}(A) = \{y \in Y \mid g(y) \in A\}$ . We have  $\operatorname{id}_X^{-1}(A) = A$  and  $(f_2f_1)^{-1}(A) = f_1^{-1}(f_2^{-1}(A))$  (the order of  $f_1$  and  $f_2$  is reversed because of the contravariance). Thus we indeed have defined a functor  $U:\operatorname{Set}^{\operatorname{op}} \to \operatorname{Set}$ . If for any set  $A \subset X$  we have  $f^{-1}(A) = g^{-1}(A)$  for two functions  $f: Y \to X$  and  $g: Y \to X$ , then f = g, so the functor is faithful. This proves that  $\operatorname{Set}^{\operatorname{op}}$  is concrete.

**Example 6.39.** We could try to turn the category Meas into a concrete category in the most naive way. Assign to a measurable space (X, M, N) the set X and to a morphism of measurable spaces  $(X, M, N) \rightarrow (X', M', N')$  coming from some function  $f: X \rightarrow X'$  this function f. This does not give us a functor Meas  $\rightarrow$  Set for the very simple reason: a morphism  $(X, M, N) \rightarrow (X', M', N')$  is an equivalence class containing many different functions, and it is unclear which one we should take so that composition is respected. The concreteness of Meas can be established by virtue of a faithful functor MeasAlg: Meas<sup>op</sup>  $\rightarrow$  BoolAlg constructed in Example 7.35 and the fact that BoolAlg (and hence BoolAlg<sup>op</sup>) is concrete.

Example 6.40. A theorem of Freyd says that HoTop is not concretizable.

#### 7 Contravariance and duality

The following construction, despite its apparent simplicity, plays a very important role in category theory.

**Definition 7.1.** Given a category C its *opposite category*  $C^{op}$  has the same objects as C and  $Mor_{C^{op}}(X, Y) = Mor_{C}(Y, X)$ . Composition is the map

$$\begin{aligned} \operatorname{Mor}_{\mathsf{C}^{\operatorname{op}}}(Y,Z) \times \operatorname{Mor}_{\mathsf{C}^{\operatorname{op}}}(X,Y) &= \operatorname{Mor}_{\mathsf{C}}(Z,Y) \times \operatorname{Mor}_{\mathsf{C}}(Y,X) \\ &\cong \operatorname{Mor}_{\mathsf{C}}(Y,X) \times \operatorname{Mor}_{\mathsf{C}}(Z,Y) \to \operatorname{Mor}_{\mathsf{C}}(Z,X) = \operatorname{Mor}_{\mathsf{C}^{\operatorname{op}}}(X,Z). \end{aligned}$$

Used in 6.33, 6.38.

**Definition 7.2.** Given a functor  $F: C \to D$ , its *opposite functor*  $F^{op}: C^{op} \to D^{op}$  has the same object function  $Ob(F^{op}): Ob(C^{op}) \to Ob(D^{op})$  as F, using  $Ob(C^{op}) = Ob(C)$  and  $Ob(D^{op}) = Ob(D)$ , so  $Ob(F^{op}) = Ob(F)$  makes sense. The morphism function  $F^{op}_{X,X'}: Mor_{C^{op}}(X,X') \to Mor_{D^{op}}(Y,Y')$  is the same function as  $F_{X',X}: Mor_{C}(X',X) \to Mor_{D}(Y',Y)$ .

The above two definitions combine together in a single functor.

 $\begin{array}{l} \textbf{Definition 7.3.} \ \ The \ functor \ \textbf{op: Cat} \rightarrow \textbf{Cat} \ \ sends \ a \ category \ \textbf{C} \ \ to \ the \ category \ \textbf{C}^{op} \ and \ a \ functor \ \textbf{F: C} \rightarrow \textbf{D} \ to \ the \ functor \ \textbf{F}^{op}: \ \textbf{C}^{op} \ \rightarrow \ \textbf{D}^{op}. \ \ u_{sed \ in \ 6.33, \ 6.38, \ 6.39, \ 7.1, \ 7.2, \ 7.3, \ 7.4, \ 7.5, \ 7.5^*, \ 7.6, \ 7.7, \ 7.10, \ 7.11, \ 7.12, \ 7.13, \ 7.15, \ 7.16, \ 7.18, \ 7.23, \ 7.24, \ 7.26, \ 7.30, \ 7.31, \ 7.35, \ 7.44, \ 7.45, \ 7.47, \ 7.48, \ 7.49, \ 8.2, \ 8.16, \ 9.9, \ 9.10, \ 9.12, \ 9.13, \ 9.32, \ 10.5, \ 10.7, \ 10.14, \ 10.19, \ 10.22, \ 12.1, \ 14.0^*, \ 17.4, \ 20.3^*, \ 25.1, \ 25.2, \ 25.4, \ 26.0^*, \ 26.1, \ 26.2, \ 26.5^*, \ 26.6, \ 26.7, \ 26.7^*, \ 27.0^*, \ 27.1, \ 27.2, \ 27.4, \ 27.7, \ 29.2, \ 30.6, \ 30.7, \ 30.11^*, \ 30.11^*, \ 30.12, \ 31.1, \ 32.5. \end{array}$ 

**Remark 7.4.** We have  $op \circ op = id_{CAT}$ .

**Remark 7.5.** Abusing the language, we may also talk about a "functor" op:  $CAT \rightarrow CAT$  defined in the same way. (It is a "functor" and not a functor because CAT is a "category" and not a category.) However,  $C^{op}$  and  $F^{op}$  always make sense for an individual category C or a functor F.

Recall that an object  $X \in C$  can be identified with a functor  $X: 1 \to C$ , where 1 denotes any category with one object and one morphism. If 1 is such a category, then so is  $1^{\circ p}$ . In particular,  $X^{\circ p}: 1^{\circ p} \to C^{\circ p}$  itself specifies an object in  $C^{\circ p}$ , which we again denote by  $X^{\circ p}$ . We have  $(X^{\circ p})^{\circ p} = X$ .

Likewise, a morphism  $f: X \to Y$  in C can be identified with a functor  $f: 2 \to C$ , where 2 denotes any category with two objects and a single nonidentity morphism, which goes from one object to the other. If 2 is such a category, then so is  $2^{op}$ . In particular,  $f^{op}: 2^{op} \to C^{op}$  itself specifies a morphism in  $C^{op}$ , which we again denote by  $f^{op}$ . We have  $(f^{op})^{op} = f$ .

Below we will see many examples when some category C is equivalent (term defined below) to the opposite category of some other category D. Such an equivalence is implemented by a *contravariant functor*.

**Definition 7.6.** A contravariant functor from C to D is a functor  $C^{op} \rightarrow D$ , or, equivalently,  $C \rightarrow D^{op}$ .

**Remark 7.7.** One way to see the equivalence between functors  $F': C^{op} \to D$  and  $F'': C \to D^{op}$  is to expand the definition: a contravariant functor F from C to D assigns to every object  $X \in C$  an object  $F(X) = F'(X^{op}) = (F''(X))^{op} \in D$  and to every morphism  $f: X \to Y$  in C a morphism  $F(f) = F'(f^{op}) = (F''(f))^{op}$ :  $F(Y) \to F(X)$  in D. Composition and identity morphisms must be respected.

**Remark 7.8.** Of course, contravariant functors are just a particular case of the general notion of functor (sometimes referred to as a *covariant* functor). The justification for introducing this new bit of terminology is that many functors naturally arise as contravariant functors, i.e., their domain or codomain is the opposite category of some previously defined category.

We now give several examples of contravariant functors.

#### 7.9. Algebra

**Example 7.10.** The functor  $O_{Set}$ : Set<sup>op</sup>  $\rightarrow$  Alg<sub>R</sub> sends a set S to the real algebra of functions on S and a function  $f: S \rightarrow T$  to the homomorphism of real algebras given by precomposition with f, i.e.,  $S \rightarrow T \rightarrow \mathbf{R}$ .

**Example 7.11.** Fix a field k. The dual vector space functor DVS:  $\operatorname{Vect}_k^{\operatorname{op}} \to \operatorname{Vect}_k$  sends a vector space V to  $V^* := \operatorname{Hom}(V, k)$  and a linear map  $V \to V'$  to the induced map  $\operatorname{Hom}(V', k) \to \operatorname{Hom}(V, k)$ . Used in 10.5.

**Example 7.12.** The functor  $2^-: \mathsf{Set}^{\mathsf{op}} \to \mathsf{ComplAtomBoolAlg}$  sends a set S to the complete atomic Boolean algebra  $2^S$  of functions  $S \to 2 = \{0, 1\}$  (equivalently, the complete atomic Boolean algebra of subsets of S) and a function  $f: S \to T$  to the homomorphism of complete atomic Boolean algebras  $f^{-1}: 2^T \to 2^S$ .

**Example 7.13.** The functor  $\text{Spec}_{A \text{tom}}$ : ComplAtomBoolAlg<sup>op</sup>  $\rightarrow$  Set sends a complete atomic Boolean algebra A to its set of atoms (which can be defined as morphisms  $A \rightarrow 2$ ) and a continuous homomorphism  $B \rightarrow A$  to the induced function given by the composition  $B \rightarrow A \rightarrow 2$ .

## 7.14. General topology

**Example 7.15.** The functor  $O_{Ban}$ :  $\mathsf{Top}^{\mathsf{op}} \to \mathsf{Ban}_{\mathbf{R}}$  sends a topological space X to the Banach space of bounded continuous functions  $X \to \mathbf{R}$  with the pointwise operations. A continuous map  $X \to Y$  is mapped to the contractive linear map of Banach spaces  $O_{Ban}(Y) \to O_{Ban}(X)$  given by the composition  $X \to Y \to \mathbf{R}$ . Used in 7.15, 10.7, 10.11\*.

**Example 7.16.** The functor  $O_{Conv}$ : CompHausGroup<sup>op</sup>  $\rightarrow Alg_R$  sends a compact Hausdorff topological group G to the real algebra of continuous functions on G, with the multiplication given by the *convolution* of functions with respect to the unique Haar measure  $\mu$  on G such that  $\mu(G) = 1$ . A continuous homomorphism of groups  $f: G \rightarrow H$  is mapped to the homomorphism of real algebras  $O_{Conv}(H) \rightarrow O_{Conv}(G)$  that sends a function p on H to its *pullback*  $f^*p$  along f, defined as  $f^*p = p \circ f$ . This example should be contrasted with the *covariant* functor of Example 6.21, which was defined on measures instead of functions. This distinction is essential: we can pushforward measures and pullback functions, but not vice versa. Measures cannot be pulled back unless we have additional data (such as a relative measure on f). The pushforward of a function can be defined as a measure, which need not be a function, e.g., it can be the Dirac  $\delta$ -measure. Used in 7.16.

#### 7.17. Topological algebra

**Example 7.18.** The functor PD: LocCompHausAb<sup>op</sup>  $\rightarrow$  LocCompHausAb sends a locally compact Hausdorff abelian topological group G to the topological group Hom(G, U(1)), whose elements are continuous homomorphisms  $G \rightarrow U(1)$ , equipped with the compact-open topology (whose subbasis consists of functions that map a given compact subset  $K \subset G$  to a given open subset  $V \subset U(1)$ . A continuous homomorphism of group  $G \rightarrow G'$  induces a continuous homomorphism Hom $(G', U(1)) \rightarrow$  Hom(G, U(1)). Used in 9.32.

#### 7.19. Banach spaces

We give some examples related to the Hahn–Banach theorem. We start by defining one of the categories involved. Everything below can be done either for real or complex spaces.

**Definition 7.20.** The category Ball has *unit balls* as objects, defined as pairs (V, B) consisting of a Hausdorff locally convex topological vector space V and a Hausdorff topological subspace  $B \subset V$  such that B is *balanced* (i.e.,  $0 \in B$  and for any  $x \in B$  and number t such that  $|t| \leq 1$  we have  $tx \in B$ ), and B is *convex* (i.e., for any  $x, y \in B$  and real numbers  $r \geq 0$  and  $s \geq 0$  such that  $r + s \leq 1$  we have  $rx + sy \in B$ ). Morphisms  $(V, B) \to (V', B')$  are continuous linear maps  $V \to V'$  that send B to B'. The category Ball is also known as the category of *Saks spaces*. Used in 7.20, 7.21, 7.22.

**Definition 7.21.** CompBall is the full subcategory of Ball consisting of balls (V, B) such that B is compact. It is also known as the category of *Waelbroeck spaces*. Used in 7.24, 7.26, 7.27, 9.9, 9.10, 10.11\*.

**Example 7.22.** We have a functor  $\mathsf{Ban}_1 \to \mathsf{Ball}$  that sends a Banach space X to the (typically noncompact) unit ball  $(X, X_{\leq 1})$ , where  $X_{\leq 1}$  denotes the subset of X consisting of elements of norm at most 1. A contractive map  $X \to X'$  of Banach spaces is sent to the induced map  $(X, X_{\leq 1}) \to (X', X'_{\leq 1})$ , the contractivity property guaranteeing that the unit ball is preserved.

**Example 7.23.** We have functors  $DBS: Ban_1^{op} \to Ban_1$  and  $DTVS: Ban_1^{op} \to TopVect$ . Given some  $X \in Ban_1^{op}$ , the Banach space DBS(X) is the Banach space of bounded linear functionals on X equipped with the induced Banach norm, whereas DTVS(X) is the Hausdorff locally convex topological vector space of bounded linear functionals on X equipped with the weak-\* topology induced by X. Both functors are

typically denoted  $X \mapsto X^*$ , and the ambiguity must be resolved from the context. On morphisms both functors are defined using precomposition, as usual. Used in 7.23.

**Example 7.24.** The dual unit ball functor DUB:  $\operatorname{Ban}_{1}^{\operatorname{op}} \to \operatorname{CompBall}$  is defined as follows. Given a Banach space X consider the vector space  $X^*$  of continuous linear functionals on X equipped with the weak-\* topology, i.e., the coarsest topology in which every function on  $X^*$  given by evaluation on some fixed element  $x \in X$  is continuous. We define  $\operatorname{DUB}(X) = (X^*, X^*_{\leq 1})$ , where  $X^*_{\leq 1}$  denotes the set of functionals of norm at most 1. A continuous linear map  $X \to Y$  of Banach spaces induces a continuous linear map  $Y^* \to X^*$ , which restricts to  $Y^*_{\leq 1} \to X^*_{\leq 1}$ . Used in 7.24, 7.25, 7.27, 9.0\*, 9.9, 10.11\*.

**Remark 7.25.** The traditional Hahn–Banach theorem can be interpreted as saying that an inclusion  $A \subset B$  of Banach spaces is sent by the functor DUB to a surjective map of unit balls.

**Example 7.26.** The functor  $O_{Ball}$ : CompBall<sup>op</sup>  $\rightarrow$  Ban<sub>1</sub> is defined as follows. A unit ball (V, B) is sent to the Banach space of linear functionals f on V. The norm of a functional is defined as the supremum of its absolute value on B. A morphism  $(V, B) \rightarrow (V', B')$  induces a contractive map from linear functionals on V' to linear functionals on V given by the composition  $V \rightarrow V' \rightarrow \mathbf{C}$ . Used in 7.27, 9.0<sup>\*</sup>, 9.9.

**Remark 7.27.** Below we will see that DUB and  $O_{Ball}$  are mutually inverse to each other in the appropriate sense and identify (in the appropriate sense)  $Ban_1$  and CompBall. This is a strengthening of the traditional Hahn–Banach theorem.

#### 7.28. Operator algebras

We now define the two functors that together form the famous  $Gelfand \ duality$  for commutative C\*algebras and compact Hausdorff spaces.

**Definition 7.29.** The category  $C^*$  of  $C^*$ -algebras is defined as follows. Its objects are  $C^*$ -algebras, i.e., complex algebras A equipped with an involution (i.e., a morphism of abelian groups  $*: A \to A$  such that  $a^{**} = a, 1^* = 1, (ab)^* = b^*a^*$ , and  $(\lambda a)^* = \overline{\lambda}a^*$  for any  $\lambda \in \mathbb{C}$ ) and a norm that is compatible with the involution and multiplication (i.e.,  $||1|| = 1, ||ab|| \leq ||a|| \cdot ||b||, ||a^*a|| = ||a^*|| \cdot ||a||$ ) such that the underlying normed vector space is complete, i.e., a Banach space. Morphisms  $f: A \to B$  are morphisms of complex algebras that preserve the involution, i.e.,  $f(a^*) = f(a)^*$ . (One can prove that f is *automatically* contractive.) The category  $\mathbb{CC}^*$  of *commutative*  $\mathbb{C}^*$ -algebras is the full subcategory of  $\mathbb{C}^*$  consisting of  $\mathbb{C}^*$ -algebras that are commutative, i.e., ab = ba.

**Definition 7.30.** The functor  $O_{Cont}$ : CompHaus<sup>op</sup>  $\rightarrow CC^*$  sends a compact Hausdorff space X to the commutative C\*-algebra  $O_{Cont}(X)$  of complex-valued continuous functions on X equipped with the pointwise algebra structure, involution given by the complex conjugation, and norm given by the supremum of the absolute value. A continuous map of compact Hausdorff spaces  $f: X \rightarrow Y$  is sent to the morphism of commutative C\*-algebras  $O_{Cont}(Y) \rightarrow O_{Cont}(X)$  given by precomposition with f, i.e., a continuous function  $Y \rightarrow \mathbf{C}$  is mapped to the continuous function  $X \rightarrow Y \rightarrow \mathbf{C}$ . Used in 7.30, 7.32, 9.12, 9.15.

**Definition 7.31.** The *Gelfand spectrum* functor  $\operatorname{Spec}_{CC}: (\operatorname{CC}^*)^{\operatorname{op}} \to \operatorname{CompHaus}$  sends a commutative C\*algebra A to the compact Hausdorff topological space  $\operatorname{Spec}_{CC}(A)$  whose points are homomorphisms of C\*-algebras  $A \to \mathbb{C}$  and a set  $S \subset \operatorname{Spec}_{CC}(A)$  is *closed* if there is a morphism of C\*-algebras  $g: A \to B$  (for some B) such that  $s: A \to \mathbb{C}$  is in S if and only if s = tg for some morphism  $t: B \to \mathbb{C}$ . (Of course, one must show that the resulting object is a compact Hausdorff topological space.) A morphism  $f: A \to B$  of commutative C\*-algebras is sent to the continuous map  $\operatorname{Spec}_{CC}(f): \operatorname{Spec}_{CC}(B) \to \operatorname{Spec}_{CC}(A)$  that sends a point  $b \in \operatorname{Spec}_{CC}(B)$  (i.e., a morphism  $B \to \mathbb{C}$ ) to the composition  $A \to B \to \mathbb{C}$ , which is a point in  $\operatorname{Spec}_{CC}(A)$ . (Again, one must show that this function is a continuous map.) Used in 7.31, 7.32, 9.12, 9.15, 9.16.

**Remark 7.32.** Once again, we will see below that  $\text{Spec}_{CC}$  and  $O_{Cont}$  form an *equivalence* of categories between CompHaus and CC<sup>\*</sup>.

#### 7.33. Measure theory

We recall that the category Meas has triples (X, M, N) as objects and equivalence classes of measurable maps as morphisms.

The following functor exhibits sets as discrete measurable spaces.

**Example 7.34.** The functor Disc: Set  $\rightarrow$  Meas sends a set S to  $(S, 2^S, \emptyset)$  and a function  $f: S \rightarrow T$  to the morphism  $(S, 2^S, \emptyset) \rightarrow (T, 2^T, \emptyset)$  given by the equivalence class of f (which in this case contains only f itself). Used in 7.43, 9.17.

We establish a connection between measurable spaces and Boolean algebras.

**Example 7.35.** The functor MeasAlg: Meas<sup>op</sup>  $\rightarrow$  BoolAlg sends a measurable space (X, M, N) to the Boolean algebra M/N of equivalences classes of measurable sets modulo negligible sets and a morphism of measurable spaces  $f:(X, M, N) \rightarrow (X', M', N')$  to the induced morphism  $M'/N' \rightarrow M/N$  of Boolean algebras given by the preimage map  $f^{-1}$ . Recall that f is an equivalence class of measurable functions and equivalent functions induce the same morphism  $M'/N' \rightarrow M/N$ , see Theorem 324A in Fremlin's Measure Theory for more details. Used in 6.39, 8.16.

We now explain how to get measurable spaces from topological spaces.

**Definition 7.36.** The functor Borel: Top  $\rightarrow$  Meas sends a topological space X to the measurable space  $(X, \operatorname{Borel}_X, \{\emptyset\})$ , where  $\operatorname{Borel}_X$  is the  $\sigma$ -algebra of Borel subsets of X (i.e., the  $\sigma$ -algebra generated by open subsets of X). A continuous map  $f: X \rightarrow Y$  is sent to the equivalence class of f. (Any equivalence class consists of a single element, and measurable functions are precisely continuous functions.) The functor Baire: Top  $\rightarrow$  Meas sends a topological space X to the measurable space  $(X, \operatorname{Baire}_X, \{\emptyset\})$ , where  $\operatorname{Baire}_X$  is the  $\sigma$ -algebra of Baire subsets of X, which is generated by functionally open subsets of X, i.e., sets of the form  $f^{-1}(0, \infty)$  for some continuous function  $f: X \rightarrow \mathbf{R}$ . Used in 7.36, 7.38, 7.44, 9.16, 10.7, 10.11\*.

For us, the following *different* construction of a measurable space from a topological space will also be of use. It formalizes the well-known set of analogies between negligible sets and meager sets, see Oxtoby's *Measure and Category*. (*Meager* sets are defined as countable unions of nowhere dense sets, i.e., sets whose closure has empty interior.)

**Definition 7.37.** The category Top<sub>Open</sub> has topological spaces as objects and continuous *open* maps as morphisms. A map is *open* if the image of any open set is an open set. Used in 7.38, 7.42, 7.45, 28.5.

**Definition 7.38.** The functor BorelMeager:  $\operatorname{Top}_{\mathsf{Open}} \to \mathsf{Meas}$  turns a topological space X into a measurable space  $(X, \operatorname{BorelMeager}_X, \operatorname{Meager}_X)$ , where  $\operatorname{BorelMeager}_X$  is the  $\sigma$ -algebra generated by open and meager subsets of X and  $\operatorname{Meager}_X$  is the  $\sigma$ -ideal of meager subsets of X. A continuous map  $X \to Y$  is sent to the equivalence class of the underlying function, which is measurable because preimages of meager subsets are meager, which in its turn follows from the fact that preimages of closed subsets with empty interior again have empty interior because the map is open. The functor BaireMeager is defined in exactly the same way, but with Borel sets replaced by Baire sets as defined in Definition 7.36. Used in 7.38, 7.42.

We now establish a connection to functional analysis and operator algebras. In practice, Meas has extremely pathological objects that make most of the familiar theorems of measure theory false. The condition of  $\sigma$ -finiteness is often used to remedy this problem, but we use a less restrictive property.

**Definition 7.39.** The category LocMeas of *localizable* measurable spaces is the full subcategory of Meas consisting of measurable spaces (X, M, N) such that the factoralgebra M/N is *complete*: every subset  $S \subset M/N$  has a supremum in M/N. Used in 6.18, 7.43, 7.44, 7.45, 9.13, 9.15, 11.20\*, 33.6.

**Remark 7.40.** Translated in the language of underlying sets, a measurable space is localizable if for any subset  $F \subset M$  (not necessarily countable) there is  $X \in M$  such that for all  $Y \in F$  we have  $Y \setminus X \in N$  and if  $X' \in M$  is another element of M with the same property, then  $X \setminus X' \in N$ . Such X is known as the *essential supremum* of F. If F is finite or countable, then X must be equivalent (up to a negligible set) to  $\bigcup F$ , the union of all elements in F, which is guaranteed to be an element of M. If F is uncountable, there is no relation between X and the union of all elements in F. For example, suppose  $X = \mathbf{R}$ , the  $\sigma$ -algebra M consists of Lebesgue measurable sets, and the  $\sigma$ -ideal N consists of sets of Lebesgue measure 0. Take as F all singleton subsets of X. Then  $X = \emptyset$ . Indeed, for any  $Y \in F$  we have  $Y \setminus X = Y \in N$  because all singleton sets have measure 0.

**Example 7.41.** The measurable space  $\text{Lebesgue}(\mathbf{C}) = (\mathbf{C}, \text{Lebesgue}_{\mathbf{C}}, \text{Null}_{\mathbf{C}})$ , where  $\text{Lebesgue}_{\mathbf{C}}$  is the  $\sigma$ -algebra of Lebesgue measurable subsets of  $\mathbf{C}$  and  $\text{Null}_{\mathbf{C}}$  is the  $\sigma$ -ideal of sets of Lebesgue measure 0, is

localizable. In fact, any measurable space that admits a finite measure that does not vanish on  $M \setminus N$  is localizable. Used in 7.41, 7.43, 11.20\*.

**Example 7.42.** The restriction of the functor BaireMeager:  $Top_{Open} \rightarrow Meas$  to the full subcategory consisting of topological spaces with a countable base lands in localizable measurable spaces. This is true because any equivalence class of measurable sets in this case contains an open set, so the supremum can be computed as the union.

**Example 7.43.** The measurable space  $Borel(\mathbf{C}) = (\mathbf{C}, Borel_{\mathbf{C}}, \{\emptyset\})$  is *not* localizable. (Take as F the uncountable family of singleton subsets a nonmeasurable subset of  $\mathbf{C}$ .) One can replace it with a certain localizable measurable space  $\hat{\mathbf{C}}$ , which is *not* Lebesgue( $\mathbf{C}$ ). Indeed, there are *no* morphisms  $Disc\{*\} \rightarrow \mathbf{C}$ . (The preimage of every point in  $\mathbf{C}$  would have to be negligible, i.e., empty, a contradiction.) On the other hand, morphisms  $Disc\{*\} \rightarrow \hat{\mathbf{C}}$  are in bijection with morphisms  $Disc\{*\} \rightarrow Borel(\mathbf{C})$ , i.e., complex numbers. The existence of  $\hat{\mathbf{C}}$  can be demonstrated most easily using the tools of category theory developed below: the category LocMeas is a *reflective subcategory* of Meas and  $\hat{\mathbf{C}}$  can be defined as the *reflection* of Borel( $\mathbf{C}$ ).

**Definition 7.44.** The functor  $L^{\infty} = O_{Meas}: LocMeas^{op} \to CW^*$  sends a localizable measurable space (X, M, N) to the von Neumann algebra of bounded measurable functions on X. The latter can be defined as the set of all morphisms  $(X, M, N) \to (\mathbf{C}, Borel_{\mathbf{C}}, \{\emptyset\})$ , where  $Borel_{\mathbf{C}}$  denotes the  $\sigma$ -algebra of Borel subsets of  $\mathbf{C}$ . The structure of a complex \*-algebra is induced from  $\mathbf{C}$ . It remains to show (by definition of a von Neumann algebra) that the underlying Banach space is the dual of some other Banach space, the *predual*. In our case we take the Banach space of *complex-valued finite measures* on X, defined as  $\sigma$ -additive functions  $M \to \mathbf{C}$  that vanish on N. (This space is also denoted by  $L^1(X, M, N)$  and is an example of  $L^p$ -spaces that are important in analysis.) Used in 9.13, 9.15, 9.16, 9.24.

**Definition 7.45.** The von Neumann spectrum functor  $\text{Spec}_{\text{Meas}}: (CW^*)^{op} \to \text{LocMeas}$  is defined as the composition of three functors: the forgetful functor  $(CW^*)^{op} \to (CC^*)^{op}$ , the Gelfand spectrum functor  $(CC^*)^{op} \to \text{CompHaus}$ , and the functor  $\text{Top}_{\text{Open}} \to \text{Meas}$ . The composition of the first two functors lands in the subcategory of  $\text{Top}_{\text{Open}}$  consisting of hyperstonean topological spaces and open maps, which is why composing with the third functor makes sense. The resulting composition itself lands in localizable measurable spaces. See §III.1, in particular, Theorem III.1.18 in Takesaki's *Theory of Operator Algebras* for the relevant facts about hyperstonean spaces used in this definition. Used in 9.13, 9.15, 9.16, 9.17, 9.25, 9.27, 9.28, 9.30.

## 7.46. Algebraic geometry

Fix an algebraically closed field k.

**Example 7.47.** The functor  $O_{Sch}$ : AffVar<sup>op</sup><sub>k</sub>  $\rightarrow$  CAlg<sub>k</sub> sends an affine variety V to the commutative k-algebra of regular maps  $V \rightarrow k$  and a regular map f of affine varieties  $V' \rightarrow V$  to the induced homomorphism  $O_{Sch}(V) \rightarrow O_{Sch}(V')$  given by the precomposition with f. This functor lands in the full subcategory CAlgAff<sub>k</sub> of finitely generated k-algebras without nilpotent elements. Used in 7.47, 7.48, 7.48\*.

**Example 7.48.** The functor  $\operatorname{Spec}_{\operatorname{Sch}}$ : CAlgAffGen<sup>op</sup><sub>k</sub>  $\to$  AffVar<sub>k</sub> has as its source the category CAlgAffGen<sup>op</sup><sub>k</sub> that is defined just like CAlgAff<sup>op</sup><sub>k</sub> except that its objects (i.e., algebras) are equipped with a finite set of generators (but morphisms need not preserve the generators). (This rather awkward kludge is necessary because we defined affine varieties as subsets of  $k^n$ . It can be eliminated by passing to *abstract* algebraic varieties, whose category is equivalent to our category.) We define the variety  $\operatorname{Spec}_{\operatorname{Sch}}(A, G)$ , where G is a finite set of generators of A (which induces a homomorphism of algebras  $k[G] \to A$ ), as the subset of  $k^G$  consisting of those points for which the associated evaluation map  $k[G] \to k$  factors as the composition  $k[G] \to A \to k$  for some homomorphism  $A \to k$  (which, if it exists, uniquely determines the corresponding point in  $k^G$ ). The regular map of varieties  $\operatorname{Spec}_{\operatorname{Sch}}(A, G) \to \operatorname{Spec}_{\operatorname{Sch}}(A', G')$  for a homomorphism  $A' \to A$  is defined by sending the point corresponding to a homomorphism  $A \to k$  to the point corresponding to the homomorphism  $A' \to A \to k$ . Used in 7.48, 7.48\*.

Below we will see that  $O_{Sch}$  and  $Spec_{Sch}$  define an equivalence of categories between AffVar<sub>k</sub> and CAlgAff<sup>op</sup><sub>k</sub> (and CAlgAffGen<sup>op</sup><sub>k</sub>). In other words, one could *define* the category AffVar<sub>k</sub> as CAlgAff<sup>op</sup><sub>k</sub> and dispose of our definition above.

One may ask how we can define morphisms between varieties defined over *different* fields k and k'. Furthermore, how one can perform operations such products and disjoint unions on such varieties? The resulting objects would have to be more general than varieties. The following definition represents a fundamental breakthrough by Grothendieck. (The definition was already used in some form by Wolfgang Krull, but it was Grothendieck who developed the associated theory systematically.)

Definition 7.49. The category AffSch of affine schemes is defined as CRing<sup>op</sup>. Used in 26.4.

Below we will see how one can define (nonaffine) schemes (which are in the same relation to affine schemes as varieties are to affine varieties) using a very powerful formalism known as the *functor of points*.

#### 8 Monomorphisms and epimorphisms

**Definition 8.1.** A monomorphism in a category C is a morphism  $f: X \to Y$  such that for any object W and any pair of morphisms  $g_1, g_2: W \to X$  such that  $fg_1 = fg_2$  we have  $g_1 = g_2$ .

**Definition 8.2.** An *epimorphism* in a category C is a morphism  $f: X \to Y$  that is a monomorphism in the category  $C^{op}$ .

**Remark 8.3.** It is instructive to unfold the above definition: an epimorphism in a category C is a morphism  $f: X \to Y$  such that for any object Z and any pair of morphisms  $h_1, h_2: Y \to Z$  such that  $h_1f = h_2f$  we have  $h_1 = h_2$ .

**Example 8.4.** In Set monomorphisms are injective functions (take  $W = \{*\}$ ) and epimorphisms are surjective functions (take  $Z = \{0, 1\}$ ).

## 8.5. Algebra

**Example 8.6.** In any category of algebraic objects, such as Group, Ab, Ring,  $Mod_k$ ,  $Alg_k$ , etc., monomorphisms are injective homomorphisms. It suffices to take as W the free object on one generator and the remainder of the argument is identical to Set using the fact that morphisms out of such a free object are in bijection with the elements of target. Furthermore, all surjective homomorphisms are epimorphisms, however, some categories may have nonsurjective epimorphisms (see below).

**Example 8.7.** In Group all epimorphisms are surjective homomorphisms. This can be established most easily using Schreier's theorem: every subgroup  $H \subset G$  equals  $\{x \in G \mid g(x) = h(x)\}$  for some homomorphisms  $g, h: G \to G'$  (observe that g = h if and only if H = G). Indeed, the image of any epimorphism  $f: F \to G$  is a subgroup  $H \subset G$ , then Schreier's theorem supplies g and h such that gf = hf, hence g = h by definition of an epimorphism, and therefore H = G, i.e., f is surjective.

**Example 8.8.** The same argument as for Group also shows that epimorphisms coincide with surjective maps in the categories Ab,  $\operatorname{Vect}_k$ , and  $\operatorname{Mod}_R$ . However, the analog of Schreier's theorem is trivial here: given an epimorphism  $f: X \to Y$  we take Z = Y/f(X), the map  $h_1$  is the canonical quotient map  $Y \to Y/f(X) = Z$ , and  $h_2$  is the zero map. (For nonabelian groups we can only make sense of Y/f(X) as a set of cosets, not as a group. However, one can use the symmetric group of the set  $* \sqcup Y/f(X)$  to a similar effect, as shown by Linderholm.)

**Example 8.9.** In Ring not all epimorphisms are surjective. For instance,  $\mathbf{Z} \to \mathbf{Q}$  is a nonsurjective epimorphism: if two ring homomorphism  $g, h: \mathbf{Q} \to R$  coincide on  $\mathbf{Z}$ , then for any integer p and  $q \neq 0$  we have  $g(p/q) = g(p)g(q)^{-1} = h(p)h(q)^{-1} = h(p/q)$ . Here we used the fact that homomorphisms of rings preserve inverses of invertible elements, which we also used to show that  $-\times$ : Ring  $\to$  Group is a functor.

One can characterize epimorphisms of rings in more familiar terms. This is a nontrivial result due to Cohn, Isbell, Mazet, and Silver. A morphism of rings  $f: A \to B$  is an epimorphism if and only if the *dominion* of the subring f(A) in B coincides with B. Here the *dominion* of a subring  $S \subset R$  is the set of all elements of R that can be represented as the product of matrices XPY (with coefficients in R), where X, P, and Y have size  $1 \times m, m \times n$ , and  $n \times 1$  respectively, and P, XP, and PY have coefficients in S.

## 8.10. Functional analysis

**Example 8.11.** In Ban and Ban<sub>1</sub> monomorphisms are injective maps (take  $W = \mathbf{R}$  or  $W = \mathbf{C}$ ). Epimorphisms are morphisms with dense image (take  $Z = Y/\overline{f(X)}$  and then proceed as for  $\operatorname{Vect}_k$ ; for Ban<sub>1</sub> note that the quotient map is contractive).

## 8.12. General topology

**Example 8.13.** In Top mononomorphisms are injective continuous maps (take  $W = \{*\}$ ) and epimorphisms are surjective continuous maps (take  $W = \{0, 1\}$  with the antidiscrete topology). In Haus (the full subcategory of Top on Hausdorff spaces) mononomorphisms are injective continuous maps (take  $W = \{*\}$ ) and epimorphisms are continuous maps with dense image (two continuous functions with Hausdorff codomains that coincide on some subset must also coincide on its closure). In CompHaus (the full subcategory of Haus on compact spaces) monomorphisms are continuous injections for the same reason, whereas epimorphisms are (once again) continuous surjective maps (continuous maps with dense image between compact Hausdorff spaces are automatically surjective).

**Example 8.14.** In the category TopGroup of topological groups and continuous group homomorphisms monomorphisms are injections (take  $W = \mathbf{Z}$ ) and epimorphisms are surjections (given an epimorphism  $f: G \to H$ , use Schreier's theorem to construct a homomorphism of discrete groups  $H \to K$  whose kernel is precisely the image of f, and equip K with the antidiscrete topology so that  $H \to K$  is continuos). In the full subcategory HausGroup monomorphisms are precisely injections (take  $W = \mathbf{Z}$ ). Any morphism with a dense image is an epimorphism, and judging by what happens for categories Group and Haus one could be led to conjecture that all epimorphisms have dense image, but a counterexample was constructed by Uspenskij. This statement is true, however, for compact topological groups (a theorem of Poguntke) as well as Hausdorff abelian topological groups (the same argument as for Ab).

#### 8.15. Measure theory

**Example 8.16.** In the category Meas monomorphisms and epimorphisms can be most easily described in terms of the functor MeasAlg: Meas<sup>op</sup>  $\rightarrow$  BoolAlg: they are precisely those morphisms that are mapped by MeasAlg to an epimorphism (i.e., surjection) respectively monomorphism (i.e., injection) of Boolean algebras. (The two classes of maps are exchanged because of op.) In more concrete terms, monomorphisms of measurable spaces are morphisms  $(X, M, N) \rightarrow (X', M', N')$  such that any element of M is equivalent to the preimage of some element of M'. Likewise, epimorphisms are characterized by the property that any element  $m' \in M'$  whose preimage belongs to N itself belongs to N', i.e., only negligible sets have negligible preimages.

#### 9 Equivalences of categories

Previously we defined the category Cat of small categories and the "huge" category CAT of categories. Any category has a built-in notion of isomorphism. In particular, we can talk about isomorphisms of categories. These are functors  $F: C \to D$  such that there is a functor  $G: D \to C$  and  $G \circ F = id_C$ ,  $F \circ G = id_D$ .

This is a perfectly good definition except that it fails to exhibit many categories as equivalent even though we consider them to be the "same". This is entirely analogous to how "same" groups need not be *equal*, but only *isomorphic*. There are many groups (in fact, a proper class of groups) with one element, but only one isomorphism class of groups with one element.

In the above definition,  $G \circ F = id_C$  means that for any object  $X \in C$  we have G(F(X)) = X. This is hardly ever true for any of the constructions that we considered above, e.g.,  $O_{Ball}(DUB(X)) \neq X$  for a Banach space X, even though these two Banach spaces are isomorphic.

We can consider functors  $F: C \to D$  for which there is a functor  $G: D \to C$  such that X is *isomorphic* (but not necessarily equal) to G(F(X)) for any object  $X \in C$  and Y is isomorphic to F(G(Y)) for any object  $Y \in D$ . This can also be formulated by saying that the functions  $\pi_0(F): \pi_0(C) \to \pi_0(D)$  and  $\pi_0(G): \pi_0(D) \to \pi_0(C)$ are mutually inverse to each other.

This modified definition is far too expansive. For instance, recall that any group G gives rise to a category BG that has a single object \* whose endomorphisms are elements of G and composition is given by multiplication. The above definition makes BG and BH the "same" for any pair of groups G and H.

Even more so, consider any groupoid C and a discrete category  $\pi_0(C)$  (all morphisms are identity morphisms) on the set of isomorphism classes of C. We have a canonical functor  $C \to \pi_0(C)$ . We can also choose an inclusion  $\pi_0(C) \to C$  that choose a representative for each equivalence class. The composition  $\pi_0(C) \to C \to \pi_0(C)$  is the identity functor  $\pi_0(C) \to \pi_0(C)$ . The other composition  $C \to \pi_0(C) \to C$  sends any object in C to an isomorphic object.

These examples show us what is wrong with out last attempt: we should take morphisms into account.

**Definition 9.1.** An *equivalence* of categories is a functor  $F: C \to D$  for which there is a functor  $G: D \to C$ such that the induced maps  $F_{X,X'}: Mor_C(X,X') \to Mor_C(F(X),F(X'))$  for any objects  $X, X' \in C$  and  $G_{Y,Y'}: Mor_D(Y,Y') \to Mor_D(G(Y),G(Y'))$  for any objects  $Y,Y' \in D$  are isomorphisms and the functions  $\pi_0(F): \pi_0(C) \to \pi_0(D)$  and  $\pi_0(G): \pi_0(D) \to \pi_0(C)$  are mutually inverse to each other.

There is a simple practical criterion for equivalence.

**Definition 9.2.** Given a functor  $F: C \to D$ , we have an induced function

$$F_{X,X'}$$
: Mor<sub>C</sub> $(X,X') \rightarrow Mor_D(F(X),F(X'))$ 

for any pair of objects  $X, X' \in C$ . We say that the functor F is

- faithful if  $F_{X,X'}$  is injective for any X, X';
- full if  $F_{X,X'}$  is surjective for any X, X';
- fully faithful if  $F_{X,X'}$  is bijective for any X, X', in which case we denote its inverse by  $F_{X,X'}^{-1}$ .

**Remark 9.3.** Suppose C is a full subcategory of D. Then the canonical inclusion  $C \rightarrow D$  is a fully faithful functor.

**Definition 9.4.** An essentially surjective functor is a functor  $F: C \to D$  such that the induced function  $\pi_0(F): \pi_0(C) \to \pi_0(D)$  is surjective. (In other words, for any object  $Y \in D$  there is an object  $X \in C$  such that F(X) is isomorphic to Y.)

**Proposition 9.5.** A functor  $F: C \to D$  is an equivalence if and only if it is fully faithful and essentially surjective.

*Proof.* Necessity of these two properties follows immediately from the definition of an equivalence. To show sufficiency, we start by constructing the functor  $G: D \to C$ . On objects, we invoke the axiom of choice and construct a function  $Ob(G): Ob(D) \to Ob(C)$  by choosing for every object  $Y \in D$  an object  $G(Y) \in C$  such that there is an isomorphism  $\epsilon_Y: F(G(Y)) \to Y$ . We also record a choice of an isomorphism  $\epsilon_Y$ .

Given a morphism  $g: Y \to Y'$  in D we define  $G(g) = F_{G(Y), G(Y')}^{-1}(\epsilon_{Y'}^{-1} \circ g \circ \epsilon_Y)$ . Here  $\epsilon_Y: F(G(Y)) \to Y$ ,  $g: Y \to Y'$ , and  $\epsilon_{Y'}^{-1}: Y' \to F(G(Y'))$  compose together into a morphism  $F(G(Y)) \to F(G(Y'))$ . By the fully

faithfulness of F the function  $F_{G(Y),G(Y')}$ :  $Mor_{C}(G(Y),G(Y')) \rightarrow Mor_{D}(F(G(Y)),F(G(Y')))$  is a bijection, so it has an inverse, which we used in the formula for G(g).

Given morphisms  $g: Y \to Y'$  and  $g': Y' \to Y''$ , we immediately compute

$$\begin{split} \mathsf{G}(g') \circ \mathsf{G}(g) &= \mathsf{F}_{\mathsf{G}(Y'),\mathsf{G}(Y'')}^{-1} (\epsilon_{Y''}^{-1} \circ g' \circ \epsilon_Y') \circ \mathsf{F}_{\mathsf{G}(Y),\mathsf{G}(Y')}^{-1} (\epsilon_{Y'}^{-1} \circ g \circ \epsilon_Y) \\ &= \mathsf{F}_{\mathsf{G}(Y),\mathsf{G}(Y'')}^{-1} (\epsilon_{Y''}^{-1} \circ g' \circ \epsilon_Y' \circ \epsilon_{Y'}^{-1} \circ g \circ \epsilon_Y) \\ &= \mathsf{F}_{\mathsf{G}(Y),\mathsf{G}(Y'')}^{-1} (\epsilon_{Y''}^{-1} \circ g' \circ g \circ \epsilon_Y) = \mathsf{G}(g' \circ g). \end{split}$$

(The second equality follows from the preservation of composition by F.) Likewise,

$$\mathsf{G}(\mathrm{id}_Y) = \mathsf{F}^{-1}(\epsilon_Y^{-1} \circ \mathrm{id}_Y \circ \epsilon_Y) = \mathsf{F}^{-1}(\mathrm{id}_{\mathsf{F}(\mathsf{G}(Y))}) = \mathrm{id}_{\mathsf{G}(Y)}.$$

Thus G is a functor.

Finally, the functions  $\pi_0(F): \pi_0(C) \to \pi_0(D)$  and  $\pi_0(G): \pi_0(D) \to \pi_0(C)$  are mutually inverse to each other once we show that  $\pi_0(F)$  is injective, which by definition of G implies that  $\pi_0(G)$  is its inverse. Thus we have to show that for any objects  $X, X' \in C$  such that F(X) and F(X') are isomorphic objects in D, the objects X and X' are themselves isomorphic in C. Indeed, if  $g: F(X) \to F(X')$  is an isomorphism, then so is  $F_{X,X'}^{-1}(g): X \to X'$ .

We illustrate this abstract theorem with two simple examples.

**Example 9.6.** Suppose C is a category and D is a full subcategory of C such that any isomorphism class in C has exactly one representative that belongs to D. Then the inclusion  $D \rightarrow C$  is an equivalence of categories. The category D has an interesting (and unusual) property: two objects are isomorphic if and only if they are equal. Such categories are known as *skeletal* categories and the above shows that any category is equivalent to a skeletal category.

**Example 9.7.** We can leverage the above example to obtain a classification of groupoids in terms of groups. Given any groupoid G, we can take the full subcategory of G formed by some fixed isomorphism class of objects. The collection of all such full subcategories contains all objects of G, and there are no morphisms between objects that belong to different subcategories. Thus G splits as a disjoint union (later to be formalized as the *coproduct*) of its connected components (indexed by  $\pi_0(G)$ ), so it suffices to classify *connected* groupoids, i.e.,  $\pi_0(G)$  is a singleton. If we pick any object  $* \in G$ , then the inclusion of the full subcategory on \* into G is an equivalence. Thus any connected groupoid is equivalent to a groupoid with one object. The latter is completely determined by the group  $\operatorname{Aut}_G(*)$ , whose elements form morphisms  $* \to *$  and the group operations determine composition. Vice versa, for any group G one constructs a groupoid BG with one object, which has  $\operatorname{Aut}_{BG}(*) \cong G$ . Furthermore, functors  $\operatorname{B}G \to \operatorname{B}H$  can be identified with group homomorphisms  $G \to H$ . Thus, up to an equivalence, every groupoid G can be thought of as a collection of groups indexed by the elements of  $\pi_0(G)$ . Used in 10.34\*.

### 9.8. Functional analysis

**Example 9.9.** The functors DUB:  $\operatorname{Ban}_{1}^{\operatorname{op}} \to \operatorname{CompBall}$  and  $\operatorname{O}_{\operatorname{Ball}}$ :  $\operatorname{CompBall} \to \operatorname{Ban}_{1}^{\operatorname{op}}$  form an equivalence of categories. This is the *Hahn-Banach theorem*. Given  $X \in \operatorname{Ban}_{1}$ , the isomorphism ev:  $X \to \operatorname{O}_{\operatorname{Ball}}(\operatorname{DUB}(X))$  sends an element  $x \in X$  to the linear function  $\operatorname{ev}(x): X^* \to \mathbb{C}$  that sends an element  $f \in X^*$  to  $f(x) \in \mathbb{C}$ , i.e.,  $\operatorname{ev}(x)(f) = f(x)$ . (We have

$$\|\operatorname{ev}(x)\| = \sup_{f \in X^*_{\leq 1}} |\operatorname{ev}(x)(f)| = \sup_{f \in X^*_{\leq 1}} |f(x)| = \|x\|$$

so ev is indeed a contractive map.) Given  $Y \in \mathsf{CompBall}$ , the isomorphism  $\operatorname{ev}: Y \to \mathsf{DUB}(\mathsf{O}_{\mathsf{Ball}}(Y))$ is a morphism of compact balls  $Y = (V, B) \to ((\mathsf{O}_{\mathsf{Ball}}(V, B))^*, (\mathsf{O}_{\mathsf{Ball}}(V, B))^*_{\leq 1})$  that sends  $v \in V$  to the continuous linear map  $\mathsf{O}_{\mathsf{Ball}}(V, B) \to \mathbf{C}$  that sends an element  $f \in \mathsf{O}_{\mathsf{Ball}}(V, B)$  to  $f(v) \in \mathbf{C}$ , i.e.,  $\operatorname{ev}(v)(f) = f(v)$ . (The norm of  $\operatorname{ev}(v)$  equals the norm of v because  $|f(v)| \leq ||f|| \cdot ||v||$ , so  $\operatorname{ev}(v)$  is indeed a morphism of balls.) Used in 9.9, 10.5, 10.11\*.

**Remark 9.10.** The traditional formulation of the Hahn–Banach theorem states that any functional  $B \to \mathbb{C}$  on a Banach subspace  $B \subset A$  can be extended to a functional  $A \to \mathbb{C}$  with the same norm. We can deduce it

from the above stronger version as follows. The inclusion  $B \subset A$  is a monomorphism in  $\mathsf{Ban}_1$ , equivalently, an epimorphism in  $\mathsf{Ban}_1^{\mathsf{op}}$ . The above equivalence sends it to an epimorphism in  $\mathsf{CompBall}$ . The forgetful functor  $\mathsf{CompBall} \to \mathsf{CompHaus}$  preserves epimorphisms. The resulting epimorphism in  $\mathsf{CompHaus}$  is a map that sends a contractive functional  $A \to \mathbf{C}$  to its restriction  $B \to \mathbf{C}$ . Epimorphisms of compact Hausdorff spaces are precisely surjective maps, which in our case means that any contractive functional  $B \to \mathbf{C}$  extends to a contractive functional  $A \to \mathbf{C}$ , which immediately implies the original statement.

#### 9.11. Gelfand and von Neumann dualities

**Example 9.12.** The functors  $\operatorname{Spec}_{CC}: (\mathbb{CC}^*)^{\operatorname{op}} \to \operatorname{CompHaus}$  and  $\operatorname{O}_{\operatorname{Cont}}: \operatorname{CompHaus}^{\operatorname{op}} \to \mathbb{CC}^*$  form an equivalence of categories. This is the *Gelfand duality theorem for commutative*  $C^*$ -algebras. We describe the involved isomorphisms. Given  $X \in \operatorname{CompHaus}$ , the isomorphism  $X \to \operatorname{Spec}_{CC}(\operatorname{O}_{\operatorname{Cont}}(X))$  sends a point  $x \in X$  to the morphism  $\operatorname{O}_{\operatorname{Cont}}(X) \to \mathbb{C}$  that evaluates on X. Given  $A \in \operatorname{CC}^*$ , the isomorphism  $A \to \operatorname{O}_{\operatorname{Cont}}(\operatorname{Spec}_{\operatorname{CC}}(A))$  sends an element  $a \in A$  to the evaluation map  $\operatorname{Spec}_{\operatorname{CC}}(A) \to \mathbb{C}$  that sends an element  $f: A \to \mathbb{C}$  of  $\operatorname{Spec}_{\operatorname{CC}}(A)$  to  $f(a) \in \mathbb{C}$ .

**Example 9.13.** The functors  $\operatorname{Spec}_{\operatorname{Meas}}: (\operatorname{CW}^*)^{\operatorname{op}} \to \operatorname{LocMeas}$  and  $\operatorname{L}^\infty: \operatorname{LocMeas}^{\operatorname{op}} \to \operatorname{CW}^*$  form an equivalence of categories. This is result is due to von Neumann. Given  $A \in \operatorname{CW}^*$ , the isomorphism  $A \to \operatorname{L}^\infty(\operatorname{Spec}_{\operatorname{Meas}}(A))$  is defined by sending an element  $a \in A$  to the equivalence class of a bounded measurable function  $\operatorname{Spec}_{\operatorname{Meas}}(A) \to \mathbb{C}$  that sends a point  $p \in \operatorname{Spec}_{\operatorname{Meas}}(A)$  (i.e., a morphism of C\*-algebras  $A \to \mathbb{C}$ ) to p(a). Given  $X \in \operatorname{LocMeas}$ , the isomorphism  $X \to \operatorname{Spec}_{\operatorname{Meas}}(\mathbb{L}^\infty(X))$  is defined as follows.

#### 9.14. Spectra of operators and functional calculus

**Example 9.15.** The continuous functional calculus and Borel functional calculus receive an easy interpretation in terms of the Gelfand and von Neumann dualities. Consider a bounded operator  $P: H \to H$  on a complex Hilbert space H. Denote by A the C\*-subalgebra of B(H) (the C\*-algebra of bounded linear operators on a Hilbert space H) generated by the operator P. Likewise, denote by B the von Neumann subalgebra of B(H) generated by P. The following statements are equivalent:

- A is commutative;
- *B* is commutative;
- P is normal:  $P^*P = PP^*$ .

In this case, we have  $\operatorname{Spec}_{CC}(A) \in \operatorname{CompHaus}$  and  $\operatorname{Spec}_{Meas}(B) \in \operatorname{LocMeas}$ . We refer to them as the spectrum of P taken as a topological space or as a measurable space respectively, and denote them  $\operatorname{Spec}_{CC}(P)$  and  $\operatorname{Spec}_{Meas}(P)$ . Thus given  $f \in O_{\operatorname{Cont}}(\operatorname{Spec}_{CC}(A)) \cong A \subset B(H)$  or  $g \in \operatorname{L}^{\infty}(\operatorname{Spec}_{Meas}(B)) \cong B \subset B(H)$  we can send f and g to operators  $f(P) \in B(H)$  and  $g(P) \in B(H)$  using the above isomorphisms provided to us by the equivalence of categories above. The morphisms  $O_{\operatorname{Cont}}(\operatorname{Spec}_{CC}(A)) \to B(H)$  and  $\operatorname{L}^{\infty}(\operatorname{Spec}_{Meas}(B)) \to B(H)$  are known as the continuous and Borel functional calculus respectively.

**Remark 9.16.** The operator  $P \in B$  corresponds to an element in  $L^{\infty}(\operatorname{Spec}_{\operatorname{Meas}}(B)) \cong B$ , i.e., a morphism of measurable spaces  $f: \operatorname{Spec}_{\operatorname{Meas}}(B) \to \operatorname{Borel}(\mathbb{C})$ . Similarly, we have a morphism  $g: \operatorname{Spec}_{\mathbb{CC}}(A) \to \mathbb{C}$  of locally compact Hausdorff spaces. These two embeddings are related to the traditional notion of a spectrum of P as a subset of  $\mathbb{C}$ , namely  $\{\lambda \in \mathbb{C} \mid P - \lambda \cdot \operatorname{id}_H \notin \mathbb{B}(H)^{\times}\}$ , where  $\mathbb{B}(H)^{\times}$  denotes the set of invertible elements of B(H) ( $Q \in \mathbb{B}(H)^{\times}$  if there is  $R \in \mathbb{B}(H)$  such that  $QR = RQ = \operatorname{id}_H$ ). This set equals the essential range of f, i.e.,  $\{\lambda \in \mathbb{C} \mid \forall U \ni \lambda: f^{-1}(U) \notin N\}$ . In our setting we have more information about the spectrum of P: we have a topological space and a measurable space, not merely a set.

**Remark 9.17.** Any measurable space splits as a disjoint union of a discrete measurable space (i.e., in the image of the functor Disc: Set  $\rightarrow$  Meas) and a *diffuse* measurable space (which has no isolated points). When applied to Spec<sub>Meas</sub>(P), the former is known as the set of *eigenvalues* or the *point spectrum* of P and the latter is known as the *continuous spectrum* of P.

**Remark 9.18.** Functional calculi for a family of commuting normal operators can be defined in exactly the same fashion.

**Remark 9.19.** One can also define a Borel functional calculus for unbounded normal operators P and unbounded measurable functions f. The operator f(P) can be defined in the same way as above, but

commutative von Neumann algebras must be replaced by commutative *extended von Neumann algebras* (alias EW\*-algebras) defined by Dixon in his paper *Unbounded operator algebras*.

## 9.20. Spectral theory

**Definition 9.21.** The category  $\operatorname{Rep}_A$  of *representations* of a von Neumann algebra A is defined as follows. Objects are pairs  $(H, \rho)$ , where  $H \in \operatorname{Hilb}_{\mathbb{C}}$  and  $\rho: A \to B(H)$  is a morphism of von Neumann algebras. Morphisms  $(H, \rho) \to (H', \rho')$  are bounded linear maps  $f: H \to H'$  such that for any  $a \in A$  and  $h \in H$  we have  $f(\rho(a)(h)) = \rho'(a)(f(h))$ . Used in 9.22, 9.24, 9.25.

**Remark 9.22.** The category  $\operatorname{Rep}_A$  is equivalent to the category  $W^*Mod_A$  of *Hilbert W\*-modules*, which are defined like Hilbert spaces, but with the inner product taking values in A instead of C.

**Definition 9.23.** The category HilbBun<sub>M</sub> of measurable fields of Hilbert spaces (alias Hilbert bundles) over a measurable space M = (X, M, N) is defined as follows. Used in 9.24, 9.25.

**Definition 9.24.** The functor  $L^2$ : HilbBun<sub>M</sub>  $\to \operatorname{Rep}_{L^{\infty}(M)}$  sends a Hilbert bundle *B* over *M* to a representation of  $L^{\infty}(M)$  whose underlying Hilbert space is the Hilbert space  $L^{\infty}(B) \otimes_{L^{\infty}(M)} L^2(M)$ , where  $L^{\infty}(B)$  denotes the  $L^{\infty}(M)$ -module of equivalence classes of bounded measurable sections of *B*. The action of  $L^{\infty}(M)$  is given by its right action on  $L^2(M)$ . A morphism of Hilbert bundles is mapped to the morphism of representations given by the fiberwise action. Used in 9.24, 9.26, 9.27.

**Definition 9.25.** The spectral decomposition functor  $\mathsf{Decomp}: \mathsf{Rep}_A \to \mathsf{Hilb}\mathsf{Bun}_{\mathsf{Spec}_{\mathsf{Meas}}(A)}$  sends a representation  $(H, \rho)$  of A to the Hilbert bundle whose fiber over some point  $p \in \mathsf{Spec}_{\mathsf{Meas}}(A)$  (i.e., a homomorphism of C\*-algebras  $p: A \to \mathbf{C}$ ) is  $H \otimes_A \mathbf{C}$ , where the right action of A on H is given by  $\rho$  and the left action of A on  $\mathbf{C}$  is given by p. Used in 9.26.

**Example 9.26.** The functors  $L^2$  and **Decomp** form an equivalence of categories.

**Example 9.27.** Continuing the discussion about functional calculus, consider a normal operator  $P \in B(H)$ . The (commutative) von Neumann algebra B generated by B has a canonical inclusion  $\rho: B \to B(H)$ , hence  $(H, \rho)$  is a representation B. Accordingly, we have a Hilbert bundle T over  $\operatorname{Spec}_{\operatorname{Meas}}(P)$  such that  $(H, \rho) \cong L^2(T)$ . This is the Hahn-Hellinger spectral theorem for bounded operators on a Hilbert space.

**Remark 9.28.** Previously we subdivided  $\operatorname{Spec}_{\operatorname{Meas}}(P)$  into its discrete part, the point spectrum, and its diffuse part, the continuous spectrum. The subset of the point spectrum consisting of those points  $\lambda \in \mathbb{C}$  that have finite-dimensional fibers (i.e., eigenspaces) and  $\lambda$  does not belong to the continuous spectrum is known as the discrete spectrum of P. Its complement in  $\operatorname{Spec}_{\operatorname{Meas}}(P)$  is the essential spectrum of P.

**Remark 9.29.** In complete analogy with the Borel functional calculus, one can deduce the spectral theorem for commuting families of normal operators. Likewise, passing to extended von Neumann algebras allows one to treat unbounded normal operators.

**Remark 9.30.** Analagous results (due to Irving Segal) can be established for representations of commutative von Neumann algebras in arbitrary von Neumann algebras, i.e., morphisms  $A \to B$ , where A is commutative. In this case, the analog of spectral theory (known as *reduction theory*) produces a bundle of von Neumann algebras over  $\operatorname{Spec}_{\operatorname{Meas}}(A)$ . In particular, if we take A to be the center of B, this identifies B with the algebra of bounded measurable sections of a bundle of von Neumann algebras with trivial center (known as *factors*) over  $\operatorname{Spec}_{\operatorname{Meas}}(A)$ .

## 9.31. Topological groups

**Example 9.32.** The functor PD: LocCompHausAb<sup>op</sup>  $\rightarrow$  LocCompHausAb constructed in Example 7.18 is an equivalence of categories. This is the *Pontryagin duality for locally compact topological groups*. Its inverse is the *same* functor, now regarded as a functor LocCompHausAb  $\rightarrow$  LocCompHausAb<sup>op</sup>. Both relevant isomorphisms are also the same: the map  $G \rightarrow PD(PD(G))$  sends  $g \in G$  to the continuous homomorphism  $PD(G) \rightarrow U(1)$  that maps  $f: G \rightarrow U(1)$  to  $f(g) \in U(1)$ .

#### 9.33. Lie groups

**Example 9.34.** The functor  $\text{LieGroup} \rightarrow \text{LieAlg}_{\mathbf{R}}$  can be restricted to simply connected Lie groups, in which case it becomes an equivalence of categories. This is *Lie's third theorem*.

#### 9.35. Posets

**Example 9.36.** The category Proset of sets equipped with a reflexive transitive binary relation (henceforth *prosets*) is equivalent to the full subcategory PC of Cat consisting of those categories C such that for any pair of morphisms  $f, g: X \to Y$  in C we have f = g. The functor Proset  $\to$  PC sends a proset (X, R) to the category whose set of objects is X and the set of morphisms  $x \to y$  consists of a single element if xRy or is empty otherwise. Composition and identity maps are uniquely defined in the obvious way. A morphism f of prosets is sent to the unique functor whose object function is f. The functor PC  $\to$  Proset sends a small category C in PC to the proset (Ob(C), R), where xRy if and only if there is a (unique) morphism  $x \to y$ . Used in 9.36.

**Example 9.37.** In particular, any poset gives rises to a category. Identity morphisms are the only isomorphisms in this category. Used in 10.13, 17.4, 23.2.

#### 10 Natural transformations

In our discussion of equivalences of categories we mentioned that it is a bad idea to talk about equalities of functors such as  $id_C = G \circ F$  because such an equality would in particular state that two objects in a category are equal, whereas typically we can only hope for an isomorphism, not equality.

Thus it seems resonable to try to define isomorphisms of functors in such a way that  $id_C$  and  $G \circ F$  are isomorphic functors if F and G form an equivalence of categories.

Isomorphisms are a particular case of morphisms, so we start by defining morphisms of functors.

**Definition 10.1.** Suppose C and D are categories and  $F: C \to D$  and  $G: C \to D$  are functors (with the same sources and targets). A *natural transformation* (alias *morphism of functors*) from F to G (notation:  $t: F \to G$  or  $t: F \Rightarrow G$ ) is given by a collection of morphisms  $t_X: F(X) \to G(X)$  for any  $X \in C$  such that the following diagram commutes for any morphism  $f: X \to Y$  in C:

$$F(X) \xrightarrow{F(f)} F(Y)$$

$$\downarrow^{t_X} \qquad \downarrow^{t_Y}$$

$$G(X) \xrightarrow{G(f)} G(Y).$$

In other words, for any  $f: X \to Y$  we have  $G(f) \circ t_X = t_Y \circ F(f)$ .

**Definition 10.2.** Suppose C and D are categories. The *category of functors* Fun(C, D) has functors F:  $C \to D$  as objects and natural transformations  $t: F \Rightarrow G$  as objects. The identity natural transformation  $id_F: F \to F$  has  $(id_F)_X = id_X$  for any  $X \in C$ . The composition of  $u: G \Rightarrow H$  and  $t: F \Rightarrow G$  (where F, G, H:  $C \to D$ ) has  $(u \circ t)_X = u_X \circ t_X$  for all  $X \in C$ . Associativity and unitality follow from the associativity and unitality of D. Used in 10.3, 10.14, 10.19, 10.31, 10.34^\*, 26.0\*, 26.1, 26.2, 26.3, 26.4, 26.5, 26.5\*, 26.6, 26.7\*, 27.4.

**Observation 10.3.** Isomorphisms in Fun(C, D) are precisely those natural transformations  $t: F \to G$  for which  $t_X$  is an isomorphism for all  $X \in C$ . These are known as *natural isomorphisms*.

## 10.4. Algebra

**Example 10.5.** Recall that the dual vector space functor DVS:  $\operatorname{Vect}_k^{\operatorname{op}} \to \operatorname{Vect}_k$  sends a vector space V to  $V^* = \operatorname{Hom}(V, k)$  and a linear map  $V \to V'$  to the induced map  $\operatorname{Hom}(V', k) \to \operatorname{Hom}(V, k)$ . The composition  $\operatorname{DVS} \circ \operatorname{DVS}^{\operatorname{op}}$ :  $\operatorname{Vect}_k \to \operatorname{Vect}_k$  sends a vector space V to  $V^{**} = \operatorname{Hom}(\operatorname{Hom}(V, k))$  and a linear map  $h: U \to V$  to the induced linear map  $h^{**}: U^{**} \to V^{**}$ , defined as  $(h^{**}(\hat{u}))(g) = \hat{u}(g \circ h)$ , where  $\hat{u} \in U^{**}$ ,  $g \in V^*$ ,  $g \circ h \in U^*$ . We have a natural transformation  $\iota: \operatorname{id}_{\operatorname{Vect}_k} \Rightarrow \operatorname{DVS} \circ \operatorname{DVS}^{\operatorname{op}}$  whose value on a vector space V is the linear map  $V \to V^{**}$  that sends  $v \in V$  to the linear map  $\operatorname{ev}_v: V^* \to k$  that sends  $g \in V^*$  to  $g(v) \in k$ . In

formulas:  $(\iota_V(v))(g) = g(v)$ . The naturality property requires that the following diagram commutes for any linear map  $h: U \to V$ :

$$U \xrightarrow{n} V$$

$$\downarrow^{\iota_U} \qquad \downarrow^{\iota_V}$$

$$U^{**} \xrightarrow{h^{**}} V^{**}.$$

In othr words, we must have  $h^{**}\iota_U = \iota_V h$ . We compute for any  $u \in U$ 

$$(h^{**}(\iota_U(u)))(g) = (\iota_U(u))(g \circ h) = (g \circ h)(u) = g(h(u))$$

and

$$(\iota_V(h(u)))(g) = g(h(u)),$$

both by definition of  $\iota$  and  $h^{**}$ .

#### 10.6. Measure theory

Definition 10.7. Recall the functors

 $Meas: CompHaus \xrightarrow{Baire|_{CompHaus}} Meas \xrightarrow{L^1} Ban_1$ 

and

DualCont: CompHaus 
$$\xrightarrow{O_{Ban}} Ban_1^{op} \xrightarrow{*} Ban_1$$

We have a natural transformation  $\iota: Meas \to DualCont$  such that for any  $X \in CompHaus$  the morphism  $\iota_X: Meas(X) \to DualCont(X)$  sends a measure  $\mu$  to the functional  $f \mapsto \int f d\mu$ . Used in 10.7, 10.11\*.

**Theorem 10.8.** (Riesz 1909, Markoff 1938, Kakutani 1941.) The natural transformation  $\iota$  is an isomorphism. Used in 10.11\*.

It follows formally from the definitions that for any  $X \in \mathsf{CompHaus}$  the map  $\iota_X$  preserves the norm, in particular, it is injective. The difficult part is to show that  $\iota_X$  is surjective, i.e., every continuous linear functional on X arises from some measure. Following Garling and Hartig, we explain how the naturality of  $\iota$ can be exploited to reduce this problem to the case of very special topological spaces.

**Definition 10.9.** A topological space X is *extremally disconnected* if the closure of any open set is an open set.

Recall the following fact from general topology. (Later we will offer a categorical perspective on the Stone–Čech compactification.)

**Proposition 10.10.** The Stone–Čech compactification of a discrete topological space X is extremally disconnected. In particular, for any compact Hausdorff topological space X the Stone–Čech compactification of its underlying set  $X_0$  equipped with the discrete topology is an extremally disconnected compact Hausdorff space  $\hat{X}_0$  equipped with a canonical surjective continuous map  $\hat{X}_0 \to X$ , which arises from the universal property of the Stone–Čech compactification by extending the canonical map  $X_0 \to X$  to  $\hat{X}_0$ .

We can now use the naturality of  $\iota$  to reduce the case of general compact Hausdorff space to the case of extremally disconnected compact Hausdorff spaces.

**Proposition 10.11.** If  $\iota_X$  is an epimorphism for all extremally disconnected X, then  $\iota$  itself is an epimorphism, i.e.,  $\iota_X$  is an epimorphism for all compact Hausdorff X.

*Proof.* Consider the naturality square of the morphism  $c: X_0 \to X$ :

$$\begin{array}{ccc} \operatorname{Meas}(\hat{X}_{0}) & \xrightarrow{\operatorname{Meas}(c)} & \operatorname{Meas}(X) \\ & & \downarrow^{\iota_{\hat{X}_{0}}} & & \downarrow^{\iota_{X}} \\ \operatorname{DualCont}(\hat{X}_{0}) & \xrightarrow{\operatorname{DualCont}(c)} & \operatorname{DualCont}(X). \end{array}$$

The morphism  $\hat{X}_0 \to X$  is an epimorphism in CompHaus. The induced morphism  $O_{Ban}(X) \to O_{Ban}(\hat{X}_0)$  is a monomorphism in Ban<sub>1</sub>. By the Hahn–Banach theorem (Example 9.9) the induced morphism

$$DUB(O_{Ban}(\hat{X}_0)) \rightarrow DUB(O_{Ban}(X))$$

is an epimorphism in CompBall, hence the underlying morphism

 $DualCont(c): DualCont(\hat{X}_0) \rightarrow DualCont(X)$ 

in the category  $Ban_1$  is an epimorphism. The morphism  $\iota_{\hat{X}_0}$  is an epimorphism by assumption, hence so is the composition  $DualCont(c) \circ \iota_{\hat{X}_0}$ . Hence  $\iota_X$  is an epimorphism.

We conclude this example by proving that  $\iota_X$  is an epimorphism for any extremally disconnected compact Hausdorff space X. For such X the  $\sigma$ -algebra Baire<sub>X</sub> is generated by clopen (closed and open) subsets of X (the nontrivial implication presents a functionally closed subset  $f^{-1}(-\infty, 0]$  of X as a countable intersection of clopen subsets given by the closures of  $f^{-1}(-\infty, \epsilon)$  for all rational  $\epsilon > 0$ ). The characteristic function of a clopen subset is continuous, and by restricting the given continuous linear functional f on  $O_{Ban}(X)$  to characteristic functions of clopen subsets of X we obtain a finite premeasure on the Boolean algebra of clopen subsets, i.e., a function on the algebra of clopen subsets that satisfies the countable additivity property for any disjoint family of clopen sets whose union is clopen (by compactness such a family is necessarily finite). By the Hahn–Kolmogorov extension theorem such a premeasure extends uniquely to a finite measure  $\mu$  on Baire<sub>X</sub>. The linear functional  $\iota_X(\mu)$  coincides with f on the characteristic functions of clopen subsets of X, and the linear span of such functions is dense in  $O_{Ban}(X)$ , hence  $\iota_X(\mu) = f$ , which completes the proof of Theorem 10.8.

#### 10.12. Sheaves of sets on topological spaces

In analysis and geometry we commonly study various spaces through functions on them. The only holomorphic functions on the Riemann sphere (alias complex projective line) are constant functions. On the other hand, open subsets of the Riemann sphere have plenty of holomorphic functions on them. Thus if we intend to study the Riemann sphere through functions on it, we must consider functions that are defined on an open subset. The relevant framework for such considerations is provided by the theory of *sheaves*.

**Definition 10.13.** Given a topological set X, the category Open(X) is the category associated to the poset of open subsets of X as explained in Example 9.37. Used in 10.14, 10.19, 10.22, 10.29, 10.33.

**Definition 10.14.** The category  $\mathsf{PreSh}(X)$  of *presheaves* on a topological space X is defined as

$$\operatorname{Fun}(\operatorname{Open}(X)^{\operatorname{op}},\operatorname{Set}).$$

Used in 10.16, 10.20.

Thus a presheaf F assigns a set F(U) to any open set  $U \subset X$  and a restriction map  $F(\iota): F(V) \to F(U)$ to any inclusion  $\iota: U \to V$  of open sets. Elements of F(V) are referred to as sections of F over V. If  $x \in F(V)$ , then its image under  $F(\iota)$  is denoted by  $x|_U$  and we refer to it as the restriction of x to U. The definition of a functor implies that for any  $x \in F(V)$  we have  $x|_V = x$  and for any  $T \subset U \subset V$  we have  $(x|_U)|_T = x|_T$ , both of which are families properties of restrictions of functions to subsets of their domains.

**Examples 10.15.** Given a topological space X, we define the following presheaves F on X (the restriction maps are given by traditional restrictions of functions):

- F(U) is the set of continuous maps  $U \to \mathbf{R}$ ;
- F(U) is the set of bounded continuous maps  $U \to \mathbf{R}$ ;
- F(U) is the set of Borel measurable maps  $U \to \mathbf{R}$ .

**Example 10.16.** Given a continuous map  $p: T \to X$  of topological spaces, we define the presheaf SecPre(p) of continuous sections of the map p by setting SecPre $(p)(U) = \{s: U \to T \mid ps = \kappa\}$  for any open subset  $\kappa: U \to X$  and SecPre $(p)(\iota):$ SecPre $(p)(V) \to$ SecPre(p)(U) to the map  $s \mapsto s \circ \iota$  for any inclusion of open subsets  $\iota: U \to V$ . In fact, we have a functor SecPre: Top $/X \to$  PreSh(X) that sends p to SecPre(p) and

a morphism  $p \to q$  in Top/X (i.e., a continuous map  $f: T \to T'$  such that qf = p) to the induced natural transformation  $\text{SecPre}(p) \to \text{SecPre}(q)$  whose value on some open set  $U \subset X$  sends  $s: U \to T$  to  $fs: U \to T'$ . Used in 10.16, 10.16<sup>\*</sup>, 10.20.

The last example is very important because many practical example can be seen as particular instances for various choices of p (sometimes involving rather weird spaces T).

We would like to characterize the image of the functor SecPre. This can be accomplished using the following intermediate definition.

**Definition 10.17.** Given a presheaf F and an open subset  $U \subset X$ , we say that some sections  $x, y \in F(U)$  are *locally equal* if there is an open cover  $\{V_i\}$  of U such that for all i we have  $x|_{V_i} = y|_{V_i}$ . In this case we write  $x \sim y$ .

Of course, x = y implies  $x \sim y$ , but the opposite need not be true in general.

**Definition 10.18.** Given a presheaf F on X, an open set  $U \subset X$ , and an open cover  $\{V_i\}$  of U, a compatible family is a collection  $\{t_i\}$  of sections  $t_i \in F(V_i)$  such that for all i and j we have  $t_i|_{V_i \cap V_j} \sim t_j|_{V_i \cap V_j}$ . The set of all compatible families of sections of F over the open cover V is denoted by ComFam(F, V). Used in 10.18\*, 10.19.

We also have a canonical map  $F(U) \to \text{ComFam}(F, V)$  that sends a section  $s \in F(U)$  to the compatible family  $\{s|_{V_i}\}$ . Indeed, the compatibility condition is verified because  $s|_{V_i \cap V_i} = s|_{V_i \cap V_i}$ 

**Definition 10.19.** Given a topological space X, the category Sh(X) of *sheaves of sets on* X is defined as the full subcategory of  $Fun(Open(X)^{op}, Set)$  consisting of those functors  $F:Open(X)^{op} \to Set$  that satisfy the *gluing property*: for any open set  $U \subset X$  and for any open cover  $\{V_i\}$  of U the canonical map  $F(U) \to ComFam(F, V)$  is an isomorphism. Used in 10.20, 10.22, 10.23, 10.24, 10.25, 10.29, 10.30.

**Example 10.20.** The functor SecPre:  $\operatorname{Top}/X \to \operatorname{PreSh}(X)$  lands in sheaves on X. We denote the corestricted functor as Sec:  $\operatorname{Top}/X \to \operatorname{Sh}(X)$ . Used in 10.23, 10.30.

**Definition 10.21.** A continuous map  $p: Y \to X$  is *etale* if it is an open map and a homeomorphism locally on Y, i.e., there is an open cover  $\{U_i\}$  of Y such that the corestriction of  $p|_{U_i}$  to its image in X is a homeomorphism for any *i*. The category  $\mathsf{Et}/X$  is the full subcategory of  $\mathsf{Top}/X$  consisting of etale maps to X. Used in 10.22, 10.23, 10.24, 10.25, 10.28, 10.30.

**Definition 10.22.** The functor Et:  $Sh(X) \to Et/X$  sends a sheaf  $F: Open(X)^{op} \to Set$  to the etale map  $Et(F): T \to X$ , where T is the *etale space* of F constructed as follows. Its points are equivalence classes of triples (x, U, s), where  $x \in U$ ,  $U \in Open(X)$ ,  $s \in F(U)$ , with respect to the equivalence relation  $(x, U, s) \sim (x', U', s')$  if x = x' and there is  $V \subset U \cap U'$  such that  $x \in V$  and  $s|_V = s'|_V$ . A base of open sets is constructed by taking for any  $U \in Open(X)$  and  $s \in F(U)$  the set of equivalence classes of (x, U, s), where  $x \in U$  is arbitrary. The map  $T \to X$  that sends the equivalence class of (x, U, s) to x is a continuous map. Used in 10.22, 10.23, 10.30.

**Theorem 10.23.** The functors Sec:  $Et/X \to Sh(X)$  and  $Et: Sh(X) \to Et/X$  form an equivalence of categories.

This theorem provides us with a powerful dictionary that relates etale maps and sheaves. We illustrate this idea by giving two important constructions, one of which is easier to state using sheaves and the other one using etale spaces.

**Definition 10.24.** Given a continuous map  $f: X \to Y$ , the *pushforward* functor  $f_*: \operatorname{Sh}(X) \to \operatorname{Sh}(Y)$  sends a sheaf F to the sheaf  $f_*F$  defined as  $(f_*F)(V) = F(f^{-1}(V))$ . A morphism of sheaves (i.e., a natural transformation of functors)  $F \to G$  is sent to the induced natural transformation  $F(f^{-1}(V)) \to G(f^{-1}(V))$ . The pushforward functor  $\operatorname{Et}/X \to \operatorname{Et}/Y$  is defined using the above equivalence.

**Definition 10.25.** Given a continuous map  $f: X \to Y$ , the *pullback* functor  $f^*: Et/Y \to Et/X$  sends an etale map  $g: T \to Y$  to the etale map  $T \times_Y X \to X$ , where  $T \times_Y X := \{(t, x) \mid g(t) = f(x)\}$ . The pullback map  $Sh(Y) \to Sh(X)$  is defined using the above equivalence.

#### 10.26. Galois theory of coverings

**Definition 10.27.** Given an object  $X \in \mathsf{Top}$ , the category  $\mathsf{Cov}/X$  of covering maps (alias covering projections) is defined as follows. Objects are morphisms  $p: T \to X$  in  $\mathsf{Top}$  for which X admits a base of open sets U such that  $p^{-1}(U)$  is homeomorphic to a disjoint union of copies of X. Morphisms  $p \to p'$  are continuous maps  $f: T \to T'$  such that p'f = p. Used in 10.28, 10.30, 10.32, 10.33, 10.34, 10.34\*, 10.36, 10.36\*.

**Remark 10.28.** The category  $\operatorname{Cov}/X$  is a full subcategory of  $\operatorname{Et}/X$ . The definition of an etale map  $p: Y \to X$  requires that any point in  $p^{-1}(x)$  for some  $x \in X$  has an open neighborhood that maps homeomorphically to X. The definition of a covering then requires that we can choose these neighborhoods to be disjoint and homeomorphic to each other.

We now translation the notion of a covering into the language of sheaves.

**Definition 10.29.** The category LCSh(X) of *locally constant sheaves* over X is the full subcategory of Sh(X) consisting of *locally constant* sheaves over X, i.e., sheaves F for which there is a base B of X such that the restriction of F to B (which is a full subcategory of Open(X)) is a *constant functor*, i.e., a functor that sends any morphism to an isomorphism. Used in 10.30, 10.34, 10.34\*.

**Proposition 10.30.** The functors Sec and Et restrict to the full subcategories Cov/X of Et/X and LCSh(X) of Sh(X) and form an equivalence of these full subcategories.

**Definition 10.31.** Given an object  $X \in \mathsf{Top}$ , the category  $\mathsf{Trans}(X)$  of *transport functors* over X is the category  $\mathsf{Fun}(\pi_{\leq 1}(X), \mathsf{Set})$ . Thus an object  $F \in \mathsf{Trans}(X)$  assigns a set  $F_x$  to each point  $x \in X$  and a map of sets  $F_x \to F_y$  to each path in X from x to y. Used in 10.31, 10.32, 10.33, 10.34, 10.34\*, 10.36\*.

**Definition 10.32.** Given an object  $X \in \text{Top}$ , the functor  $\text{Monodromy: } \text{Cov}/X \to \text{Trans}(X)$  is defined as follows. For an object  $p: T \to X$  in Cov/X the functor  $\text{Monodromy}(p): \pi_{\leq 1}(X) \to \text{Set}$  sends a point  $x \in X$  to fiber  $f^{-1}(x)$ . A morphism (i.e., the homotopy equivalence class of a path  $h: [0, 1] \to X$  from  $x \in X$  to  $y \in Y$ ) is sent to the map  $f^{-1}(x) \to f^{-1}(y)$  that sends an element  $a \in f^{-1}(x)$  to the element  $b \in f^{-1}(y)$  such that there is a path  $g: [0, 1] \to T$  from a to b with pg = h. The definition of a covering space is designed in such a way as to guarantee that g is unique and the map  $f^{-1}(x) \to f^{-1}(y)$  does not depend on the choice of h in the equivalence class. Used in 10.32, 10.34.

**Definition 10.33.** Given an object  $X \in \mathsf{Top}$ , the reconstruction functor Recons:  $\mathsf{Trans}(X) \to \mathsf{Cov}/X$  sends a transport functor  $F: \pi_{\leq 1}(X) \to \mathsf{Set}$  to the covering space  $p: T \to X$  defined as follows. The underlying set of T is the disjoint union of sets F(x) for all  $x \in X$ , which is equipped with a canonical map to X. A base of open subsets of T is constructed by taking for any  $x \in X$ ,  $w \in F(x)$ , and  $U \in \mathsf{Open}(X)$  such that  $x \in U$ the set  $\{F(h)(w) \mid h: [0,1] \to U\}$ . A natural transformation  $F \to G$  induces a map  $T_F \to T_G$ , which turns out to be continuous. Used in 10.34, 10.34\*, 10.36.

To exclude pathologies we work in the category GoodTop of locally path-connected and (locally or semilocally) simply connected topological spaces. A space is *locally path-connected* if it admits a base of path-connected open subsets. A space is *locally simply connected* if it admits a base of simply connected open subsets, i.e., path-conected open subsets U such that any map  $S^1 \to U$  extends to a map  $D^2 \to U$ . A space X is *semilocally simply connected* if it admits a base of open subsets U such that  $\pi_{\leq 1}(\iota): \pi_{\leq 1}(U) \to \pi_{\leq 1}(X)$ identifies any pair of parallel morphisms.

**Theorem 10.34.** Given an object  $X \in GoodTop$ , the functors Monodromy:  $Cov/X \to Trans(X)$  and Recons:  $Trans(X) \to Cov/X$  form an equivalence of categories. In particular, both of these categories are further equivalent to the category LCSh(X) of locally constant sheaves on X. Used in 10.35.

Thus we have three different languages of talking about the same things: Cov/X, Trans(X), and LCSh(X).

Recall the classification of groupoids given in Example 9.7: any groupoid is equivalent to to a disjoint union of groupoids of the form BG for some groups G. In the case of the groupoid  $\pi_{\leq 1}(X)$  the disjoint union is indexed by the path components of X, which we can study separately. Thus for the sake of brevity, we consider only the case of connected groupoids, i.e., BG for some group G (which is  $\pi_1(X, *)$  in the case of a path-connected space X with some basepoint  $* \in X$ ). In this case the category Fun(BG, Set) is equivalent

to the category GSet, i.e., sets equipped with an action of G, as defined in Example 4.10. The equivalence  $\operatorname{Fun}(\operatorname{B} G,\operatorname{Set}) \to \operatorname{GSet}$  is implemented by the functor that sends  $F \in \operatorname{Fun}(\operatorname{B} G,\operatorname{Set})$  to the pair  $(F(*),\rho)$ , where  $* \in \operatorname{B} G$  is the only object of  $\operatorname{B} G$  and  $\rho: G = \operatorname{Mor}_{\operatorname{B} G}(*,*) \to \operatorname{Mor}_{\operatorname{Set}}(F(*),F(*))$  is induced by the functor data of F.

Coverings of a nonconnected  $X \in \text{GoodTop}$  can be treated separately for each connected component of X, so we lose no generality by assuming that X is connected. In this case, if we choose a some basepoint  $x \in X$ , then  $\pi_{\leq 1}(X)$  is canonically equivalent to  $B\pi_1(X, x)$  and Trans(X) is canonically equivalent to GSet, where  $G = \pi_1(X, x)$ .

Furthermore, for any covering  $p: T \to X$  we may decompose T into its connected components. Such a decomposition corresponds (in the category **GSet**) to the decomposition of a G-set into a disjoint union of *transitive G-sets* (a G-set X is *transitive* if for any  $x, x' \in X$  there is  $g \in G$  such that  $g \cdot x = x'$ ). Again, we lose no generality if we assume T to be connected. Thus the category of connected coverings of a connected base X is equivalent (canonically once we choose a basepoint  $x \in X$ ) to the category **GSetTrans** of transitive G-sets, where  $G = \pi_1(X, x)$ .

We now give a more concrete description of the category of transitive G-sets for any group G.

**Proposition 10.35.** Given a group G, the category GSetTrans of transitive G-sets is equivalent to the category whose objects

We now illustrate the power of the three languages explained above by giving a simple proof of the existence of universal coverings.

**Proposition 10.36.** Any object  $X \in \mathsf{GoodTop}$  admits a *universal covering*, which is an object  $p \in \mathsf{Cov}/X$  (i.e.,  $p: T \to X$  is a covering map) such that  $\pi_0(p): \pi_0(T) \to \pi_0(X)$  is an isomorphism and whose total space T has a property that any continuous map  $S^1 \to T$  extends to a continuous map  $D^2 \to T$ , i.e., any circle can be filled by a disk. (If X is path-connected, this amounts to saying that T is simply connected.)

*Proof.* It suffices to treat the case  $\pi_0(X) = \{*\}$ , so  $\pi_{\leq 1}(X)$  is (noncanonically) equivalent to BG for some G (which can be taken to be the fundamental group of X with respect to any basepoint of X). In this case, the category  $\operatorname{Trans}(X)$  is equivalent to GSet. The latter category contains the *left regular action of* G (on itself), i.e., the pair  $(G, \rho)$ , where  $\rho: G \to \Sigma_G$  acts via  $g \cdot s := gs$  for any  $g \in G$  and  $s \in G$ . This G-set gives us an object in  $\operatorname{Trans}(X)$ , hence an object in  $\operatorname{Cov}/X$ , which is the universal cover of X.

# Limits and colimits

## 11 Products

We are familiar with such constructions as the product of sets, groups, topological spaces. As it turns out, these are all instances of a single categorical construction.

**Definition 11.1.** Given a category C and a family  $\{X_i\}_{i \in I}$  of objects in C, the *product* of  $\{X_i\}$  (if it exists) is the following collection of data that satisfies the following properties.

- An object  $Y \in \mathsf{C}$ , commonly denoted  $\prod_{i \in I} X_i$ .
- For each *i*, a projection morphism  $p_i: Y \to X_i$ .
- For any object  $Y' \in \mathbb{C}$  and any family of morphisms  $\{p'_i: Y' \to X_i\}$  there is a unique morphism  $f: Y' \to Y$  such that  $p_i f = p'_i$  for all i.

**Example 11.2.** Suppose  $I = \{0, 1\}$ . The product of  $\{X_0, X_1\}$  is typically denoted  $X_0 \times X_1$ . It has two projection maps  $p_0: X_0 \times X_1 \to X_0$  and  $p_1: X_0 \times X_1 \to X_1$ . If C = Set, then  $X_0 \times X_1 = \{(x_0, x_1) \mid x_i \in X_i\}$  is the set of all ordered pairs of elements of  $X_0$  and  $X_1$ . We have  $p_0(x_0, x_1) = x_0$  and  $p_1(x_0, x_1) = x_1$ . Used in <sup>11.7.</sup>

Special indexing classes I give rise to special types of products:

- If I is a set, we talk about *small* products.
- If *I* is a finite, infinite, countable, uncountable set, we talk about finite, infinite, countable, uncountable products.
- If I has two elements, we talk about binary products, or sometimes simply products.
- If I has a single element, we will see below that  $p: Y \to X$  is an isomorphism, so this case is trivial.
- If I is empty, we talk about *terminal objects* (see below).

**Remark 11.3.** There is no reason why products must exist, and below we will see that the category of fields does not have any products, excluding the case when I has a single element.

**Example 11.4.** If the indexing class I is empty, then the definition of an I-indexed product boils down to saying that the product of an empty family is an object  $Y \in C$  such that for any object  $Z \in C$  there is a unique morphism  $Z \to Y$ . We refer to such object as the *terminal object* of C and denote it by 1.

**Remark 11.5.** Suppose Y and Y' are terminal objects in the same category C. By definition of terminal object, there is exactly one morphism of the form  $y: Y \to Y$ ,  $f: Y \to Y'$ ,  $g: Y' \to Y$ ,  $y': Y' \to Y'$ . On the other hand,  $\operatorname{id}_Y: Y \to Y$ ,  $\operatorname{id}_{Y'}: Y' \to Y'$ ,  $gf: Y \to Y$ ,  $fg: Y' \to Y'$  are morphisms, so we must have  $\operatorname{id}_Y = y = gf$  and  $\operatorname{id}_{Y'} = y' = fg$ . In other words, f and g are mutually inverse isomorphisms. Thus any two terminal objects in the same category are isomorphic, and the isomorphism itself is unique. This is the reason why we talk about the terminal object of C above: although the terminal object is not unique, it is unique up to a unique isomorphism, which is all what we really care about. For instance, in the category of groups the terminal object is the group with one element. Even though there is more than such group because there are many sets with one element, they are all uniquely isomorphic to each other. There is no reason for terminal object to exist, though: for instance, the category of fields has no terminal object.

**Remark 11.6.** Suppose C is a category and  $\{X_i\}_{i \in I}$  is an *I*-indexed family of objects in C. Consider the category  $\operatorname{Cone}_X$ , whose objects are pairs  $(Y, \{p_i\})$ , where  $Y \in \mathsf{C}$  and  $p_i: Y \to X_i$ , and morphisms  $(Y', \{p'_i\}) \to (Y, \{p_i\})$  are morphisms  $f: Y' \to Y$  such that  $p'_i = p_i f$  for all  $i \in I$ . The product of  $\{X_i\}$ , if it exists, is the terminal object in this category. In particular, in light of the above remark, the product of  $\{X_i\}$  is unique up to a unique isomorphism.

**Remark 11.7.** We return to Example 11.2. We glossed over the definition of ordered pair and the existence of  $X_0 \times X_1$ . In traditional Zermelo-style set theory all objects are sets, so ordered pairs must be encoded as sets. There are many different definitions of ordered pairs in terms of sets. For instance, one can take  $(x_0, x_1) = \{\{x_0\}, \emptyset\}, \{\{x_1\}\}\}$  (Wiener) or  $(x_0, x_1) = \{\{x_0\}, \{x_0, x_1\}\}$  (Kuratowski). Both of these definitions satisfy the following property of ordered pairs:  $(x_0, x_1) = (x'_0, x'_1)$  if and only if  $x_0 = x'_0$  and  $x_1 = x'_1$ . As for existence, using the first definition we see that  $(x_0, x_1) \subset 2^{2^{X_0 \cup X_1}}$ , whereas for the second definition we have  $(x_0, x_1) \subset 2^{X_0 \cup X_1}$ . This allows us to define  $X_0 \times X_1 = \{z \in 2^{2^{2^{X_0 \cup X_1}}} \mid \exists x_0, x_1: z = (x_0, x_1) \land x_i \in X_i\}$  (using the first definition; for the second definition we take  $z \in 2^{2^{X_0 \cup X_1}}$ ). Thus the existence of binary products of sets follows from the axioms of separation, power set, pair, and extensionality.

**Remark 11.8.** The case of finite products of sets can be treated similarly to the case of binary products. The product of  $\{X_0, \ldots, X_{n-1}\}$  is denoted  $X_0 \times \cdots \times X_{n-1}$ . It can be written as  $\{(x_0, \ldots, x_{n-1}) \mid x_i \in X_i\}$ , and the projection map  $p_i$  sends  $(x_0, \ldots, x_{n-1})$  to  $x_i$ . Of particular interest is the case n = 0: the product of the empty family of sets is a set consisting of a single element, namely, the empty tuple (). This singleton set is the terminal object in the category of sets.

**Remark 11.9.** Suppose we have no knowledge of ordered pairs. How could one guess what  $X_0 \times X_1$  should be just from its universal property? We have already seen that elements of any set Y are in canonical bijection with morphisms  $\{*\} \to Y$  in the category of sets, where  $\{*\}$  denotes some arbitrary fixed singleton set. Using the universal property of  $X_0 \times X_1$  we see that elements of  $X_0 \times X_1$  are in bijection with morphisms  $\{*\} \to X_0 \times X_1$ , which themselves are in bijection with morphisms  $\{*\} \to X_0$  and  $\{*\} \to X_1$ , which themselves can be identified with elements of  $X_0$  and  $X_1$ . This determines the set  $X_0 \times X_1$ .

**Example 11.10.** Infinite (small) products of sets are constructed in a similar fashion. Given a family  $\{X_i\}$  of sets, we set  $\prod_i X_i = \{f: I \to \bigcup_i X_i \mid \forall i \in I: f(i) \in X_i\}$ . The projection maps  $p_k: \prod_i X_i \to X_k$  are given by evaluation:  $p_k(f) = f(k)$ . Given a family of functions  $p'_i: Y \to X_i$ , the (unique) function  $f: Y \to \prod_i X_i$  such that  $p_i f = p'_i$  is given by  $f(y)(i) = p'_i(y)$ .

**Example 11.11.** Suppose  $X_i \neq \emptyset$  for all  $i \in I$ . Then  $\prod_i X_i \neq \emptyset$  if and only if the axiom of choice is satisfied for the family  $\{X_i\}$ .

**Example 11.12.** If the indexing family I is a proper class, the product  $\prod_i X_i$  of an I-indexed family  $\{X_i\}$  need not exist. For instance, take  $X_i = \{0, 1\}$ . The elements of the set  $\prod_i X_i$ , if it exists, should be in bijection with I-indexed families of elements of  $\{0, 1\}$ , which can be identified with subclasses of I.

Subclasses of the proper class I themselves form a proper class, therefore  $\prod_i X_i$  does not exist. If I is a proper class, then  $\prod_i X_i$  does exist in some situations. If  $X_i = \emptyset$  for some i, then  $\prod_i X_i = \emptyset$ . If  $X_i$  is a singleton set away from some subset (and not just a subclass)  $I' \subset I$ , then  $\prod_i X_i \cong \prod_{I'} X_i$ .

From now on we will concentrate on small products, whereas the case of products indexed by a proper class can be treated similarly to the above remark.

## 11.13. Algebra

**Example 11.14.** In the category of groups small products always exist. Indeed, homomorphisms of groups  $\mathbf{Z} \to G$  can be identified with elements of G. Consider a family  $\{G_i\}$  of groups with the underlying sets  $U(G_i)$ , where  $U: \operatorname{Group} \to \operatorname{Set}$  is the forgetful functor. By the universal property of products, the product  $\prod_i G_i$ , if it exists, must satisfy  $U(\prod_i G_i) \cong \prod_i U(G_i)$ , i.e., the underlying set of product is the product of underlying sets. Using the same trick with homomorphisms from  $\mathbf{Z}$ , we see that  $U(p_i)$  must be the projection map  $\prod_i U(G_i) \to U(G_i)$ . The projection maps are homomorphisms of groups, and this *forces* the group operations on  $\prod_i U(G_i)$  to be defined indexwise. Once again, we managed to discover what the product must be via simple applications of the universal property.

Two special cases deserve to be mentioned: of course, the product of a two-element family of groups  $\{G_0, G_1\}$  is their usual group-theoretic product  $G_0 \times G_1$ . The product of the empty family is the terminal group, i.e., the group with one element.

Nothing in the above example is specific to groups. Exactly the same argument works for rings, modules, monoids, and other algebraic structures that are defined using only algebraic identities. Such structures are known as *varieties of algebras*.

Fields do *not* form a variety of algebras. Nonzero elements in a field must form a group with respect to multiplication, and it is not possible to have an algebraic identity that is true only for some elements. More formally, in any variety of algebras products must exist, which is false for the category of fields.

Nonexample 11.15. In the category of fields only products of one-element families exist.

#### 11.16. General topology

**Example 11.17.** In the category of topological spaces products can be computed as follows. First, observe that continuous functions  $* \to X$ , where  $X \in \mathsf{Top}$  and \* denotes the singleton topological space, are in bijection with the points of X. Thus the points of  $X_0 \times X_1$  are in bijection with continuous functions  $* \to X_0 \times X_1$ . The latter are in bijection with pairs of continuous functions  $* \to X_0$  and  $* \to X_1$ , i.e., pairs of points in  $X_0$  and  $X_1$ . Thus  $U(X_0 \times X_1) \cong U(X_0) \times U(X_1)$ , where  $U:\mathsf{Top} \to \mathsf{Set}$  is the forgetful functor. Likewise, U sends the projection morphisms  $X_0 \times X_1 \to X_0$  and  $X_0 \times X_1 \to X_1$  to the corresponding projection functions. Furthermore, the projection maps are continuous, so sets of the form  $U_0 \times U_1$ , where  $U_0 \subset X_0$  and  $U_1 \subset X_1$  are open, must be open in  $X_0 \times X_1$ . If we take these open sets as a base of a topology for  $X_0 \times X_1$ , the resulting topological space does satisfy the universal property of a product, which in this case boils down to saying that for any continuous maps  $Y \to X_0$  and  $Y \to X_1$  form a base of  $X_0 \times X_1$ .

#### 11.18. Functional analysis

**Example 11.19.** In the category  $\text{Ban}_1$  small products exist and can be computed as follows. Observe that morphisms  $\mathbb{C} \to X$  for some  $X \in \text{Ban}_1$  are in bijection with the elements of the unit ball of X. If  $\prod_i X_i$  exists, its unit ball must be isomorphic to  $\text{Mor}_{\text{Ban}_1}(\mathbb{C}, \prod_i X_i) \cong \prod_i \text{Mor}_{\text{Ban}_1}(\mathbb{C}, X_i) \cong \prod_i (X_i)_{\leq 1}$ . In particular, we have  $U(\prod_i X_i) \subset \prod_i U(X_i)$ , where  $U: \text{Ban}_1 \to \text{Set}$  is the forgetful functor, and the inclusion is proper. Reconstructing a Banach space from its unit ball  $\prod_i (X_i)_{\leq 1}$  we get that  $\prod_i X_i$  is a Banach space such that  $U(\prod_i X_i)$  is the subset of  $\prod_i U(X_i)$  consisting of those tuples f such that  $i \mapsto ||f(i)||$  is a bounded function. Algebraic operations are defined indexwise. The norm of  $f \in \prod_i X_i$  is defined as  $\sup_{i \in I} ||f(i)||$ , which in fact is forced upon us by the fact that the projection functions  $p_k$  are contractive maps.

#### 11.20. Measure theory

The previous examples were all based on the same scheme: use some simple object S (such as a singleton set or a one-dimensional vector space) such that morphisms  $S \to X$  detect various information about X, such as its underlying set, algebraic operations on this set, projection maps, etc. Further, the fact that the projection maps must be morphisms in the category allowed us to recover any remaining structure (this was the case with topological spaces and Banach spaces).

The situation with the categories Meas and LocMeas is quite different. For starters, there is no measurable space that could play the role of S. Of course, one could try to take  $S = (\{*\}, \{\emptyset, \{*\}\}, \{\emptyset\}, \{\emptyset\}\}, \{\emptyset\})$  or  $S = (\{*\}, \{\emptyset, \{*\}\}, \{\emptyset, \{*\}\}, \{\emptyset, \{*\}\}\})$ . The first takes the discrete singleton measurable space. Morphisms  $S \to X$  can be identified with *atoms* of X, i.e., points  $x \in X$  such that  $\{x\} \notin N$ . This tells us that the set of atoms of a product of measurable spaces is the product of sets of atoms of individual factors. However, the set of atoms tells us very little about the measurable space, for instance, Lebesgue( $\mathbb{R}$ ) has no atoms. The second choice of S is even worse: it is isomorphic to  $(\emptyset, \{\emptyset\}, \{\emptyset\})$ , and there is exactly one morphism  $S \to X$  for any X, which reveals no information about X whatsoever.

However, we can still guess that one can construct the product  $(X_0, M_0, N_0) \times (X_1, M_1, N_1)$  in the form  $(X_0 \times X_1, -, -)$  and then try to guess what the blanks should be assuming that the projection morphisms are represented by the set-theoretical projection functions. Indeed, this immediately tells us that sets of the form  $m_0 \times m_1$  must be measurable for any  $m_0 \in M_0$  and  $m_1 \in M_1$ , whereas sets of the form  $n_0 \times m_1$  and  $m_0 \times n_1$  must be negligible for any  $n_0 \in N_0$  and  $n_1 \in N_1$ .

**Proposition 11.21.** The product of a countable family  $\{(X_i, M_i, N_i)\}_{i \in I}$  of measurable spaces can be computed as  $(\prod_i X_i, M, N)$  with projection maps  $p_i: \prod_k X_k \to X_i$ , where N is the  $\sigma$ -ideal generated by the sets  $p_i^{-1}(n)$  for all  $n \in N_i$  and  $i \in I$  and M is the  $\sigma$ -algebra generated by N and the sets  $p_i^{-1}(m)$  for all  $m \in M_i$  and  $i \in I$ .

Proof. The maps  $p_i$  are morphisms of measurable spaces by definition of M and N. We now verify the universal property for a given  $(Y, M_Y, N_Y)$  and morphisms  $f_i: (Y, M_Y, N_Y) \to (X_i, M_i, N_i)$ . For existence, observe that the tuple  $(f_i): (Y, M_Y, N_Y) \to \prod_i (X_i, M_i, N_i)$  is a morphism by definition of M and N. For uniqueness, suppose  $f, g: (Y, M_Y, N_Y) \to \prod_i (X_i, M_i, N_i)$  are morphisms such that  $p_i f = p_i g$  for all  $i \in I$ . If we fix some representatives for f and g, this means that some  $p_i f = p_i g$  on a conegligible set, whose preimage under  $p_i$  is also conegligible. Now f = g on the (countable) intersection of these conegligible sets, hence f and g are representatives of the same morphism.

## 12 Coproducts

Coproducts are defined as products in the opposite category.

**Definition 12.1.** Given a category C and a family  $\{X_i\}_{i \in I}$  of objects in C, their coproduct  $\coprod_{i \in I} X_i$  is the product of  $\{X_i\}$  in C<sup>op</sup>.

We unfold this definition to make it easier to see what is going on. The coproduct consists of an object  $Y = \coprod_{i \in I} X_i$  in C together with a family of *injection morphisms*  $\iota_i \colon X_i \to Y$  (which need not be injections in the sense of set theory) such that the following universal property is satisfied: for any object  $Y' \in \mathsf{C}$  with morphisms  $\iota'_i \colon X_i \to Y'$  there is a unique morphism  $f \colon Y \to Y'$  such that  $f\iota_i = \iota'_i$  for all  $i \in I$ .

**Definition 12.2.** The *initial object* in a category C is the coproduct of the empty family, i.e., an object  $0 \in C$  such that for any object  $Y \in C$  there is a unique morphism  $0 \to Y$ .

**Example 12.3.** Suppose C = Set. Then we can take  $\coprod_i X_i = \bigcup_i X_i \times \{i\}$  and  $\iota_i(x) = (x, i)$  for any  $x \in X_i$ . In this case the coproduct is known as the *disjoint union* of  $X_i$ .

**Example 12.4.** If C = Top, then  $U(\coprod_i X_i) = \coprod_i U(X_i)$ . The topology on  $\coprod_i U(X_i)$  can be recovered by taking Y to be the Sierpiński space, i.e.,  $\{0, 1\}$  with the base of open sets  $\{\{1\}\}$ . Maps into Y are in bijection with open subsets of the source, which tells us that open subsets in  $\coprod_i U(X_i)$  are precisely unions of  $\iota_i(U)$ , where  $U \subset X_i$  is open.

**Example 12.5.** If C = Ab, more generally,  $Mod_R$  for some ring R, then  $\coprod_i X_i$  can be computed as the R-module of finitely supported functions  $I \to \coprod_i U(X_i)$  with indexwise operations. If I is finite, then  $\coprod_i X_i$  is in fact isomorphic to  $\prod_i X_i$ .

**Example 12.6.** If C = Group, then  $\coprod_i X_i$  can be described as a group whose elements are finite tuples of elements in  $X_i$  and their formal inverses with composition given by concatenation, modulo the relations  $x \cdot x^{-1} = e$ . In particular,  $\coprod_i \mathbf{Z}$  is known as the *free group* on *I*.

**Example 12.7.** If C = CRing, then  $\coprod_i X_i$  is also known as the (infinite) tensor product of  $X_i$ .

Nonexample 12.8. If C = Field, no nonsingleton coproducts exist, essentially for the same reason as products. If we restrict to the full subcategory of fields of some fixed characteristic p, the resulting category does have an initial object: the prime field of characteristic p, i.e.,  $\mathbf{Z}/p\mathbf{Z}$  for  $p \neq 0$  and  $\mathbf{Q}$  for p = 0.

### 13 Equalizers

Equalizers generalizes notions such as kernels of groups and fixed points of maps.

**Definition 13.1.** Suppose C is a category and  $f, g: X \to Y$  is a pair of parallel morphisms in C. The equalizer of f and g is a morphism  $e: W \to X$  such that fe = ge and for any other morphism  $e': W' \to X$  such that fe' = ge' there is a unique morphism  $h: W' \to W$  such that eh = e'.

**Remark 13.2.** Just like in the case of products, we can define the equalizer of f and g as the terminal object in a certain category  $\mathsf{Fork}_{f,g}$ , whose objects are morphisms  $e: W \to X$  in  $\mathsf{C}$  such that fe = ge (i.e., forks  $W \xrightarrow{e} X \xrightarrow{f} \xrightarrow{g} Y$ ) and morphisms from  $e: W \to X$  to  $e': W' \to X$  are morphisms  $h: W \to W'$  in  $\mathsf{C}$  such that e'h = e. Used in 13.3, 14.2.

**Example 13.3.** The equalizer of f and g in the category C = Set can be computed as  $W = \{x \in X \mid f(x) = g(x)\}$  with its canonical inclusion map into X. Indeed, if  $e': W' \to X$  is such that fe' = ge', then f and g are equal on the image of e', or in other words, the image of e' is contained in  $\{x \in X \mid f(x) = g(x)\} = W$ . Therefore, there is a unique map  $h: W' \to W$  such that eh = e', namely, the corestriction of e' to  $W \subset X$ . In terms of forks, we see that the category  $\mathsf{Fork}_{f,g}$  is equivalent to the category  $\mathsf{Set}/W$  (whose terminal object is id:  $W \to W$ ).

**Example 13.4.** The set of fixed points of a function  $f: X \to X$  is the equalizer of f and  $id_X$ , i.e.,  $\{x \in X \mid f(x) = x\}$ .

**Example 13.5.** The equalizer of f and g in the category C = Group can be computed as the subgroup  $W = \{x \in X \mid f(x) = g(x)\}$  with its canonical inclusion map into X. The same is true for any other varieties of algebras e.g., Ab,  $\text{Mod}_R$ ,  $\text{Vect}_k$ , Ring, CRing,  $\text{Alg}_k$ .

**Example 13.6.** The equalizer of f and g in the category C = Ab,  $Vect_k$ , or  $Mod_R$  can be computed as the set  $W = \{x \in X \mid f(x) = g(x)\}$  with its induced operations and the canonical inclusion map into X. Indeed,  $W \in C$  because W is closed under all operations like addition and multiplication by scalars because f and g are homomorphisms. The remainder of the argument proceeds in the same way as for C = Set.

**Example 13.7.** The equalizer of f and g in the category C = Top can be computed as the subspace  $W = \{x \in X \mid f(x) = g(x)\}$  (with the induced topology) with its canonical inclusion map into X.

**Example 13.8.** Suppose C is an arbitrary category,  $X \in C$ , and  $f: X \to X$ . The equalizer W of f and  $id_X$  is the *fixed points object* of f. If C = Set, then  $W = \{x \in X \mid x = f(x)\}$ .

**Example 13.9.** Suppose C is Ab,  $Vect_k$ , or  $Mod_R$ . The equalizer of  $f: X \to Y$  and the zero morphism  $0: X \to Y$  is the *kernel* of f.

#### 14 Coequalizers

Coequalizers in C are precisely equalizers in  $C^{op}$ . As usual, we unfold the definition.

**Definition 14.1.** Suppose C is a category and  $f, g: X \to Y$  is a pair of parallel morphisms in C. The *coequalizer* of f and g is a morphism  $q: Y \to Z$  such that qf = qg and for any other morphism  $q': Y \to Z'$  such that q'f = q'g there is a unique morphism  $h: Z \to Z'$  such that hq = q'.

**Remark 14.2.** Coequalizers are precisely initial objects in the category  $\mathsf{Cofork}_{f,g}$ , defined entirely analogously to  $\mathsf{Fork}_{f,g}$ .

**Example 14.3.** Any equivalence relation R on a set X gives rise to a coequalizer diagram

$$R \xrightarrow[p_1]{p_1} X \longrightarrow X/R.$$

The universal property of coequalizers in this case boils down to precisely the universal property of X/R, i.e., functions  $X/R \to Z$  can be identified with functions  $f: X \to Z$  such that f(x) = f(x') whenever  $x \sim x'$  in R.

**Example 14.4.** In general, in the category Set the coequalizer of f and g can be computed as the quotient map  $Y \to Y/R$ , where R is the equivalence relation on Y generated by  $f(x) \sim g(x)$  for all  $x \in X$ . The universal property of the coequalizer in this case coincides with the universal property of the quotient map.

**Example 14.5.** The quotient topological space X/R of a topological space X with respect to an equivalence relation R on X is defined as the set X/R equipped with the topology whose open subsets are those subsets of X/R whose preimage under the quotient map  $X \to X/R$  is open. The cofork

$$R \xrightarrow[p_1]{p_0} X \longrightarrow X/R$$

is a coequalizer cofork in the category Top. Here R on the left is equipped with the discrete topology.

**Example 14.6.** In general, in the category Top the coequalizer of f and g can be computed as the quotient topological space  $Y \to Y/R$ , where R is the equivalence relation on Y generated by  $f(x) \sim g(x)$  for all  $x \in X$ . The universal property of the coequalizer again coincides with the universal property of the quotient map.

**Example 14.7.** In the category Group, the coequalizer of f and g can be computed as follows. First, observe that the homomorphism  $q: Y \to Z$  must satisfy q(f(x)) = q(g(x)) for any  $x \in X$ , in particular,  $q(f(x)g(x)^{-1}) = 1$ , i.e.,  $f(x)g(x)^{-1}$  is in the kernel of q. Thus the kernel of q must contain the normal subgroup N generated by the elements  $f(x)g(x)^{-1}$ , i.e., the closure of the set  $\{f(x)g(x)^{-1} \mid x \in X\}$  under multiplication, inverses, and conjugations by arbitrary elements of the group Y. We claim that the quotient homomorphism  $Y \to Y/N$  is the desired coequalizer. Indeed, by construction, homorphisms  $Y/N \to Z'$  are precisely those homomorphisms  $q': Y \to Z'$  that are identity on N, equivalently, identity on  $f(x)g(x)^{-1}$ , or equivalently, q'(f(x)) = q'(g(x)).

**Example 14.8.** Using an entirely similar argument, the coequalizer of f and g in the categories  $Ban_1$  and Ban can be computed as the quotient map  $q: Y \to Y/N$ , where N is the norm closure of the linear span of f(x) - g(x) for all  $x \in X$ .

#### 15 Sequential colimits

Recall that  $\mathbf{N} = \{0, 1, 2, ...\}$  denotes the set of natural numbers.

**Definition 15.1.** Suppose C is a category,  $\{X_i\}_{i \in \mathbb{N}}$  is an infinite family of objects in C and  $t_i: X_i \to X_{i+1}$  are morphisms in C (sometimes referred to as *transition maps*). The above data can also be written as  $X_0 \xrightarrow{t_0} X_1 \xrightarrow{t_1} X_2 \xrightarrow{t_2} \cdots$ . The sequential colimit of (X, t) (denoted  $\operatorname{colim}_{i \in I} X_i$ ) is an object  $U \in C$  equipped with a family of injection morphisms  $\iota_i: X_i \to U$  such that  $\iota_{i+1}t_i = \iota_i$  for all  $i \in \mathbb{N}$  and the following universal property is satisfied: if  $U' \in C$  and  $\iota'_i: X_i \to U'$  satisfy  $\iota'_{i+1}t_i = \iota'_i$ , then there is a unique morphism  $f: U \to U'$  such that  $f\iota_i = \iota'_i$  for all  $i \in \mathbb{N}$ .

**Remark 15.2.** In complete analogy with coproducts and coequalizers, the sequential colimit of (X, t) can be described as the initial cocone over (X, t). Here a cocone over (X, t) is a pair  $(U, \iota)$ , where  $U \in \mathsf{C}$  and  $\iota_i: X_i \to U$  are such that  $\iota_{i+1}t_i = \iota_i$  for all  $i \in \mathbf{N}$ . A morphism of cocones  $(U, \iota) \to (U', \iota')$  is a morphism  $f: U \to U'$  in  $\mathsf{C}$  such that  $f_{\iota_i} = \iota'_i$  for all  $i \in \mathbf{N}$ .

**Example 15.3.** In the category Set the sequential colimit of (X, t) can be computed as the quotient of  $\coprod_i X_i$  with respect to the equivalence relation generated by  $(i, x) \sim (i + 1, t_i(x))$ , where  $x \in X_i$  and  $i \in \mathbb{N}$ . In other words, any element of  $X_i$  for any  $i \in \mathbb{N}$  gives an element in the sequential colimit (we say that  $x \in X_i$  lives at stage i), and two such elements (i, x) and (j, y) are equivalent if we can send both of them to a later stage k ( $k \ge i$  and  $k \ge j$ ) so that  $(k, t(t(\cdots x))) = (k, t(t(t(\cdots y))))$ .

**Example 15.4.** Continuing the previous example, if all transition maps  $t_i: X_i \to X_{i+1}$  are inclusions of sets, then  $\coprod_i X_i$  can be computed as  $\bigcup_i X_i$ , with inclusions as the injection maps.

**Example 15.5.** In the category Group (or any other variety of algebras) the sequential colimit of (X, t) can be computed by computing the sequential colimit of the underlying sets, i.e., (U(X), U(t)), where U:Group  $\rightarrow$  Set is the forgetful functor and equipping it with group operations as follows. Given two elements (i, x) and (j, y), we first replace them by equivalent elements (k, x') and (k, y') as explained above. Then the product of the equivalence classes of (i, x) and (j, y) is defined as the equivalence class of (k, x'y'). A different choice of k produces the same answer if we use the same trick one more time.

**Example 15.6.** In the category Top the sequential colimit of (X, t) can be computed by taking the sequential colimit of underlying sets in Set and equipping it with the topology in which a subset is open if and only if all of its preimages under the injection maps are open.

#### 16 Sequential limits

Sequential limits are sequential colimits in the opposite category. As usual, we provide an unfolded definition.

**Definition 16.1.** Suppose C is a category,  $\{X_i\}_{i \in \mathbb{N}}$  is an infinite family of objects in C and  $t_i: X_{i+1} \to X_i$  are morphisms in C (sometimes referred to as *transition maps*). The above data can also be written as  $X_0 \xleftarrow{t_0} X_1 \xleftarrow{t_1} X_2 \xleftarrow{t_2} \cdots$ . The sequential limit of (X, t) (denoted  $\lim_{i \in I} X_i$ ) is the terminal object in the category  $\operatorname{Cone}_{X,t}$  of cones over (X, t). Objects in  $\operatorname{Cone}_{X,t}$  are pairs (U, p), where  $U \in C$  and  $p_i: X_i \leftarrow U$  are such that  $p_i = t_i p_{i+1}$  for all  $i \in \mathbb{N}$ . Morphisms  $(U, p) \to (U', p')$  are morphisms  $f: U \to U'$  in C such that  $p_i f = p'_i$  for all  $i \in \mathbb{N}$ .

**Example 16.2.** In the category Set the sequential limit of (X, t) can be computed as the subset of  $\coprod_i X_i$  consisting of those families  $(x_i)_{i \in I}$  for which  $t_i(x_{i+1}) = x_i$  for all  $i \in I$ .

**Remark 16.3.** If all transition maps are inclusions of sets, then the sequential limit can be computed as the intersection.

**Example 16.4.** In the category Top of topological spaces the sequential limit of (X, t) can be computed as the sequential limit of (U(X), U(t)) in sets (here  $U: \text{Top} \to \text{Set}$  is the forgetful functor) equipped with the topology generated by a subbase of sets given by the preimages of open subsets of  $X_i$  under the projection maps  $p_i$ .

We illustrate the power of sequential limits and colimits by giving a conceptual explanation of the topology on compactly supported smooth functions.

**Example 16.5.** Consider the category TopVect of topological vector spaces. Sequential limits and colimits are computed like for vector spaces and topological spaces (taking closures of subspaces whenever quotients are needed). The topological vector space of compactly supported smooth functions on a smooth manifold M (e.g., an open subset of  $\mathbb{R}^n$ ) is defined as  $\operatorname{colim}_{K \subset M} \lim_{k \in \mathbb{N}} C_K^k(M, \mathbb{R})$ . Here  $C_K^k(M, \mathbb{R})$  is the topological vector space of k times differentiable functions on M whose support is a subset of K. The set K runs over a countable system of compact subsets of M that covers M. (We assume M to be second countable here. Below we will see that the above sequential colimit can be replaced by a *filtered colimit* over all compact subsets  $K \subset M$ .) The maps  $t_k: C_K^{k+1}(M, \mathbb{R}) \to C_K^k(M, \mathbb{R})$  are inclusions. The limit  $\lim_{k \in \mathbb{N}} C_K^k(M, \mathbb{R})$  has  $C_K^\infty(M, \mathbb{R})$  as its underlying vector space. The topology is generated from a local subbase of open neighborhoods of zero given by  $\{f \in C_K^\infty(M, \mathbb{R}) \mid \|f\|_k < \epsilon\}$  for all  $k \in \mathbb{N}$  and  $\epsilon > 0$ , where  $\|-\|_k$  is the norm in  $\mathbb{C}^k$ . The topological vector space  $C_K^\infty(M, \mathbb{R})$  is not a Banach space even though  $C_K^k(M, \mathbb{R})$  is a Banach space for all  $k \in \mathbb{N}$ . The maps  $C_{cs}^\infty(M, \mathbb{R}) \to C_{C'}^\infty(M, \mathbb{R})$  for  $K \subset K'$  are inclusions. The colimit colim $_{K \subset M} C_K^\infty(M, \mathbb{R})$  has  $C_{cs}^\infty(M, \mathbb{R})$ , the space of compactly supported smooth functions on M, as its underlying vector space. A subset of  $\mathbb{C}_{cs}^\infty(M, \mathbb{R})$  is open if its intersections with  $\mathbb{C}_K^\infty(M, \mathbb{R})$  are open for all compact  $K \subset M$ .

#### 17 Limits and colimits

**Definition 17.1.** A *diagram* in a category C is a functor  $D: I \to C$ , where I is the *indexing category*. If I is small, finite, and so on, we talk about small diagrams, finite diagrams, and so on.

**Example 17.2.** Consider the discrete category on a set I: its objects are elements of I and the only morphisms are identity morphisms. An I-indexed diagram in C is precisely the data we used to define products and coproducts in C, i.e., an I-indexed family of objects in C.

**Example 17.3.** Consider the category  $\{0 \implies 1\}$  with two objects 0 and 1 and two nonidentity morphisms from 0 to 1. An *I*-indexed diagram in C is precisely the data we used to define equalizers and coequalizers in C, i.e., two objects  $X, Y \in C$  and two parallel morphisms  $f, g: X \to Y$ .

**Example 17.4.** Turn the poset  $\mathbf{N} = \{0 < 1 < 2 < 3 < \cdots\}$  of natural numbers into a category as explained in Example 9.37. An N-indexed diagram in C is precisely the data we used to define sequential colimits in C, i.e.,  $X_0 \xrightarrow{t_0} X_1 \xrightarrow{t_1} X_2 \xrightarrow{t_2} \cdots$ . Likewise, an N<sup>op</sup>-indexed diagram in C is the data used to define sequential limits in C, i.e.,  $X_0 \xleftarrow{t_0} X_1 \xleftarrow{t_1} X_2 \xleftarrow{t_2} \cdots$ .

**Definition 17.5.** Given a diagram  $D: I \to \mathsf{C}$ , the category  $\mathsf{Cone}_D$  of *cones* over D is defined as follows. The objects of  $\mathsf{Cone}_D$  are pairs  $(U, \{p_i\}_{i \in I})$ , where  $U \in \mathsf{C}$  and for any object  $i \in I$  the projection morphism  $p_i: U \to D(i)$  is a morphism in  $\mathsf{C}$  such that for any morphism  $t: i \to j$  in I we have  $D(t) \circ p_i = p_j$ . A morphism  $(U, p) \to (U', p')$  is a morphism  $f: U \to U'$  in  $\mathsf{C}$  such that  $p'_i f = p_i$  for all  $i \in I$ . Used in 11.6, 16.1, 17.5, 17.7.

**Definition 17.6.** The category  $\mathsf{Cocone}_D$  of *cocones* over a diagram D is defined analogously, using injection morphisms  $\iota_i: D(i) \to U$  such that  $\iota_i = \iota_j \circ D(t)$  and morphisms of cocones satisfying  $f\iota_i = \iota'_i$ . Used in 17.7.

**Definition 17.7.** The *limit* (if it exists) of a diagram  $D: I \to C$  in a category C is the terminal object in the category  $Cone_D$ , typically denoted as  $\lim_I D$  and by abuse of notation identified with its underlying object U in C. Likewise, the *colimit* of D is the initial object in the category  $Cocone_D$ , typically denoted as  $colim_I D$ . Used in 15.1, 16.1, 16.5, 20.2, 20.3, 20.5, 21.2, 21.3, 21.4, 23.7, 23.7\*, 27.0\*, 27.7, 28.2.

**Example 17.8.** The three types of diagrams considered above yield (co)products, (co)equalizers, and sequential (co)limits.

#### 18 Pullbacks

Consider the indexing category  $I = \{1 \rightarrow 0 \leftarrow 2\}$ . *I*-indexed limits are known as *pullbacks*.

**Definition 18.1.** The pullback  $X_1 \times_{X_0} X_2$  of a diagram  $X_1 \xrightarrow{t_1} X_0 \xleftarrow{t_2} X_2$  in a category C is the limit of associated functor  $I \to C$ .

**Remark 18.2.** Unfolding the above definition, one notices some redundancy. Specifically, the definition of a cone involves three morphisms  $p_0: Y \to X_0$ ,  $p_1: Y \to X_1$ , and  $p_2: Y \to X_2$  such that  $t_1p_1 = p_0 = t_2p_2$ . This tells us that the data of  $p_0$  is redundant and it suffices to consider  $p_1$  and  $p_2$  such that  $t_1p_1 = t_2p_2$ . This can be presented as a commutative square

$$Y \xrightarrow{p_1} X_1$$
$$\downarrow^{p_2} \qquad \downarrow^{t_1}$$
$$X_2 \xrightarrow{t_2} X_0.$$

**Example 18.3.** In the category Set we have  $X_1 \times_{X_0} X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid t_1(x_1) = t_2(x_2)\}$ , the fiber product of  $X_1$  and  $X_2$  over  $X_0$ .

**Example 18.4.** Suppose  $t_2: X_2 \to X_0$  is an inclusion of sets, i.e.,  $X_2 \subset X_0$ . Then  $X_1 \times_{X_0} X_2$  can be computed as  $t_1^{-1}(X_2)$ . The map  $p_1: t_1^{-1}(X_2) \to X_1$  is the canonical inclusion, whereas  $p_2: t_1^{-1}(X_2) \to X_2$  is the restriction of  $t_1$ .

## 19 Pushouts

Pushouts are pullbacks in the opposite category. Consider the indexing category  $I = \{1 \leftarrow 0 \rightarrow 2\}$ .

**Definition 19.1.** The pushout  $X_1 \sqcup_{X_0} X_2$  of a diagram  $X_1 \xleftarrow{t_1} X_0 \xrightarrow{t_2} X_2$  in a category C is the colimit of associated functor  $I \to C$ .

**Example 19.2.** In the category Set we have  $X_1 \sqcup_{X_0} X_2 = (X_1 \sqcup X_2)/\langle t_1(x_1) \sim t_2(x_2) \mid (x_1, x_2) \in X_1 \times X_2 \rangle$ , where  $\langle \cdots \mid \cdots \rangle$  denotes the equivalence relation generated by a given set of pairs.

**Example 19.3.** In the category Group we have  $X_1 \sqcup_{X_0} X_2 = (X_1 \sqcup X_2)/\langle t_1(x_1)^{-1}t_2(x_2) | (x_1, x_2) \in X_1 \times X_2 \rangle$ , where  $\langle \cdots | \cdots \rangle$  denotes the normal subgroup generated by given elements.

**Example 19.4.** In the category CRing we have  $X_1 \sqcup_{X_0} X_2 = X_1 \otimes_{X_0} X_2$ , the tensor product of  $X_1$  and  $X_2$  over  $X_0$ .

#### 20 Expressing limits via equalizers and products

**Proposition 20.1.** For any diagram  $D: I \to C$ , the equalizer E of  $\prod_{k \in I} D(k) \xrightarrow{=f \longrightarrow g} \prod_{\gamma: i \to j} D(j)$ , if it exists, is the limit of D. Here the components of f and g are  $f_{\gamma} = p_j: \prod_{k \in I} D(k) \to D(j)$  and  $g_{\gamma} = D(\gamma) \circ p_i: \prod_{k \in I} D(k) \to D(j)$ . The projection maps are given by projecting the equalizer first to  $\prod_{k \in I} D(k)$ , and then using the projection maps for the product.

Proof. Morphisms  $X \to E$  can be identified with morphisms  $h: X \to \prod_{k \in I} D(k)$  such that fh = gh. Morphisms  $h: X \to \prod_{k \in I} D(k)$  can be identified with a family of morphisms  $h_k: X \to D(k)$ . The equality fh = gh can be likewise decomposed into a family of equalities  $(fh)_{\gamma} = (gh)_{\gamma}$  for each  $\gamma: i \to j$ . Unfolding the definitions of f and g, this becomes  $p_jh = D(\gamma)p_ih$ . Decomposing h into its components, this becomes  $h_j = D(\gamma)h_i$ . Assembling all of this together, a morphism  $X \to E$  is the same data as a family of morphisms  $h_k: X \to D(k)$  such that  $h_j = D(\gamma)h_i$  for all  $\gamma: i \to j$ . This is precisely what a cone over D is. This completes the proof of the fact that E with its projection maps satisfies the universal property of a limit.

**Corollary 20.2.** In the category Set we have  $\lim D = \{x \in \prod_{k \in I} D(k) \mid D(\gamma)(x_i) = x_j\}$ , where  $\gamma: i \to j$  is an arbitrary morphism in *I*. The projection maps are obtained by restricting the projection maps of  $\prod_{k \in I} D(k)$ .

**Corollary 20.3.** In any variety of algebras, e.g., Ab,  $Vect_k$ ,  $Mod_R$ , Group,  $Alg_k$  (but not Field), we can computed  $\lim D$  by computing  $\lim U \circ D$ , i.e., the limit of the underlying sets, and equipping it with the indexwise algebraic operations.

By applying the above results to  $C^{op}$ , we get analogous descriptions of colimits.

**Proposition 20.4.** For any diagram  $D: I \to C$ , the coequalizer E of  $\coprod_{\gamma:i\to j} D(i) \xrightarrow{-f \to j} \coprod_{k\in I} D(k)$ , if it exists, is the colimit of D. Here the components of f and g are  $f_{\gamma} = \iota_i: D(i) \to \coprod_{k\in I} D(k)$  and  $g_{\gamma} = \iota_j \circ D(\gamma): D(i) \to \coprod_{k\in I} D(k)$ . The injection maps are given by the injection maps for the coproduct that are further composed with the canonical injection map to the coequalizer.

**Corollary 20.5.** In the category Set we have  $\operatorname{colim} D = \coprod_{k \in I} D(k) / \langle D(\gamma)(x_i) = x_j | \gamma: i \to j \rangle$ , where  $\langle \cdots | \cdots \rangle$  denotes the equivalence relation generated by the given set of pairs. The injection maps are obtained from the injection maps of  $\coprod_{k \in I} D(k)$  by composing with the quotient map.

### 21 Complete and cocomplete categories

**Definition 21.1.** A category C is *(co)complete* if any small diagram  $D: I \to C$  has a (co)limit **Example 21.2.** The category Set is (co)complete. Indeed, previously we have seen that

$$\lim D = \left\{ x \in \prod_{i \in I} D(i) \mid \forall \gamma : i \to j : D(\gamma)(x_i) = x_j \right\}$$

and

$$\operatorname{colim} D = \left( \prod_{i \in I} D(i) \right) / \langle \iota_i(s) \sim \iota_j(D(\gamma)(s)) \mid \gamma: i \to j, s \in D(i) \rangle.$$

**Example 21.3.** The category Group and any other variety of algebras, like Ring,  $\mathsf{CRing}$ ,  $\mathsf{Vect}_k$ ,  $\mathsf{Mod}_R$ ,  $\mathsf{Alg}_R$  (but not Field) is (co)complete. Indeed, previously we have seen that  $\lim D$  can be computed as  $\lim(U \circ D)$  (here  $U: \mathsf{Group} \to \mathsf{Set}$  is the forgetful functor) and equipping the result with indexwise algebraic operations. Likewise, colim D can be computed as the quotient of the free object on  $\operatorname{colim}(U \circ D)$  by all possible relations present in the original objects D(i).

**Example 21.4.** The category Top of topological spaces is (co)complete. Both  $\lim D$  and  $\operatorname{colim} D$  can be computed by equipping  $\lim(U \circ D)$  and  $\operatorname{colim}(U \circ D)$  with a certain topology. Here  $U: \operatorname{Top} \to \operatorname{Set}$  is the forgetful functor. For  $\lim D$  we take the topology generated by the subbase of preimages of open subsets of U(D(i)) under the projection map  $p_i: \lim(U \circ D) \to U(D(i))$ . For  $\operatorname{colim} D$  we declare the open subsets to be those subsets whose preimages under all injection maps  $\iota_i: U(D(i)) \to \operatorname{colim}(U \circ D)$  are open in D(i).

**Example 21.5.** The category  $Ban_1$  of Banach spaces and contractive maps is (co)complete. On the other hand, the category Ban of Banach spaces and bounded maps is *finitely* (co)complete, i.e., admits finite (co)limits, but does not admit infinite (co)products, so is not complete or cocomplete.

**Example 21.6.** The category Field of fields is not complete or complete, in fact, it has no initial or terminal object and no binary (co)products.

**Remark 21.7.** The above examples may create an impression that if a category is complete or cocomplete, then it is both complete and cocomplete. This is not true, but counterexamples are rather convoluted. The reason for this is that requiring the category to be *accessible*, a very mild restriction, does ensure that one of the properties implies both of them.

## 22 Preservation of limits and colimits by functors

Limits and colimits are the most common operations that one can perform on objects and morphisms of a category. Thus it is natural to look at functors that preserve these operation.

**Definition 22.1.** Suppose  $D: I \to C$  is a diagram that admits a (co)limit. We say that a functor  $F: C \to D$  preserves this (co)limit if applying F to the components of a (co)limit (co)cone over D produces a (co)limit (co)cone over  $F \circ D$ .

**Example 22.2.** The forgetful functor U: Group  $\rightarrow$  Set preserves limits and filtered colimits. This boils down to saying that the underlying set of the limit or filtered colimit of a diagram of groups is the limit or filtered colimit of the diagram of underlying sets. This follows immediately from the explicit constructions of these limits and colimits.

#### 23 Filtered colimits

Filtered colimits are a convenient generalization of sequential colimits. We start by replacing natural numbers with an arbitrary poset.

**Definition 23.1.** A poset P is *directed* if any finite subset of P has an upper bound. The latter condition can be reformulated by saying that  $P \neq \emptyset$  and for any  $x, y \in P$  there is  $z \in P$  such that  $x \leq z$  and  $y \leq z$ .

**Definition 23.2.** A *directed diagram* is a diagram  $D: I \to C$  whose indexing category I is the category associated to some directed poset as in Example 9.37.

The notion of a directed diagram can be extended slightly in a way that is convenient for various applications.

**Definition 23.3.** A category *I* is *filtered* if any finite diagram  $D: J \to I$  admits a cocone. This is equivalent to the following three conditions (in which we take *J* to be empty,  $\{0, 1\}$ , or  $\{0 \rightrightarrows 1\}$ ):

- *I* is nonempty;
- for any two objects i and j in I there is an object  $k \in I$  and two morphisms  $i \to k$  and  $j \to k$ ;
- for any two objects i and j in I and any two morphisms  $f, g: i \to j$  there is an object  $k \in I$  and a morphism  $h: j \to k$  such that hf = hg.

**Example 23.4.** The category associated to any directed poset is filtered. In particular, sequential colimits are filtered colimits.

**Definition 23.5.** A diagram  $D: I \to C$  is *filtered* if I is a filtered category.

A crucial example distinguishing filtered categories from directed categories is the splitting of idempotents.

**Example 23.6.** Consider the category I with one object \* and one nonidentity morphism  $e:* \to *$  such that  $e \circ e = e$ . Such morphisms are known as *idempotents*. The category I is filtered. It is not induced by any poset because such categories can have at most one morphism between any pair objects, whereas here we have two. The (filtered) colimit of a  $D: I \to C$  is a *splitting* of D(e), i.e., an object  $X \in C$  with an injection map  $\iota: D(*) \to X$  and a (retraction) map  $r: X \to D(*)$  (induced by the universal property of a colimit) such that  $\iota r = \operatorname{id}_X$  and  $r\iota = e$ . In fact, such a triple  $(X, r, \iota)$  exists if and only if D admits a colimit. This allows us to show that I-indexed colimits are preserved by *any* functor because the conditions  $\iota r = \operatorname{id}_X$  and  $r\iota = e$  are preserved by functoriality. Furthermore, X is also the *limit* of the same diagram, if we take  $r: X \to D(*)$  to be the projection map.

Another important property of filtered colimits is that they commute with finite limits in many categories, such as the category of sets, and, as explained, *locally finitely presentable categories*, a large class of categories that includes varieties of algebras.

**Proposition 23.7.** Suppose  $D: I \times J \to \text{Set}$  is a diagram in the category of sets, where I is a filtered category and J is a finite category. The canonical map  $M: \operatorname{colim}_I \lim_J D \to \lim_J \operatorname{colim}_I D$  induced by the universal property of limits and colimits is an isomorphism.

*Proof.* To show surjectivity we pick an arbitrary element in the right side and show it comes from the left side. Suppose x is such an element, with components  $x_j \in \operatorname{colim}_I D(-, j)$ . An element in  $\operatorname{colim}_I D(-, j)$  is an equivalence class of some pair  $(i, y \in D(i, j))$ , where i can be replaced by any i' such that  $i \leq i'$ . Two such elements are equal if they can be replaced by equivalent equal elements. In particular, any finite collection of elements can be replaced by equivalent elements with the same i, using the first two properties of filtered categories listed above. Since J is finite, all  $x_j$  simultaneously can be represented as  $x_j = [(i, y_j)]$ , where  $i \in I$  does not depend on j. If we increase i further, we can also assume that for any morphism  $f: j \to j'$  in J we have  $D(-, f)(y_j) = y_{j'}$ . Thus the elements  $y_j$  assemble together into  $y \in \lim_J D(-, j)$ . The image of (i, y) under M is precisely x.

For injectivity suppose (i, y) and (i', y') are two elements in  $\operatorname{colim}_I \lim_J D$  with the same image under M. As before, we may assume that i = i'. Having the same image under M means the individual J-components are equal, i.e.,  $(i, y_j) = (i, y'_j)$  in  $\operatorname{colim}_I D$  for all  $j \in J$ .

#### 24 Sifted colimits

## Representable functor theorem

#### 25 Representable functors

**Definition 25.1.** Suppose C is a category and  $X \in C$ . The represented functor of X is a functor  $\operatorname{Hom}(-, X)$ :  $C^{op} \to \operatorname{Set}$  that sends  $P \in C$  to  $\operatorname{Hom}(P, X) \in \operatorname{Set}$  and  $f: P \to Q$  to  $\operatorname{Hom}(f, X)$ :  $\operatorname{Hom}(Q, X) \to \operatorname{Hom}(P, X)$ . A functor F:  $C^{op} \to \operatorname{Set}$  is representable if it is isomorphic to a functor  $\operatorname{Hom}(-, X)$  for some object  $X \in C$ , which will be shown below to be unique up to an isomorphism, and which is referred to as the representing object of F. Dually, we have the corepresented functor of X, namely,  $\operatorname{Hom}(X, -)$ :  $C \to \operatorname{Set}$ , and we say that a functor F:  $C \to \operatorname{Set}$  is corepresentable if it is isomorphic to  $\operatorname{Hom}(X, -)$  for some corepresenting object  $X \in C$ .

**Example 25.2.** A representable functor  $C^{op} \to Set$  or a corepresentable functor  $C \to Set$  always preserve limits. This is an immediate consequence of the universal property of limits in  $C^{op}$  (i.e., colimits in C) and limits in C. For instance, for binary (co)products we have bijections  $Mor(X, A \times A') \to Mor(X, A) \times$ Mor(X, A') and  $Mor(A \sqcup A', X) \to Mor(A, X) \times Mor(A', X)$ .

**Example 25.3.** Suppose V and W are two vector spaces, i.e., objects in  $\operatorname{Vect}_k$ . (More generally, we can work with  $\operatorname{Mod}_R$  for any commutative ring R.) We define a functor F:  $\operatorname{Vect}_k \to \operatorname{Set}$  as follows. For any  $A \in \operatorname{Vect}_k$  we set F(A) to be the set of k-bilinear maps  $V, W \to A$ , i.e., functions  $b: \operatorname{U}(V) \times \operatorname{U}(W) \to \operatorname{U}(A)$  such that for any  $v \in \operatorname{U}(V)$  the function  $b(v, -): \operatorname{U}(W) \to \operatorname{U}(A)$  that maps  $w \mapsto b(v, w)$  is k-linear and for any  $w \in \operatorname{U}(W)$  the function  $b(-, w): \operatorname{U}(V) \to \operatorname{U}(A)$  that maps  $v \mapsto b(v, w)$  is also k-linear. For any morphism  $f: A \to A'$  we set  $F(f): F(A) \to F(A')$  to be the function that sends  $b: V, W \to A$  to its composition with f, i.e.,  $v, w \mapsto f(b(v, w))$ . We will show below that the functor F is corepresentable. Its corepresenting object is denoted by  $V \otimes_k W$  (or simply  $V \otimes W$  if no ambiguity can arise) and is known as the *tensor product of* V and W over k. Elements of  $V \otimes_k W$  are known as *tensors*. The fact that  $V \otimes_k W$  is the corepresenting object of F can be reformulated by saying that k-linear maps  $V \otimes_k W \to A$  are in natural bijective correspondence with k-bilinear maps  $V, W \to A$ . In particular, the identity map  $V \otimes_k W \to V \otimes_k W$  corresponds to a bilinear map  $V, W \to V \otimes_k W$ , which is known as the *universal bilinear map*. The image of (v, w) under this map is denoted by  $v \otimes_k w$ , or simply  $v \otimes w$  if no ambiguity can arise.

**Example 25.4.** Suppose X and Y are objects in Top. Consider the functor F: Top<sup>op</sup>  $\rightarrow$  Set defined as follows. For any  $A \in$  Top the set F(A) consists of continuous maps  $A \times X \rightarrow Y$ . For any morphism  $A \rightarrow A'$  in Top the induced function sends  $A \times X \rightarrow Y$  to its precomposition with  $A \times X \rightarrow A' \times X$ . Below we will see that the functor F preserves limits if and only if the functor  $- \times X$ : Top  $\rightarrow$  Top,  $A \mapsto A \times X$  preserves colimits (equivalently, coequalizers). The latter in its turn holds if and only if X is *core-compact*: for any open  $U \subset X$  we have  $U = \bigcup_{V \ll U} V$ , where  $V \ll U$  means that any open cover of U admits a finite subcover of V. If X is Hausdorff, it is core-compact if and only if it is locally compact. The representing object is usually denoted  $Y^X$ . By definition, continuous maps  $A \rightarrow Y^X$  are in bijection with continuous maps  $A \times X \rightarrow Y$ . In particular, points of  $Y^X$  are continuous maps  $X \rightarrow Y$ . The topology on  $Y^X$  is generated by the subbase consisting of sets  $O_{U,V} = \{f: X \rightarrow Y \mid U \ll f^{-1}(V)\}$ , where  $U \subset X$  and  $V \subset X$  are open subsets. If X and Y are Hausdorff, then this topology coincides with the compact-open topology. Thus one can see the above universal property of  $Y^X$  as a motivation for the compact-open topology.

#### 26 The Yoneda lemma

Suppose we have a representable functor  $F: C^{op} \to Set$ . How could we construct its representing object? To answer this question, we first investigate how C embeds into  $Fun(C^{op}, Set)$ .

We start by introducing a new notation that will make it easier to write down formulas with representable functors.

**Definition 26.1.** The Yoneda embedding functor  $Y: C \to Fun(C^{op}, Set)$  sends an object  $X \in C$  to its represented functor  $Mor(-, X): C^{op} \to Set$  and a morphism  $X \to X'$  in C to the induced natural transformation of represented functors  $Mor(-, X) \to Mor(-, X')$  whose components are the maps  $Mor(A, X) \to Mor(A, X')$  given by composition with  $X \to X'$ .

We now turn to examples of Yoneda embeddings. The examples below all have a single unifying theme: starting from an arbitrary category C, one can construct a category of "generalized objects glued from objects of C". Formally, such a generalized object G is specified by defining the abstract set of maps  $X \to G$  for any  $X \in C$  and specifying how a map  $X \to G$  can be composed with a morphism  $X' \to X$  in C to give a map  $X' \to G$ . The Yoneda embedding then sends an object  $W \in C$  to itself, considered now as such a generalized object: the abstract set of maps  $X \to W$  can be defined as morphisms  $X \to W$  in C. Of course, not every generalized object need to come from C.

**Example 26.2.** If C is the category of *simplices*, i.e., finite nonempty linearly ordered sets and orderpreserving maps (objects are  $\{0 < 1 < 2 < \cdots < n\}$  for all  $n \ge 0$  and morphisms are nondecreasing functions), then  $\operatorname{Fun}(C^{\operatorname{op}}, \operatorname{Set})$  is known as sSet, the category of *simplicial setsm* which should be thought of as a bunch of simplices glued together. The Yoneda embedding sends a simplex to the corresponding simplicial set consisting of that simplex alone.

**Example 26.3.** If C is the category of finite-dimensional real vector spaces and infinitely-differentiable functions between them, then  $Fun(C^{op}, Set)$  is the category of *generalized manifolds* or *smooth spaces*. It contains the category of smooth manifolds as a full subcategory. The Yoneda embedding sends a vector space to itself, now considered as a generalized manifold.

**Example 26.4.** If C is the category  $AffSch = CRing^{op}$  of affine schemes, then the category  $Fun(C^{op}, Set) = Fun(CRing, Set)$  is the category of "generalized schemes". It contains the categories of schemes and algebraic spaces as full subcategories. The Yoneda embedding sends an affine scheme to itself, considered as a generalized scheme.

**Example 26.5.** If C is the category of free commutative rings on finitely many generators, then the category  $Fun(C^{op}, Set)$  is closely related to the category CRing of all commutative rings, which can be identified with functors  $C^{op} \rightarrow Set$  that preserve finite products. Commutative rings can be replaced with any variety of algebras.

Consider an object  $X \in C$  and its incarnation  $Y(X) \in Fun(C^{op}, Set)$  as a generalized object. Given another object  $W \in C$ , we can come up with two different types of "maps" from W to Y(X), namely (Y(X))(W) = Mor(W, X) and  $Y(W) \to Y(X)$ . A priori, these two types of maps may be different, but the Yoneda lemma shows that they are in fact exactly the same.

**Lemma 26.6.** (The Yoneda lemma.) For any category C, functor  $F: C^{op} \to Set$ , and an object  $X \in C$  there is an isomorphism of sets (natural in X)  $Mor_{Fun(C^{op}, Set)}(Y(X), F) \to F(X)$ .

*Proof.* A natural transformation  $Y(X) = Mor(-, X) \rightarrow F$  has components  $Mor(Y, X) \rightarrow F(Y)$  for any  $Y \in C$ . In particular, taking Y = X yields a function  $Mor(X, X) \rightarrow F(X)$ , which we can evaluate on  $id_X \in Mor(X, X)$ , obtaining an element of F(X). This defines the morphism from the statement.

Vice versa, an element  $p \in F(X)$  yields a natural transformation  $Mor(-, X) \to F$  with components  $Mor(Y, X) \to F(Y)$  for any  $Y \in C$  that send a map  $h: Y \to X$  to the element  $F(h)(p) \in F(Y)$ .

By inspection, the two natural transformations constructed above are mutually inverse, which completes the proof.  $\blacksquare$ 

**Proposition 26.7.** For any category C the Yoneda embedding functor  $Y: C \to Fun(C^{op}, Set)$  is a fully faithful functor.

*Proof.* Fully faithfulness means that for any  $X, Y \in C$  the induced map

$$Mor_{C}(X, Y) \rightarrow Mor_{Fun(C^{op}, Set)}(Y(X), Y(Y))$$

is an isomorphism. By the Yoneda lemma, the right side can be evaluated as Y(Y)(X) = Mor(X, Y), which completes the proof.

## 27 The canonical diagram of a presheaf of sets

In this section we construct a certain diagram  $D: I \to C$  from a functor  $F: \mathbb{C}^{op} \to \mathsf{Set}$  such that colim D exists if and only if F is representable, in which case colim D is a representing object of F.

**Definition 27.1.** The category of elements EI(F) of a functor  $F: C^{op} \to Set$  is defined as follows. Objects are pairs (X, e), where  $X \in C$  and  $e \in F(X)$ . Morphisms  $(X, e) \to (X', e')$  are morphisms  $f: X \to X'$  in C such that e = F(f)(e'). Associativity and unitality follow from the same properties of C and preservation of compositions by F. Used in 27.2, 27.4.

**Definition 27.2.** The *Grothendieck construction*  $\int \mathsf{F}$  of a functor  $\mathsf{F}: \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$  is a functor  $\int \mathsf{F}: \mathsf{El}(\mathsf{F}) \to \mathsf{C}$  that sends (X, e) to X and  $(X, e) \to (X', e')$  to  $X \to X'$ .

**Example 27.3.** If F = Mor(-, X), then  $\int F: C/X \to C$  is the forgetful functor.

**Remark 27.4.** Another way to define El(F) is to say that El(F) is the *comma category* (defined below) C/F with respect to the Yoneda embedding  $Y: C \to Fun(C^{op}, Set)$ . In other words, objects in El(F) are maps  $Y(X) \to F$  and morphisms are maps  $X \to X'$  that make the corresponding triangle commute.

**Definition 27.5.** The comma category D/A of a diagram  $D: I \to C$  over an object  $A \in C$  is defined as follows. Objects are pairs (i, f), where  $i \in I$  and  $f: D(i) \to A$ . Morphisms  $(i, f) \to (i', f')$  are morphisms  $\gamma: i \to j$  in I such that  $f'D(\gamma) = f$ .

**Remark 27.6.** If  $D = id_{\mathsf{C}}$ , then the comma category A/D is simply the slice category  $A/\mathsf{C}$ .

**Proposition 27.7.** Suppose C is a category and  $F: C^{op} \to Set$  is a presheaf of sets on C. The functor F is representable if and only if colim  $\int F$  exists. In the latter case, colim  $\int F$  is a representing object of F and the cocone over colim  $\int F$  encodes precisely an isomorphism  $F \to Y(\operatorname{colim} \int F)$ .

*Proof.* If F is represented by X, then  $\int F: C/X \to C$  is the forgetful functor, whose colimit can be computed by evaluating  $\int F$  on the terminal object of C/X, i.e.,  $id_X: X \to X$ .

#### 28 Compact objects

**Definition 28.1.** An object  $X \in C$  is *compact* if the corepresentable functor  $Hom(X, -): C \to Set$  preserves filtered colimits (equivalently, directed colimits).

**Remark 28.2.** Unfolding the definition, an object  $X \in C$  is compact if for any morphism  $f: X \to \operatorname{colim} D$ , where  $D: I \to C$  is a directed diagram, we can find some  $i \in I$  and a morphism  $g: X \to D(i)$  such that f equals the composition of  $g: X \to D(i)$  and  $\iota_i: D(i) \to \operatorname{colim} D$ .

Example 28.3. In the category Set, compact objects are precisely finite sets.

**Example 28.4.** In the category Group (or any variety of algebras), compact objects are precisely finitely presented objects, i.e., objects specified using finitely many generators and finitely many relations between them.

**Example 28.5.** In the category Top<sub>Open</sub> of topological spaces and open maps, compact objects are precisely compact topological spaces. This explains the choice of terminology.

#### 29 Locally presentable categories and the corepresentable functor theorem

**Definition 29.1.** A category C is *locally presentable* if it is cocomplete and there is a set G of small objects such that any object in C can be presented as a colimit of a diagram  $D: I \to C$  such that  $D(i) \in G$  for all  $i \in I$ .

**Remark 29.2.** If C is a category such that  $C^{op}$  is locally presentable, we say that C is *locally copresentable*. This is a fairly rare condition in practice.

**Theorem 29.3.** Suppose C is a locally presentable category. A functor  $F: C \to Set$  is corepresentable if and only if it is accessible (i.e., preserves  $\lambda$ -filtered (or  $\lambda$ -directed) colimits for some  $\lambda$ ) and continuous, i.e., preserves all small limits.

**Example 29.4.** Arbitrary small limits exist in any locally presentable category.

**Example 29.5.** Many (if not most) forgetful functors from locally presentable categories preserve limits, hence are representable. For instance,  $\text{Group} \rightarrow \text{Set}$  is representable by Z.

**Example 29.6.** Commutative differential graded algebras and commutative differential graded  $C^{\infty}$ -rings form locally presentable categories.

Example 29.7. Sheafification.

#### 30 Total categories and the representable functor theorem

For the representable functor theorem we need a slight strengthening of the cocompleteness condition for categories. Recall that a category C is cocomplete if any small diagram  $D: I \to C$  admits a colimit. Small means that the class of objects in C is a set, as opposed to a proper class. Total categories, in addition to having all small colimits, are also required to have certain (but not all) large colimits, i.e., colimits indexed by a category whose class of objects is a proper class, not a set.

**Definition 30.1.** The class of connected components  $\Pi_0(C)$  of a category C is defined as the coequalizer of classes<sup>\*</sup>  $Mor(C) \xrightarrow{s}{t} Ob(C)$ , where  $s, t: Mor(C) \to Ob(C)$  are the source and target maps of C. In other words, we identify objects in C that are connected by a morphism or a chain of morphisms going in any direction. Used in 30.2, 30.3.

**Definition 30.2.** A category C is *total* if it admits colimits for all diagrams  $D: I \to C$  such that for each  $A \in C$  the class  $\prod_0 (A/D)$  is a set.

**Remark 30.3.** Of course, if I is a small category, so is A/D and therefore  $\Pi_0(A/D)$  is automatically a set. Hence total categories are, in particular, cocomplete.

**Remark 30.4.** Every locally presentable category is total, but not every total category is locally presentable. Thus the theorem below also holds for locally presentable categories. In this case arbitrary limits can be replaced by small limits, i.e., F should be a continuous functor.

Example 30.5. The general linear group as an algebraic group.

**Theorem 30.6.** Suppose C is a total category. A functor  $F: C^{op} \to Set$  is representable if and only if it preserves all limits (not necessarily small).

**Example 30.7.** The functor  $\mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$  that sends  $A \mapsto 2^A$  and  $f \mapsto f^{-1}$  is represented by the set  $\{0, 1\}$ .

Example 30.8. The above example can be generalized to define *subobject classifiers*.

**Example 30.9.** The category of topological spaces is total. In particular, this allows us to conclude the following. If X and Y are topological space, then a space  $Y^X$  with a universal property that maps  $A \to Y^X$ 

<sup>\*</sup> Coequalizers of *classes* (as opposed to sets) always exist, and can be constructed using the so-called *Scott's trick* due to Dana Scott. The traditional construction with equivalence classes no longer works because equivalences classes can be proper classes and proper classes cannot themselves be elements of other classes.

are in bijection with maps  $A \times X \to Y$  exists if and only if the functor  $A \mapsto A \times X$  preserves colimits (equivalently, coequalizers, equivalently, quotient spaces). As we have seen above, this happens if and only if X is core-compact, e.g., locally compact and Hausdorff.

**Example 30.10.** The category of topological groups is total, and, more generally, groups can be replaced by any variety of algebras. Any category monadic over the category of sets is total. (Monads will be explained later.)

If we apply the above theorem to  $C^{op}$ , we get the following result.

**Corollary 30.11.** If D is a cototal category (i.e.,  $D^{op}$  is total), then  $F: D \to Set$  is corepresentable if and only if it preserves all limits (not necessarily small).

*Proof.* By the above theorem for  $C = D^{op}$ , the functor  $F: C^{op} \to Set$  is representable by an object X in C, i.e.,  $F(A) = Mor_{C}(A, X) = Mor_{D}(X, A)$ , i.e., F is corepresentable by X as an object in D.

**Example 30.12.** The Stone-Čech compactification of a topological space X is a continuous map  $X \to K$ , where X is compact Hausdorff and for any other map  $X \to K'$  to a compact Hausdorff space K' there exists a unique continuous map  $K \to K'$  that makes the obvious triangle compute. We construct K using the representable functor theorem, which is applicable because compact Hausdorff spaces form a cototal category. Thus it suffices to to show that the functor F: CompHaus<sup>op</sup>  $\to$  Set that sends  $A \in$  Top to the set of continuous maps  $X \to A$  and  $A \to A'$  to the corresponding composition map, preserves limits. Indeed, this follows instantly from the universal property of limits in Top and the fact that the forgetful functor Top  $\to$  CompHaus preserves limits.

# Adjunctions

## 31 Definition of adjunction

**Definition 31.1.** Suppose C and D are categories. An *adjunction* between C and D is a triple  $(L, R, \Psi)$ , where L: C  $\rightarrow$  D is the *left adjoint functor*, R: D  $\rightarrow$  C is the *right adjoint functor*, and  $\Psi$ : Mor $(L-, -) \rightarrow$  Mor(-, R-) is a natural isomorphism of functors  $C^{op} \times D \rightarrow$  Set. Schematically we write

$$L: C \rightleftharpoons D: R$$

or

$$C \xrightarrow{L} D.$$

**Remark 31.2.** We can unfold the definition of  $\Psi$  as follows. For any  $X \in \mathsf{C}$  and  $Y \in \mathsf{D}$  we have an isomorphism of sets  $\Psi_{X,Y}$ : Mor $(\mathsf{L}X, Y) \to \mathsf{Mor}(X, \mathsf{R}Y)$ . For any morphism  $f: X \to X'$  in  $\mathsf{C}$  and  $g: Y \to Y'$  in  $\mathsf{D}$  the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Mor}(\mathsf{L}X',Y) & \xrightarrow{h \mapsto g \circ h \circ \mathsf{L}(f)} & \mathsf{Mor}(\mathsf{L}X,Y') \\ & & & \downarrow^{\Psi_{X',Y}} & & \downarrow^{\Psi_{X,Y'}} \\ \mathsf{Mor}(X',\mathsf{R}Y) & \xrightarrow{h \mapsto \mathsf{R}(g) \circ h \circ f} & \mathsf{Mor}(X,\mathsf{R}Y'). \end{array}$$

Adjunctions are omnipresent in mathematics. We start with some of the more typical examples.

**Example 31.3.** Take C = Set,  $D = Vect_k$ ,  $R: Vect_k \to Set$  is the forgetful functor,  $L: Set \to Vect_k$  is the functor that sends a set S to the vector space with basis S and a map of sets  $f: S \to S'$  to the linear map  $L(S) \to L(S')$  whose restriction to basis elements is precisely f. The isomorphism  $\Psi_{S,V}$  sends a linear map  $L(S) \to V$  to its restriction  $S \to R(V)$  to the basis elements of L(S). It is a bijection because a linear map from L(S) is determined uniquely by its values on the elements of S. Used in 32.3.

**Example 31.4.** The above example also works with D being Group, CRing, Ring,  $Mod_R$ ,  $Alg_R$ , or any other variety of algebras. Instead of a vector space with basis S we now use the free object on S, e.g., free group, free ring, etc.

**Example 31.5.** Take  $C = Vect_k$ ,  $D = CAlg_k$ ,  $R: CAlg_k \rightarrow Vect_k$  is the forgetful functor, and  $L: Vect_k \rightarrow CAlg_k$  sends a vector space V to the symmetric algebra S(V), i.e., polynomials in dim V variables. The

natural transformation  $\Psi$  restricts a homomorphism of algebras  $S(V) \to A$  to a linear map  $V \to R(A)$ . It is an isomorphism by the universal property of symmetric algebras.

**Remark 31.6.** The above situation is very typical for adjunctions: the right adjoint functor "forgets" some structure, whereas the left adjoint functor constructs a "free" object with such a structure. The category C need not be the category of sets, as illustrated by the example with symmetric algebras.

## 32 Units and counits

**Definition 32.1.** Consider an adjunction  $(C, D, L, R, \Psi)$ . The *unit* of an object  $X \in C$  is the morphism  $X \to RLX$  that is adjoint to the identity morphism  $LX \to LX$ . The *counit* of an object  $Y \in D$  is the morphism  $LRY \to Y$  that is adjoint to the identity morphism  $RY \to RY$ .

**Remark 32.2.** The naturality of  $\Psi$  implies that the unit maps can be assembled into a natural transformation  $id_{\mathsf{C}} \to \mathsf{RL}$ , whereas for counits we get a natural transformation  $\mathsf{LR} \to id_{\mathsf{D}}$ .

**Example 32.3.** In the free-forgetful adjunction of Example 31.3, the unit of a set X is the map  $X \to \mathsf{RL}(S)$  that sends an element  $x \in X$  to the element of the vector space  $\mathsf{L}(S)$  given by the indicator function  $S \to k$  of the element x:  $\chi(x) = 1$  and  $\chi(y) = 0$  if  $y \neq x$ . The counit of a vector space V is the linear map  $\mathsf{LR}(V) \to V$  from the free vector space of the underlying set of V to V given by sending a formal k-linear combination of some elements of V to the corresponding element of V given by evaluation this linear combination.

**Proposition 32.4.** Suppose  $(C, D, L, R, \Psi)$  is an adjunction. Denote by C' the full subcategory of C on objects  $X \in C$  such that the unit map  $X \to RLX$  is an isomorphism. Denote by D' the full subcategory of D on objects  $Y \in D$  such that the counit map  $LRY \to Y$  is an isomorphism. Denote by L' and R' the restriction of functors L and R to these subcategories. We have  $L'(C') \subset D'$  and  $R'(D') \subset C'$ , and the restricted adjunction  $(C', D', L', R', \Psi)$  is an equivalence of categories.

**Example 32.5.** The Nullstellensatz adjunction is as follows. The category  $C = CAlg_k^{op}$  is the opposite category of commutative algebras over an algebraically closed field k equipped with a finite set of generators. Morphisms are homomorphisms of algebras. (There is no requirement for morphisms to preserve generators.) The category  $D = AffVar_k$  is the category of affine algebraic varieties. Its objects are pairs (n, S), where  $S \subset k^n$  is a subset such that for some family p of polynomials with n variables and coefficients in k we have  $S = \{z \in k^n \mid \forall i: p_i(z) = 0\}$ . Morphisms  $(n, S) \to (n', S')$  are equivalence classes of polynomial functions  $k^n \to k^{n'}$  (i.e., n'-tuples of polynomial functions in n variables), where two functions are equivalent if they coincide on S.

#### 33 Cartesian internal hom

**Definition 33.1.** Consider a category C that admits finite products. Fix an object  $X \in C$  and consider the functor  $X \times -: C \to C$  that sends  $W \in C$  to  $X \times W \in C$  and likewise for morphisms. The *cartesian internal* hom Hom(X, Y) = [X, Y] (if it exists) is the value of the right adjoint functor of the functor  $X \times -: C \to C$  on the object Y. Used in 4.32, 7.11, 7.18, 10.5, 25.1, 28.1, 33.2, 33.4, 33.5, 33.6.

**Remark 33.2.** Unfolding the definition, the cartesian internal hom Hom(X, Y) is unquely determined by its representable functor that sends  $A \in C$  to  $Mor(X \times A, Y)$  and likewise for morphisms.

**Example 33.3.** In the category Set, Hom(X, Y) is the set of maps  $X \to Y$ , i.e., Hom(X, Y) = Mor(X, Y).

**Example 33.4.** In the category  $\mathsf{Vect}_k$  (or  $\mathsf{Mod}_R$ ),  $\mathsf{Hom}(X, Y)$  is the vector spaces of k-linear maps  $X \to Y$ , i.e.,  $\mathsf{U}(\mathsf{Hom}(X,Y)) = \mathsf{Mor}(X,Y)$  and the vector space operations are pointwise.

**Example 33.5.** In the category  $Ban_1$ , Hom(X, Y) is the Banach space of bounded linear maps  $X \to Y$ . In particular, the underlying set of the unit ball of Hom(X, Y) is precisely Mor(X, Y).

**Example 33.6.** In the category LocMeas, the cartesian internal hom Hom(X, Y) does *not* exist in many cases.