1 Preface

These notes offer an elementary introduction to category theory. Why bother writing a new text when so many exist already? Two main features distinguish this text from all others known to the author:

- The fraction of the text occupied by examples is considerably larger.
- A much larger area of mathematics is covered by examples. In particular, areas such as measure theory, functional analysis, smooth manifolds, and partial differential equations are emphasized.

2 Notation

Bold letters **Z**, **Q**, **R**, **C** denote the rings of integer, rational, real, and complex numbers. Calligraphic letters $\mathcal{O}$, $\mathcal{D}$, and $\mathcal{M}$ denote functions, distributions, and measures on a space. Sans-serif letters like Set denote categories. Euler roman font like Mor denotes functors.

3 Contents

4 Introduction

Category theory is omnipresent in such branches of mathematics as algebraic geometry, algebraic topology, number theory, complex geometry, logic, commutative algebra, K-theory. More recently, categories made their way into a variety of applied areas such as condensed matter physics, signal processing, statistics, etc.

Very roughly, categories fit into the following chain of abstractions:

- Antiquity and middle ages: numbers and figures as mathematical objects. Abstraction: some numbers and figures need not be present in nature. Operations: addition, multiplication, division of numbers; compass and straightedge constructions with geometric objects.
- 18th and 19th century: functions as mathematical objects axiomatizing sequences of operations mentioned in the previous item: polynomials, analytic functions, smooth functions, continuous functions. Abstraction: some functions might not be specified by an explicit formula. Operations on functions: addition, multiplication, limit, infinite sums, etc.
- Early 20th century: abstract mathematical structures axiomatizing the above operations on functions: sets, groups, rings, fields, vector spaces, topological spaces, Banach spaces, C*-algebras, measurable spaces, Lie groups. Abstraction: some mathematical structures might not have functions as their elements. Operations on structures: direct sum, product, direct and inverse limits, etc.
- Middle of 20th century: categories (abstract collections of mathematical structures axiomatizing the above operations): categories, abelian categories, toposes, regular categories, sites and Grothendieck topologies, etc. Abstraction: some categories need not arise as categories of mathematical structures. Operations on categories: coproducts and products, functor categories, etc.
- 21st century: higher categories (abstract collections of gadgets mentioned in the previous item): 2-categories, model categories, ∞-categories, (∞, n)-categories, etc. Abstraction: some higher categories need not arise from specific classes of categories. Operations: same as above (roughly, higher categories themselves form a higher category and higher category theory can process itself).

When trying to characterize the structure of category theory and its role in mathematics, it is useful to compare the notion of a category to that of a complex number. Both are omnipresent in mathematics: it is hard to name an area of mathematics untouched by category theory or complex numbers. Another unifying property of both notions is that there are relatively few deep theorems about categories or complex numbers *per se*, i.e., not belonging to some other field of mathematics.†

For instance, there are many theorems in other fields of mathematics for which the notion of a complex number is essential:

- the field of complex numbers is algebraically closed (the fundamental theorem of algebra);
- bounded entire functions on the complex plane are constant (Liouville's theorem in complex analysis);
- a compact Kähler manifold with vanishing first Chern class has a Kähler metric with vanishing Ricci curvature (differential geometry).

However, one cannot say that these results form a "theory of complex numbers" in the sense one normally uses the word "theory" in mathematics.

In the same way, categories are essential components of many theorems throughout mathematics:

- Pushforward along proper morphisms of locally Noetherian schemes preserves coherent sheaves (algebraic geometry);
- Čech cohomology, de Rham cohomology, and singular cohomology of a smooth manifold are isomorphic (topology);
- On a Stein manifold, the first Cousin problem is always solvable, whereas the second Cousin problem is always solvable if and only if the second integer cohomology vanishes (complex analysis).

It is important to point out, though, that these theorems are only a tiny sample of the enormous variety of results making use of categories, in the same way as with the three theorems using complex numbers above.

† It is not entirely true that "pure" category theory is devoid of deep theorems. Among nontrivial results in category theory proper one can cite the Barr-Beck monadicity theorem, the Giraud theorem, and the Freyd–Mitchell embedding theorem.

The analogy with complex numbers breaks down when we consider the learning aspects. Complex numbers can be introduced and their basic properties proven in less than an hour. In contrast, category theory requires at least two orders of magnitude more time to get acquainted with. Acquiring a working understanding of category theory resembles climbing the Tibetan Plateau: one first has to expend a substantial amount of effort simply to climb 5 kilometers (3 miles) to the top of the plateau (i.e., learn and understand the relevant notions such as categories, functors, adjunctions, Kan extensions, etc.). After this, one still has to spend a considerable amount of time acclimatizing to the high altitude of the plateau (i.e., the high level of abstraction associated with the categorical language). The first few days one is guaranteed to have altitude sickness (i.e., difficulty managing the high level of abstraction and using the associated notions and tools), which eventually disappears once one spends a sufficient amount of time on the plateau.

5 Categories

Definition 5.1. A *category* C is a collection of the following data:

- a class† $\mathsf{Ob}(\mathsf{C})$ of *objects* (we write $X \in \mathsf{C}$ instead of $X \in \mathsf{Ob}(\mathsf{C})$;
- for any $X \in \mathsf{C}$ and $Y \in \mathsf{C}$ we have a set‡ $\mathsf{Mor}_{\mathsf{C}}(X, Y)$ of *morphisms* from X to Y (alias *maps*, *arrows*), but instead of $f \in \mathsf{Mor}_{\mathsf{C}}(X, Y)$ we write $f\colon X \to Y$ or $X \xrightarrow{f} Y$;
- for any $X \in \mathsf{C}$ an *identity morphism* on X: $\mathrm{id}_X\colon X \to X$;
- for any $X, Y, Z \in \mathsf{C}$ the *composition* of morphisms $\circ\colon \mathsf{Mor}_{\mathsf{C}}(Y, Z) \times \mathsf{Mor}_{\mathsf{C}}(X, Y) \to \mathsf{Mor}_{\mathsf{C}}(X, Z)$, but instead of $\circ(g, f)$ we write $g \circ f$ or gf.

This data must satisfy the following properties:

- *unitality*: for any morphism $f\colon X \to Y$ we have $\mathrm{id}_Y f = f \,\mathrm{id}_X = f$;
- *associativity*: for any morphisms $f\colon W \to X$, $g\colon X \to Y$, $h\colon Y \to Z$ we have $(hg)f = h(gf)$.

We often write Mor instead of $\mathsf{Mor}_{\mathsf{C}}$ when no ambiguity can arise.

The primordial example of a category is the *category of sets*:

Example 5.2. The category Set has sets as objects and $\mathsf{Mor}(X, Y)$ is the set of functions from X to Y. Composition is given by the composition of functions and id_X is the identity function $X \to X$.

Example 5.3. The category Group has groups as objects and $\mathsf{Mor}(X, Y)$ is the set of group homomorphisms $X \to Y$. Composition is given by the composition of group homorphisms (which is again a group homorphism) and id_X is the identity group homorphism on a group X.

Before we continue with more examples, we introduce an important construction on categories.

Definition 5.4. The *full subcategory* of a category C on a class of objects $X \subset \mathsf{C}$ is a category that has X as its class of objects, whereas the sets of morphisms as well as identities and composition are inherited from C.

Example 5.5. The category FinSet of finite sets is the full subcategory of Set on the class of finite sets.

Example 5.6. The category Ab of abelian groups is the full subcategory of Group on the class of abelian groups.

We now give more examples of categories from various areas of mathematics.

5.7. ***Algebra***

Example 5.8. The category Ring of rings has (associative) rings as objects and homorphisms of rings as morphisms. (We require associative rings to have a unit and their homorphisms to preserve units.) It has a full subcategory CRing of commutative rings. The latter has a full subcategory Field of fields.

Example 5.9. Given a ring R, the category Mod_R has right R-modules as objects and R-linear homorphisms of modules as morphisms. If $k = R$ is a field, we denote this category by Vect_k (vector spaces over a field k).

† A class is like a set, except that it can be much bigger. For instance, there is a class of all sets, but there is no set of all sets (by Russell's paradox).

‡ Some mathematicians also allow a class here, in which case our variant is referred to as a "locally small category".

Likewise, we have the categories Alg_k (associative unital algebras over k) and LieAlg_k (Lie algebras over k). The category Alg_k has a full subcategory CAlg_k of commutative algebras.

Example 5.10. The category BoolAlg is the full subcategory of Ring on *Boolean algebras*: rings in which all elements are idempotent, i.e., $x^2 = x$. (Such rings are automatically commutative.) We will also make use of the (nonfull) subcategory ComplBoolAlg of *complete* Boolean algebras and continuous homomorphisms (a Boolean algebra A is complete if any subset of A has a supremum with respect to the order $x \leq y \equiv x = xy$ and a homomorphism of Boolean algebras is continuous if it preserves these suprema). Finally the category ComplAtomBoolAlg is the full subcategory of ComplBoolAlg consisting of complete *atomic* Boolean algebras (a Boolean algebra is *atomic* if for any nonzero $z \in A$ there is an atom $a \in A$ such that $a \leq z$, where a is an *atom* if $a \neq 0$ and for any $b \in A$ such that $0 \leq b \leq a$ either $b = 0$ or $b = a$).

Example 5.11. Other algebraic structures, far too numerous to be named here, also form categories. Morphisms are maps of underlying sets that preserve all algebraic operations. Examples include monoids, magmas, loops, heaps, rigs, G-actions for a fixed group G, division rings, algebras over a ring R, Lie algebras over a field k, k-vector spaces with an inner product, etc. Order-theoretic notions, such as posets, linearly ordered sets, ordered groups, ordered fields, etc., also form categories.

5.12. ***Combinatorics***

Example 5.13. The category Graph of (directed graphs) has graphs (i.e., pairs of functions $s, t\colon E \to V$) as objects and homomorphisms of graphs (i.e., functions $v\colon V \to V'$ and $e\colon E \to E'$ such that $vs = s'e$ and $vt = t'e$) as morphisms.

Example 5.14. The category of *species* plays an important role in combinatorics. We will define it later as a *category of functors* from $\mathsf{FinSet}^\times$ to Set.

5.15. ***General topology***

Example 5.16. The category Top has topological spaces as objects and continuous maps as morphisms. (The composition of continuous maps is again continuous.) The category Top_* has pointed topological spaces as objects and continuous maps that preserve the basepoint as morphisms. We also have the following full subcategories of Top: Haus (Hausdorff spaces) and CompHaus (compact Hausdorff spaces).

Example 5.17. The category TopGroup has topological groups as objects and continuous homomorphisms of topological groups as morphisms. The categories of topological rings, topological fields, topological modules, and topological vector spaces over a topological field k (denoted $\mathsf{TopVect}_k$ or simply TopVect if no ambiguity can arise) can be defined in an analogous fashion. We will also need the full subcategory HausGroup (Hausdorff topological groups), LocCompHausGroup (locally compact Hausdorff groups), LocCompHausAb (locally compact abelian groups), CompHausGroup (compact groups). Finally, the category LocCompHausGroupOpen has locally compact Hausdorff topological groups as objects and continuous *open* homomorphisms as morphisms.

Example 5.18. There are *two* different categories of metric spaces. The category Met_1 of metric spaces and *contractive* maps has metric spaces as objects and contractive maps as morphisms. (A map $f\colon X \to Y$ is *contractive* if $d(f(x), f(x')) \leq d(x, x')$ for any points $x, x' \in X$.) The category Met of metric spaces and *continuous* maps has metric spaces as objects and continuous maps as morphisms. (Every contractive map is continuous, but not vice versa.) These two categories have different properties and illustrate the fact that in category theory morphisms are as important as objects.

5.19. ***Functional analysis***

The spaces below can be either real or complex, but we omit this data in the notation.

Example 5.20. Continuing the examples with metric spaces, one can define two different categories of Banach spaces: Ban_1 has Banach spaces as objects and contractive linear maps as morphisms, whereas Ban has Banach spaces as objects and continuous linear maps as morphisms. One also has the categories Hilb_1 and Hilb for Hilbert spaces.

Remark 5.21. As we will see later, the categories Ban and Hilb can be identified (in the appropriate sense) with certain full subcategories of $\mathsf{TopVect}$, the category of topological vector spaces respectively. This is *not* true for Ban_1 and Hilb_1: a topological vector space contains no information about norms or inner products.

Example 5.22. The theory of operator algebras delivers many examples of categories. The category BanAlg has Banach algebras as objects and continuous homomorphisms of algebras as morphisms. The category C^* has C*-algebras as objects and *-homomorphisms as morphisms. The category W^* has von Neumann algebras as objects and *ultraweakly continuous* *-homomorphisms as morphisms. The full subcategories CC^* and CW^* of commutative algebras are also important.

Remark 5.23. The examples given so far may create an impression that objects in a category are sets with structures, whereas morphisms are functions that preserve these structures. (Such an approach is explained in Chapter IV of Bourbaki's Set Theory.) However, this is not always the case and below we define the categories Meas, HoTop, and $\Psi\mathsf{DO}^\infty_M$, none of which can be interpreted as "sets with structures". This situation is analogous to the one with groups. Groups were originally defined as sets of permutations of a fixed set S closed under composition and inverses.

5.24. ***Measure theory***

Example 5.25. A naive approach to defining an appropriate category for measure theory would take pairs (X, M) as objects, where X is a set and M is a σ-algebra of measurable subset of X. Morphisms $(X, M) \to (X', M')$ would be functions $f\colon X \to X'$ such that for any $m \in M'$ we have $f^{-1}(m) \in M$, i.e., preimages of measurable sets are measurable.

The problem with this approach is that there is not enough data to formulate any nontrivial theorem of measure theory using this category: the notion of a *negligible set* (alias set of measure 0) features prominently in all main results of measure theory. Furthermore, measure theory *identifies* different maps that differ on a set of measure 0, which is not reflected in the above category.

We modify our definition accordingly and define a category Meas whose objects are triples (X, M, N), where X and M are as above and $N \subset M$ is a σ-ideal of *negligible* sets. (A σ-ideal is a σ-algebra that is additionally closed under passage to subsets, which reflects the fact that subsets of sets of measure 0 again have measure 0.) We remark that the data of N encodes exactly the same data as a *measure class*, i.e., an equivalence class of measures on (X, M) with respect to the following equivalence relation: $\mu \sim \nu$ if $\mu \ll \nu$ and $\nu \ll \mu$.

Morphisms $(X, M, N) \to (X', M', N')$ are equivalence classes of functions $f\colon X \to X'$ such that f^{-1} sends elements of M' to M and elements of N' to N (the latter condition is motivated below). The equivalence relation says that $f \sim g$ if $\{x \in X \mid f(x) \neq g(x)\} \in N$ (two functions are identified if they differ on a negligible set).

The operation of composition descends to equivalences classes: if $f \sim g$ for some

$$f, g\colon (X_1, M_1, N_1) \to (X_2, M_2, N_2),$$

then $fe \sim ge$ for any $e\colon (X_0, M_0, N_0) \to (X_1, M_1, N_1)$ and $hf \sim hg$ for any $h\colon (X_2, M_2, N_2) \to (X_3, M_3, N_3)$. Indeed,

$$\{x_0 \in X_0 \mid f(e(x_0)) \neq g(e(x_0))\} = e^{-1}\{x_1 \in X_1 \mid f(x_1) \neq g(x_1)\},$$

and we have

$$A = \{x_1 \mid f(x_1) \neq g(x_1)\} \in N_1,$$

so $e^{-1}(A) \in N_0$ because e^{-1} sends elements of N_1 to N_0. Likewise,

$$B = \{x_1 \in X_1 \mid h(f(x_1)) \neq h(g(x_1))\} \subset \{x_1 \in X_1 \mid f(x_1) \neq g(x_1)\} \in N_1,$$

so $B \in N_1$ because N_1 is closed under passage to subsets.

As a vindication of this definition, we will see later that a subcategory of Meas consisting of *localizable* measurable spaces can be identified with CW^*, the category of commutative von Neumann algebras.

Remark 5.26. All previous categories have the following pattern: objects are sets equipped with additional structure, morphisms are functions that preserve this structure. The category Meas is *not* of this type because we identified functions that differ on a set of measure 0. In particular, one can *prove* that given a morphism in Meas, there is no way to choose a representative function in such a way that these choices respect composition (i.e., the composition of two representatives is again a representative). In other words, there *no* reasonable notion of an "underlying set" in Meas.

5.27. ***Differential geometry***

Example 5.28. The category Man has smooth manifolds as objects and smooth maps as morphisms. (Smooth means infinitely differentiable. A smooth manifold can be defined as a subset of $\mathbf{R}^n$ that is locally diffeomorphic to some coordinate inclusion $\mathbf{R}^k \to \mathbf{R}^n$.) Analysts like to work with a full subcategory of this category consisting of open subsets of $\mathbf{R}^n$ for all $n \geq 0$.

Example 5.29. The category LieGroup has Lie groups as objects and smooth homomorphisms of groups as morphisms.

Example 5.30. Given a smooth manifold M, the category VBun_M has vector bundles over M as objects and smooth linear maps of vector bundles as morphisms. As we will see later, this category can be identified with a certain subcategory of $\mathsf{Mod}_{\mathrm{C}^\infty(M)}$, where $\mathrm{C}^\infty(M)$ denotes the algebra of smooth functions on M. Analysts like to work with a full subcategory of this category consisting of *trivial vector bundles*, whose objects are $\mathbf{R}^n$ and morphisms $\mathbf{R}^n \to \mathbf{R}^{n'}$ are smooth functions $M \to \mathsf{Hom}(\mathbf{R}^n, \mathbf{R}^{n'})$.

Example 5.31. The category VBun has pairs (M, V) ($M \in \mathsf{Man}$ and $V \in \mathsf{VBun}_M$) as objects and pairs $(f, g)\colon (M, V) \to (M', V')$ ($f\colon M \to M'$ and $g\colon M \to f^*M'$) as morphisms. Composition is defined as $(f', g') \circ (f, g) = (f' \circ f, f^*g' \circ g)$.

5.32. ***Partial differential equations***

Example 5.33. Given a smooth manifold M, the category DO_M has vector bundles over M as objects and *differential operators* as morphisms. Specifically, a morphism $V \to V'$ is a linear map of real vector spaces $T\colon \mathrm{C}^\infty(V) \to \mathrm{C}^\infty(V')$ that preserves support: for any $f \in \mathrm{C}^\infty(V)$ we have $\operatorname{supp}(T(f)) \subset \operatorname{supp}(f)$, where $\operatorname{supp}(g)$ is the closure of the set $\{m \in M \mid g(m) \neq 0\}$. Equivalently, we can say that the Schwartz kernel of T is supported on the diagonal of $M \times M$. By Peetre's theorem this definition is equivalent to the coordinate definition that defines differential operators using local coordinate expressions of the form $\sum_k a_k \partial^k f$, where the sum is finite, k is a multi-index, a_k is a smooth function on M, and $\partial^k f$ denotes the partial derivative of f corresponding to the multi-index k.

Example 5.34. Given a smooth manifold M, the category $\Psi\mathsf{DO}_M$ has vector bundles over M as objects and (properly supported) *pseudodifferential operators* as morphisms. Specifically, a morphism $V \to V'$ is a linear map of real vector spaces $T\colon \mathrm{C}^\infty_{\mathrm{cs}}(V) \to \mathrm{C}^\infty_{\mathrm{cs}}(V')$ ($\mathrm{C}^\infty_{\mathrm{cs}}$ denotes smooth compactly supported sections) whose Schwartz kernel is properly supported and is a conormal distribution on $M \times M$ with respect to its diagonal.

Example 5.35. Another important category $\Psi\mathsf{DO}^\infty_M$ is obtained from $\Psi\mathsf{DO}_M$ using a quotient-type construction we already used to define Meas: we declare two pseudodifferential operators equivalent if their difference is a smoothing operator (i.e., its Schwartz kernel is a smooth function on $M \times M$). One can verify that composition respects this equivalence relation (smoothing operators composed on either side with a pseudodifferential operator give a smoothing operator), so we indeed get a category. This category is important in the *calculus of pseudodifferential operators*. For instance, elliptic differential operators become *isomorphisms* (defined below) in this category, and their inverse is known as a *parametrix*.

5.36. ***Homotopy theory***

Example 5.37. In homotopy theory and algebraic topology a key role is played by the *homotopy category of topological spaces*, sometime denoted HoTop. It is formed from Top by identifying *homotopic* continuous maps: two continuous maps $f, g\colon X \to Y$ of topological spaces are homotopic if there is a *homotopy* between them, i.e., a continuous map $h\colon X \times [0, 1] \to Y$ whose restrictions to $X \times \{0\}$ and $X \times \{1\}$ are f and g respectively. (Strictly speaking, the category that is actually used in homotopy theory is the full subcategory of HoTop consisting of *CW-complexes*, but we ignore such details for now.)

5.38. ***Sheaf theory***

Example 5.39. In set theory, categories of presheaves and sheaves (to be defined below), allow one to prove the independence of the continuum hypothesis and the axiom of choice from the Zermelo–Fraenkel axioms. Roughly speaking, some of these categories behave like the category Set, except that the continuum hypothesis or the axiom of choice fails in them.

Example 5.40. Presheaves and sheaves also play a very important role in complex analysis and algebraic geometry, for instance, they are used to define *sheaf cohomology*, which is one of the most important invariants of complex manifolds and algebraic varieties.

5.41. ***Algebraic geometry***

Example 5.42. Fix an algebraically closed field k. The category AffVar_k has *affine algebraic varieties* over k (i.e., subsets of k^n defined by polynomial equations with coefficients in k) as objects and *regular maps* (restrictions of polynomial maps $k^m \to k^n$) as morphisms.

6 Isomorphisms

Any set with one element can be turned into a group in the obvious fashion. Different sets with one element (e.g., $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\emptyset, \{\emptyset\}\}\}$, etc.) give rise to groups (e.g., G_1, G_2, G_3 for the above sets) that from a formal viewpoint are different groups: $G_1 \neq G_2$, $G_2 \neq G_3$, etc. However, in group theory we perceive these groups as the "same" group: even though they are not *equal* groups, they are *isomorphic* groups. The situation in other categories is entirely analagous, and the notion of an isomorphism in a category generalizes the notion of an isomorphism in a group.

Definition 6.1. A morphism $f\colon X \to Y$ in some category C is an *isomorphism* if there is a morphism $g\colon Y \to X$ (typically denoted f^{-1}) such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

Remark 6.2. The morphism $f^{-1} = g$ defined above is unique. Indeed, if some g' has the same property, then $g' = \mathrm{id}_X\, g' = (gf)g' = gfg' = g(fg') = g\,\mathrm{id}_Y = g$.

Examples 6.3. We list several categories and describe isomorphisms in them.

- Set: bijections (alias one-to-one and onto functions);
- Ab, Ring, Mod_R, and other categories algebraic of algebraic structures: isomorphisms;
- Top: homemorphisms;
- HoTop: homotopy equivalences of topological spaces;
- Man: diffeomorphisms;
- Ban_1: isometric isomorphisms of Banach spaces;
- Ban: homeomorphisms of Banach spaces (not necessarily norm-preserving);
- DO_M: zeroth order differential operators given by multiplication by a smooth nonvanishing function;
- $\Psi\mathsf{DO}_M^\infty$: a large class that contains all elliptic differential (and pseudodifferential) operators;
- Meas: an isomorphisms of the underlying sets with σ-algebras and σ-ideals, after possibly removing negligible sets from source and target (e.g., $\mathbf{R}$ and $\mathbf{R} \setminus \{0, 1, 2\}$ are isomorphic, if we use Lebesgue structures).

Definition 6.4. An *endomorphism* is a morphism whose source and target are the same. An *automorphism* is an endomorphism that is also an isomorphism. Given an object X in a category C, the *group of automorphisms* of X is the set of all automorphisms of X equipped with the operation of composition and is denoted by $\mathsf{Aut}_{\mathsf{C}}(X)$ or simply $\mathsf{Aut}(X)$ if C is clear from the context.

Examples 6.5.

- $X \in \mathsf{FinSet}$ (finite sets): $\mathsf{Aut}(X)$ is the *symmetric group* on X;
- $X = \mathbf{Z}^n \in \mathsf{Ab}$ (lattices): $\mathsf{Aut}(X) = \mathbf{Z}/2 \times \mathrm{SL}_n(\mathbf{Z})$ is (up to the factor $\mathbf{Z}/2$) the *unimodular group* of degree n;
- $X = k^n \in \mathsf{Vect}_k$: $\mathsf{Aut}(X) = \mathrm{GL}(k^n) = \mathrm{GL}(n, k)$ is the *general linear group* of degree n over a field k;

- $X \in \mathsf{Meas}$: $\mathtt{Aut}(X)$, the group of measurable automorphisms, plays an important role in ergodic theory.

6.6. ***Groupoids***

Definition 6.7. A *groupoid* is a category in which all morphisms are isomorphisms.

Example 6.8. The *fundamental groupoid* $\pi_{\le 1}(X)$ of a topological space X is defined as follows. Objects are points of X. Morphisms $x \to y$ are equivalences classes of continuous maps $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$ modulo the equivalence relation of *relative homotopy*: $f \sim g$ if there is a continuous map $h\colon [0,1] \times [0,1] \to X$ such that $h|_{0\times[0,1]} = f$, $h|_{1\times[0,1]} = g$, $h|_{[0,1]\times 0} = \hat{x}$, and $h|_{[0,1]\times 1} = \hat{y}$. Here $\hat{x}$ and $\hat{y}$ denote the constant maps $[0,1] \to X$ with values x and y respectively. Composition is defined by composing the underlying representative functions $f\colon [0,1] \to X$ and $g\colon [0,1] \to X$ as follows:

$$gf = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1. \end{cases}$$

One checks that this operation respects the above equivalence relation, which gives a well-defined composition. The identity morphism $x \to x$ is the equivalence class of the constant map $[0,1] \to X$ with value x. The category that we just defined is a groupoid: the inverse can be defined on a representative function $f\colon [0,1] \to X$ as $g\colon [0,1] \to X$, $g(t) = f(1-t)$.

Example 6.9. Consider a topological space X with a point $x \in X$. The group $\mathtt{Aut}_{\pi_{\le 1}(X)}(x)$ is denoted $\pi_1(X,x)$ and is referred to as the *fundamental group* of the pointed space (X,x). It is an important invariant in topology. A different choice x' of a basepoint yields a *noncanonically isomorphic* fundamental group $\pi_1(X,x')$. More precisely, a morphism $f\colon x \to x'$ (i.e., an homotopy class of paths) in $\pi_{\le 1}(X)$ gives rise to an isomorphism $\pi_1(X,x) \to \pi_1(X,x')$ (namely, $p \mapsto fpf^{-1}$, where the right side uses composition in $\pi_{\le 1}(X)$), and different morphisms $x \to x'$ can give different morphisms of groups. Used in 7.16.

Example 6.10. The (absolute) *Galois groupoid* $\mathtt{Gal}(k)$ of a field k has as objects algebraically closed (or separably closed if $\operatorname{char} k \ne 0$) extensions L/k, whereas morphisms are isomorphisms $L \to L'$ of extensions over k (i.e., the action on k is identity). The automorphism group $\mathtt{Aut}_{\mathtt{Gal}(k)}(L/k)$ of some algebraic closure L of k is known as the (absolute) *Galois group* of k. A different choice of L/k yields a *noncanonically isomorphic* Galois group. More precisely, isomorphisms $L \to L'$ produce isomorphisms $\mathtt{Gal}(L/k) \to \mathtt{Gal}(L'/k)$, and different isomorphisms can produce different isomorphisms of groups.

Remark 6.11. The above examples of fundamental groupoids and Galois groupoids seem to be analogous. Indeed, both groupoids can be defined using the same construction: the *fundamental groupoid of a topos*. For topological spaces one takes the topos of sheaves, whereas for fields one takes the etale topos.

Exercise 6.12. Generalize the above arguments and show that in any groupoid G an isomorphism $X \to X'$ induces a homomorphism of groups $\mathtt{Aut}_{\mathsf{G}}(X) \to \mathtt{Aut}_{\mathsf{G}}(X')$.

7 Functors

In the previous section we gave many examples of categories in different areas of mathematics. One glaring omission from this list is category theory itself. Morphisms between categories are known as *functors*.

Definition 7.1. A *functor* $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ from a category C to a category D is given by the following data:

- a function $\mathsf{Ob}(\mathsf{F})\colon \mathsf{Ob}(\mathsf{C}) \to \mathsf{Ob}(\mathsf{D})$ that sends objects of C to objects of D;
- for any object $X, Y \in \mathsf{C}$ we have a function $\mathsf{Mor}_{\mathsf{F}}\colon \mathsf{Mor}_{\mathsf{C}}(X,Y) \to \mathsf{Mor}_{\mathsf{D}}(\mathsf{F}(X), \mathsf{F}(Y))$, i.e., F maps a morphism $f\colon X \to Y$ to a morphism $\mathsf{F}(f)\colon \mathsf{F}(X) \to \mathsf{F}(Y)$. This data must satisfy the following properties:
- for any morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in the category C we have $\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$, i.e., F preserves composition;
- for any object $X \in \mathsf{C}$ we have $\mathsf{F}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{F}(X)}$, i.e., F preserves identity morphisms.

Remark 7.2. Thus, to define a functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ one must specify an object $\mathsf{F}(X) \in \mathsf{D}$ for any object $X \in \mathsf{C}$, a morphism $\mathsf{F}(f)\colon \mathsf{F}(X) \to \mathsf{F}(Y)$ of D for any morphism $f\colon X \to Y$ in C such that the operation of composition and identity morphisms are preserved.

Example 7.3. *Forgetful* functors are one of the easiest functors to define. For instance, the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ is defined as follows. We send an abelian group A to its underlying set with the algebraic operations discarded. A homomorphism of abelian groups $X \to Y$ is a function between the underlying sets, hence already a morphism of sets. The forgetful functor preserves compositions because morphisms of abelian groups are composed by composing their underlying functions. The identity morphism is preserved for the same reason.

Remark 7.4. Although one can give rigorous definitions of the adjective "forgetful", typically this term is used in an informal manner, with somewhat imprecise corner cases.

Example 7.5. In an entirely analogous fashion we have forgetful functors $\mathsf{Group} \to \mathsf{Set}$, $\mathsf{Ring} \to \mathsf{Set}$, $\mathsf{Top} \to \mathsf{Set}$, $\mathsf{Ban} \to \mathsf{Set}$, $\mathsf{Ban}_1 \to \mathsf{Set}$, etc. Somewhat less obviously one also has forgetful functors $\mathsf{Vect}_k \to \mathsf{Ab}$, $\mathsf{Ring} \to \mathsf{Ab}$, $\mathsf{Ban} \to \mathsf{Met}$, $\mathsf{Ban}_1 \to \mathsf{Met}_1$ (but *not* $\mathsf{Ban} \to \mathsf{Met}_1$), $\mathsf{W}^* \to \mathsf{C}^*$, $\mathsf{CW}^* \to \mathsf{CC}^*$. For instance, any vector space has an underlying abelian group, and linear maps of vector spaces are maps of abelian groups with additional properties (namely, preservation of multiplication), hence we have a forgetful functor $\mathsf{Vect}_k \to \mathsf{Ab}$.

Example 7.6. By definition of HoTop we have a functor $\mathsf{Top} \to \mathsf{HoTop}$. Applying this functor can be seen as discarding the nontopological information.

Example 7.7. We have the obvious *inclusion functors* (another informal term) $\mathsf{Met}_1 \to \mathsf{Met}$, $\mathsf{Ban}_1 \to \mathsf{Ban}$.

7.8. ***Algebra***

Example 7.9. The *free group functor* $\mathsf{Free}_{\mathsf{Group}}\colon \mathsf{Set} \to \mathsf{Group}$ sends a set S to the free group $\mathsf{Free}_{\mathsf{Group}}(S)$ on the generating set S. A function $f\colon X \to Y$ is sent to the (unique) homomorphism of free groups $\mathsf{Free}_{\mathsf{Group}}(f)\colon \mathsf{Free}_{\mathsf{Group}}(X) \to \mathsf{Free}_{\mathsf{Group}}(Y)$ that sends elements of $X \subset \mathsf{Free}_{\mathsf{Group}}(X)$ to their images in $Y \subset \mathsf{Free}_{\mathsf{Group}}(Y)$ via f. (Here we used the universal property of free groups to extend the above map to a homomorphism of groups.)

Example 7.10. The *group of units functor* $-^{\times}\colon \mathsf{Ring} \to \mathsf{Group}$ sends a ring R to its group $R^{\times}$ of invertible elements, i.e., elements $x \in R$ for which there is $y \in R$ such that $1 = xy = yx$. Any homomorphism of rings $R \to S$ preserves invertible elements and therefore induces a homomorphism of groups $U(R) \to U(S)$.

Example 7.11. The *polynomial ring functor* $-[x]\colon \mathsf{Ring} \to \mathsf{Ring}$ sends a ring R to the ring $R[x]$ of polynomials in a single variable x with coefficients in R. A homomorphism of rings $R \to S$ is sent to the homomorphism of rings $R[x] \to S[x]$ given by applying it to each coefficient.

Nonexample 7.12. The *group center* construction sends a group G to its center $Z(G)$ defined as $\{g \in G \mid \forall x \in G\colon gx = xg\}$. A homomorphism of groups $G \to H$ does *not* restrict to a homomorphism of groups $Z(G) \to Z(H)$. For instance, take $G = \mathbf{Z}/2$, $H = \Sigma_3$, and $G \to H$ sends the nontrivial element of $\mathbf{Z}/2$

to a permutation in Σ_3 that permutes two of the elements and leaves the third one untouched. We have $Z(\mathbf{Z}/2) = \mathbf{Z}/2$, but $Z(\Sigma_3) = \{1\}$.

Example 7.13. The *exterior algebra functor* $\Lambda\colon \mathsf{Vect}_k \to \mathsf{Alg}_k$ sends a k-vector space V to its exterior algebra ΛV and a linear map $V \to W$ to the induced homomorphism of algebras $\Lambda V \to \Lambda W$.

Example 7.14. Fix a field k. The *group algebra functor* $k[-]\colon \mathsf{Group} \to \mathsf{Alg}_k$ sends a group G to its group algebra $k[G]$ and a homomorphism of groups $G \to H$ to the induced homomorphism of algebras $k[G] \to k[H]$. The *group of units functor* $-^\times\colon \mathsf{Alg}_k \to \mathsf{Group}$ sends a k-algebra A to the group of its units (invertible elements) $A^\times$, with the induced multiplication.

7.15. ***Topology***

Example 7.16. The *fundamental group functor* $\mathsf{Top}_* \to \mathsf{Group}$ was defined in Example 6.9.

7.17. ***Measure theory***

Example 7.18. The *Haar measurable space functor* $\mathsf{LocCompHausGroupOpen} \to \mathsf{Meas}$ sends a locally compact Hausdorff topological group G to a measurable space (X, M, N), where X is the underlying set of G, N is the σ-ideal of sets of measure 0 with respect to some (hence all) left (or right) Haar measure on G, and M is the σ-algebra generated by N and open sets. (A left Haar measure is a left-invariant nonzero Radon measure on G, or, equivalently, a left-invariant continuous functional on the space of compactly supported continuous functions on G equipped with the topology of uniform convergence on compact subsets.) An open continuous homomorphism of locally compact groups is sent to the equivalence class of its underlying function. (Negligible sets are preserved under preimages of open maps.)

Example 7.19. The functor $\mathcal{M}\colon \mathsf{LocCompHausGroup} \to \mathsf{Alg}_{\mathbf{R}}$ sends a locally compact Hausdorff topological group G to the real algebra of bounded measures on G, with the product of μ and ν given by the *convolution* $\mu * \nu$ of measures. A continuous homomorphism of groups $f\colon G \to H$ is mapped to the homomorphism of real algebras $\mathcal{M}(G) \to \mathcal{M}(H)$ that sends a bounded measure μ on G to its *pushforward* $f_*\mu$ along f, defined as $(f_*\mu)(E) = \mu(f^{-1}(E))$ for any open set E in H. Used in 8.12.

7.20. ***Smooth manifolds and Lie groups***

Example 7.21. The functor $T\colon \mathsf{Man} \to \mathsf{VBun}$ sends a smooth manifold M to its *tangent bundle* TM and a smooth map $f\colon X \to Y$ to the induced tangent map $TM \to TN$.

Example 7.22. Given a manifold $M \in \mathsf{Man}$ with a basepoint $* \in M$, we define a functor $\mathtt{fiber}\colon \mathsf{VBun}_M \to \mathsf{Vect}_k$ by sending a vector bundle over M to its fiber over $* \in M$ and a morphism of vector bundles to the induced morphism of fibers.

Example 7.23. The functor $\mathsf{LieGroup} \to \mathsf{Vect}_{\mathbf{R}}$ is defined as the composition $\mathsf{LieGroup} \to \mathsf{Man} \to \mathsf{VBun} \to \mathsf{Vect}_k$, where the first functor is the forgetful functor, the second functor is the tangent functor T, and the third functor is the fiber functor with respect to the identity element of the Lie group. As shown in any book on Lie groups, this functor factors as the composition $\mathsf{LieGroup} \to \mathsf{LieAlg}_{\mathbf{R}} \to \mathsf{Vect}_{\mathbf{R}}$, where the second functor is the forgetful functor.

7.24. ***Category theory***

Recall that not every class is a set. For instance, by Russell's paradox, the class of all sets is not a set.

Definition 7.25. A category C is *small* if the class of its objects is a set.

Example 7.26. The category Set is not a small category. The full subcategory of Set on objects that are subsets of some fixed set X is a small category.

Definition 7.27. The category Cat of small categories has small categories as objects and functors as morphisms. Composition of morphisms $\mathsf{G}\colon \mathsf{D} \to \mathsf{E}$ and $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is given by the *composition of functors*: the functor $\mathsf{G} \circ \mathsf{F}$ sends an object $X \in \mathsf{C}$ to the object $\mathsf{G}(\mathsf{F}(X)) \in \mathsf{E}$ and a morphism $f\colon X \to Y$ in C to the morphism $\mathsf{G}(\mathsf{F}(f))\colon \mathsf{G}(\mathsf{F}(X)) \to \mathsf{G}(\mathsf{F}(Y))$ in E. The identity morphism on C is the identity functor $\mathrm{id}_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{C}$ such that $\mathrm{id}_{\mathsf{C}}(X) = X$ and $\mathrm{id}_{\mathsf{C}}(f) = f$.

Remark 7.28. The above definition restricts to *small* categories because functors between small categories form a set (as opposed to a mere class) and we require a *set* (not a class) of morphisms between any pair of objects. There is no category of categories because functors (say) $\mathsf{Set} \to \mathsf{Set}$ form a class that is not a set: this class contains *constant* functors, i.e., functors $\mathsf{Set} \to \mathsf{Set}$ that send any object $X \in \mathsf{Set}$ to some fixed set A and any morphism f in Set to id_A. Thus there are as many constant functors as there are sets, so in particular the class of functors $\mathsf{Set} \to \mathsf{Set}$ contains a subclass isomorphic to the class of all sets, and therefore cannot be a set by Russell's paradox. This problem is easily circumvented by introducing *conglomerates*, which are collections that can contain classes (and not just sets) as elements. This yields a "huge" category CAT of categories.

Definition 7.29. The functor $\mathsf{Ob}\colon \mathsf{Cat} \to \mathsf{Set}$ sends a category C to its set of objects and a functor to its underlying function on objects.

Definition 7.30. Given a category C, the functor $\mathrm{Mor}_{\mathsf{C}}\colon \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Set}$ sends a pair of objects (X, Y) in C to the set $\mathrm{Mor}_{\mathsf{C}}(X, Y)$. Morphisms $(X, Y) \to (X', Y')$ are pairs of functions $f\colon X' \to X$ and $g\colon Y \to Y'$, which are sent to the induced function $\mathrm{Mor}_{\mathsf{C}}(X, Y) \to \mathrm{Mor}_{\mathsf{C}}(X', Y')$. (In the above $\times$ denotes the product of categories C and C^{op}, a construction that will be explained below.)

Many examples of categories given above have "sets with structures" as objects and "functions that preserve the structure" as morphisms, e.g., groups and group homomorphisms, topological spaces and continuous maps, smooth manifolds and smooth maps, etc. We do not give a definition of a "structure" here, but see Chapter IV of Bourbaki's *Theory of Sets* for one possible definition. We can, however, rather easily formalize such types of categories as *concrete categories*.

Definition 7.31. A functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is *faithful* if for any pair of object $X, Y \in \mathsf{C}$ the induced function $\mathrm{Mor}_{\mathsf{C}}(X, Y) \to \mathrm{Mor}_{\mathsf{D}}(\mathsf{F}(X), \mathsf{F}(Y))$ is injective.

In other words, F is faithful if $\mathsf{F}(f) = \mathsf{F}(g)$ implies $f = g$ for any pair of morphisms $f, g\colon X \to Y$.

Definition 7.32. A *concrete category* is a pair (C, U), where C is a category and $U\colon \mathsf{C} \to \mathsf{Set}$ is a faithful functor. A category C is *concretizable* if there is U such that (C, U) is a concrete category.

Examples 7.33. The following categories are concrete for the obvious choice of the functor U, the underlying set functor:

- Set;
- Group, Ring, Vect_k, any other category of algebraic objects;
- Met, Met_1;
- Top, $\mathsf{TopGroup}$, $\mathsf{TopVect}_k$;
- Ban, Ban_1, C^*, W^*;
- Man, $\mathsf{LieGroup}$, VBun.

Example 7.34. A given category C can admit many different functors U that make it concrete. For instance, for Ban_1 apart from the underlying set functor we can take the functor $U(X) = \{x \in X \mid \|x\| \leq 1\}$.

Example 7.35. We show that $\mathsf{Set}^{\mathsf{op}}$ (see Definition 8.1 below) is concrete by defining a faithful functor $U\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$. Set $U(X) = 2^X$, the set of all subsets of X. For a morphism $q\colon X \to Y$ in $\mathsf{Set}^{\mathsf{op}}$ (i.e., a function $f\colon Y \to X$), we have to define a function $U(q)\colon U(X) \to U(Y)$, i.e., a function $2^X \to 2^Y$. We take f^{-1}, the function that sends a subset $A \subset X$ to its preimage $f^{-1}(A) = \{y \in Y \mid g(y) \in A\}$. We have $\mathrm{id}_X^{-1}(A) = A$ and $(f_2 f_1)^{-1}(A) = f_1^{-1}(f_2^{-1}(A))$ (the order of f_1 and f_2 is reversed because of the contravariance). Thus we indeed have defined a functor $U\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$. If for any set $A \subset X$ we have $f^{-1}(A) = g^{-1}(A)$ for two functions $f\colon Y \to X$ and $g\colon Y \to X$, then $f = g$, so the functor is faithful. This proves that $\mathsf{Set}^{\mathsf{op}}$ is concrete.

Example 7.36. We could try to turn the category Meas into a concrete category in the most naive way. Assign to a measurable space (X, M, N) the set X and to a morphism of measurable spaces $(X, M, N) \to (X', M', N')$ coming from some function $f\colon X \to X'$ this function f. This does *not* give us a functor $\mathsf{Meas} \to \mathsf{Set}$ for the very simple reason: a morphism $(X, M, N) \to (X', M', N')$ is an equivalence class containing many different functions, and it is unclear which one we should take so that composition is respected. The concreteness of Meas can be established by virtue of a faithful functor $\mathrm{MeasAlg}\colon \mathsf{Meas}^{\mathsf{op}} \to \mathsf{BoolAlg}$ constructed in Example 8.30 and the fact that $\mathsf{BoolAlg}$ (and hence $\mathsf{BoolAlg}^{\mathsf{op}}$) is concrete.

Example 7.37. A theorem of Freyd says that HoTop is *not* concretizable.

8 Contravariance and duality

The following construction, despite its apparent simplicity, plays a very important role in category theory.

Definition 8.1. Given a category C its *opposite category* C^{op} has the same objects as C and $\mathsf{Mor}_{\mathsf{C}^{\mathsf{op}}}(X,Y) = \mathsf{Mor}_{\mathsf{C}}(Y,X)$. Composition is the map

$$\begin{aligned}\mathsf{Mor}_{\mathsf{C}^{\mathsf{op}}}(Y,Z)\times\mathsf{Mor}_{\mathsf{C}^{\mathsf{op}}}(X,Y) &= \mathsf{Mor}_{\mathsf{C}}(Z,Y)\times\mathsf{Mor}_{\mathsf{C}}(Y,X)\\ &\cong \mathsf{Mor}_{\mathsf{C}}(Y,X)\times\mathsf{Mor}_{\mathsf{C}}(Z,Y)\to\mathsf{Mor}_{\mathsf{C}}(Z,X)=\mathsf{Mor}_{\mathsf{C}^{\mathsf{op}}}(X,Z).\end{aligned}$$

Used in 7.35.

Below we will see many examples when some category C is equivalent (term defined below) to the opposite category of some other category D. Such an equivalence is implemented by a *contravariant functor*.

Definition 8.2. A *contravariant functor* from C to D is a functor $\mathsf{C}^{\mathsf{op}}\to\mathsf{D}$, or, equivalently, $\mathsf{C}\to\mathsf{D}^{\mathsf{op}}$.

Remark 8.3. One way to see the equivalence between functors $\mathsf{C}^{\mathsf{op}}\to\mathsf{D}$ and $\mathsf{C}\to\mathsf{D}^{\mathsf{op}}$ is to expand the definition: a contravariant functor F from C to D assigns to every object $X\in\mathsf{C}$ an object $\mathsf{F}(X)\in\mathsf{D}$ and to every morphism $f\colon X\to Y$ in C a morphism $\mathsf{F}(f)\colon\mathsf{F}(Y)\to\mathsf{F}(X)$ in D. Composition and identity morphisms must be respected.

Remark 8.4. Of course, contravariant functors are just a particular case of the general notion of functor (sometimes referred to as a *covariant* functor). The justification for introducing this new bit of terminology is that many functors naturally arise as contravariant functors, i.e., their domain or codomain is the opposite category of some previously defined category.

We now give several examples of contravariant functors.

8.5. ***Algebra***

Example 8.6. The functor $\mathsf{O}_{\mathtt{Set}}\colon\mathsf{Set}^{\mathsf{op}}\to\mathsf{Alg}_{\mathbf{R}}$ sends a set S to the real algebra of functions on S and a function $f\colon S\to T$ to the homomorphism of real algebras given by precomposition with f, i.e., $S\to T\to\mathbf{R}$.

Example 8.7. Fix a field k. The *dual vector space functor* $*\colon\mathsf{Vect}_k^{\mathsf{op}}\to\mathsf{Vect}_k$ sends a vector space V to $V^*=\mathsf{Hom}(V,k)$ and a linear map $V\to V'$ to the induced map $\mathsf{Hom}(V',k)\to\mathsf{Hom}(V,k)$.

Example 8.8. The functor $2^-\colon\mathsf{Set}^{\mathsf{op}}\to\mathsf{ComplAtomBoolAlg}$ sends a set S to the complete atomic Boolean algebra 2^S of functions $S\to 2=\{0,1\}$ (equivalently, the complete atomic Boolean algebra of subsets of S) and a function $f\colon S\to T$ to the homomorphism of complete atomic Boolean algebras $f^{-1}\colon 2^T\to 2^S$.

Example 8.9. The functor $\mathsf{Spec}_{\mathtt{Atom}}\colon\mathsf{ComplAtomBoolAlg}^{\mathsf{op}}\to\mathsf{Set}$ sends a complete atomic Boolean algebra A to its set of atoms (which can be defined as morphisms $A\to 2$) and a continuous homomorphism $B\to A$ to the induced function given by the composition $B\to A\to 2$.

8.10. ***General topology***

Example 8.11. The functor $\mathsf{O}_{\mathtt{Cont}}\colon\mathsf{Top}^{\mathsf{op}}\to\mathsf{Alg}_{\mathbf{R}}$ sends a topological space X to the real algebra of continuous functions $X\to\mathbf{R}$ with the pointwise operations. A continuous map $X\to Y$ is mapped to the morphism of algebras $\mathsf{O}_{\mathtt{Cont}}(Y)\to\mathsf{O}_{\mathtt{Cont}}(X)$ given by composition $X\to Y\to\mathbf{R}$.

Example 8.12. The functor $\mathsf{O}_{\mathtt{Conv}}\colon\mathsf{CompHausGroup}^{\mathsf{op}}\to\mathsf{Alg}_{\mathbf{R}}$ sends a compact Hausdorff topological group G to the real algebra of continuous functions on G, with the multiplication given by the *convolution* of functions with respect to the unique Haar measure μ on G such that $\mu(G)=1$. A continuous homomorphism of groups $f\colon G\to H$ is mapped to the homomorphism of real algebras $\mathsf{O}_{\mathtt{Conv}}(H)\to\mathsf{O}_{\mathtt{Conv}}(G)$ that sends a function p on H to its *pullback* f^*p along f, defined as $f^*p=p\circ f$. This example should be contrasted with the *covariant* functor of Example 7.19, which was defined on measures instead of functions. This distinction is essential: we can pushforward measures and pullback functions, but not vice versa. Measures cannot be pulled back unless we have additional data (such as a relative measure on f). The pushforward of a function can be defined as a measure, which need not be a function, e.g., it can be the Dirac δ-measure.

8.13. ***Topological algebra***

Example 8.14. The functor $\mathsf{PD}\colon \mathsf{LocCompHausAb}^{\mathsf{op}} \to \mathsf{LocCompHausAb}$ sends a locally compact Hausdorff abelian topological group G to the topological group $\mathsf{Hom}(G, \mathrm{U}(1))$, whose elements are continuous homomorphisms $G \to \mathrm{U}(1)$, equipped with the compact-open topology (whose subbasis consists of functions that map a given compact subset $K \subset G$ to a given open subset $V \subset \mathrm{U}(1)$. A continuous homomorphism of group $G \to G'$ induces a continuous homomorphism $\mathsf{Hom}(G', \mathrm{U}(1)) \to \mathsf{Hom}(G, \mathrm{U}(1))$. Used in 10.9.

8.15. ***Banach spaces***

We give some examples related to the Hahn–Banach theorem. We start by defining one of the categories involved. Everything below can be done either for real or complex spaces.

Definition 8.16. The category Ball has *unit balls* as objects, defined as pairs (V, B) consisting of a Hausdorff locally convex topological vector space V and a Hausdorff topological subspace $B \subset V$ such that B is *balanced* (i.e., $0 \in B$ and for any $x \in B$ and number t such that $|t| \le 1$ we have $tx \in B$), and B is *convex* (i.e., for any $x, y \in B$ and real numbers $r \ge 0$ and $s \ge 0$ such that $r + s \le 1$ we have $rx + sy \in B$). Morphisms $(V, B) \to (V', B')$ are continuous linear maps $V \to V'$ that send B to B'. The category Ball is also known as the category of *Saks spaces.*

Definition 8.17. $\mathsf{CompBall}$ is the full subcategory of Ball consisting of balls (V, B) such that B is compact. It is also known as the category of *Waelbroeck spaces.*

Example 8.18. We have a functor $\mathsf{Ban}_1 \to \mathsf{Ball}$ that sends a Banach space X to the unit ball $(X, X_{\le 1})$, where $X_{\le 1}$ denotes the subset of X consisting of elements of norm at most 1. A contractive map $X \to X'$ of Banach spaces is sent to the induced map $(X, X_{\le 1}) \to (X', X'_{\le 1})$, the contractivity property guaranteeing that the unit ball is preserved.

Example 8.19. The *dual unit ball functor* $\mathsf{DUB}\colon \mathsf{Ban}_1^{\mathsf{op}} \to \mathsf{CompBall}$ is defined as follows. Given a Banach space X consider the vector space X^* of continuous linear functionals on X equpped with the weak-* topology, i.e., the coarsest topology in which every function on X^* given by evaluation on some fixed element $x \in X$ is continuous. We define $\mathsf{DUB}(X) = (X^*, X^*_{\le 1})$, where $X^*_{\le 1}$ denotes the set of functionals of norm at most 1. A continuous linear map $X \to Y$ of Banach spaces induces a continuous linear map $Y^* \to X^*$, which restricts to $Y^*_{\le 1} \to X^*_{\le 1}$.

Remark 8.20. The traditional Hahn–Banach theorem can be interpreted as saying that an inclusion $A \subset B$ of Banach spaces is sent by the functor DUB to a surjective map of unit balls.

Example 8.21. The functor $\mathsf{O}_{\mathsf{Ban}}\colon \mathsf{CompBall}^{\mathsf{op}} \to \mathsf{Ban}_1$ is defined as follows. Given a unit ball (V, B) to the Banach space of linear functionals f on V. The norm of a functional is defined as the supremum of its absolute value on B. A morphism $(V, B) \to (V', B')$ induces a contractive map from linear functionals on V' to linear functionals on V given by the composition $V \to V' \to \mathbf{C}$.

Remark 8.22. Below we will see that DUB and $\mathsf{O}_{\mathsf{Ban}}$ are mutually inverse to each other in the appropriate sense and identify (in the appropriate sense) Ban_1 and $\mathsf{CompBall}$. This is a strengthening of the traditional Hahn–Banach theorem.

8.23. ***Operator algebras***

We now define the two functors that together form the famous *Gelfand duality* for commutative C*-algebras and compact Hausdorff spaces.

Definition 8.24. The category C^* of C*-algebras is defined as follows. Its objects are C*-algebras, i.e., complex algebras A equipped with an involution (i.e., a morphism of abelian groups $*\colon A \to A$ such that $a^{**} = a$, $1^* = 1$, $(ab)^* = b^* a^*$, and $(\lambda a)^* = \bar{\lambda} a^*$ for any $\lambda \in \mathbf{C}$) and a norm that is compatible with the involution and multiplication (i.e., $\|1\| = 1$, $\|ab\| \le \|a\| \cdot \|b\|$, $\|a^* a\| = \|a^*\| \cdot \|a\|$) such that the underlying normed vector space is complete, i.e., a Banach space. Morphisms $f\colon A \to B$ are morphisms of complex algebras that preserve the involution, i.e., $f(a^*) = f(a)^*$. (One can prove that f is *automatically* contractive.)

The category CC^* of *commutative* C*-algebras is the full subcategory of C^* consisting of C*-algebras that are commutative, i.e., $ab = ba$.

Definition 8.25. The functor $\mathsf{O}_{\mathsf{CC}} \colon \mathsf{CompHaus}^{\mathsf{op}} \to \mathsf{CC}^*$ sends a compact Hausdorff space X to the commutative C*-algebra $\mathsf{O}_{\mathsf{CC}}(X)$ of complex-valued continuous functions on X equipped with the pointwise algebra structure, involution given by the complex conjugation, and norm given by the supremum of the absolute value. A continuous map of compact Hausdorff spaces $f \colon X \to Y$ is sent to the morphism of commutative C*-algebras $\mathsf{O}_{\mathsf{CC}}(Y) \to \mathsf{O}_{\mathsf{CC}}(X)$ given by precomposition with f, i.e., a continuous function $Y \to \mathbf{C}$ is mapped to the continuous function $X \to Y \to \mathbf{C}$.

Definition 8.26. The *Gelfand spectrum* functor $\mathsf{Spec}_{\mathsf{CC}} \colon (\mathsf{CC}^*)^{\mathsf{op}} \to \mathsf{CompHaus}$ sends a commutative C*-algebra A to the compact Hausdorff topological space $\mathsf{Spec}_{\mathsf{CC}}(A)$ whose points are homomorphisms of C*-algebras $A \to \mathbf{C}$ and a set $S \subset \mathsf{Spec}_{\mathsf{CC}}(A)$ is *closed* if there is a morphism of C*-algebras $g \colon A \to B$ (for some B) such that $s \colon A \to \mathbf{C}$ is in S if and only if $s = tg$ for some morphism $t \colon B \to \mathbf{C}$. (Of course, one must show that the resulting object is a compact Hausdorff topological space.) A morphism $f \colon A \to B$ of commutative C*-algebras is sent to the continuous map $\mathsf{Spec}_{\mathsf{CC}}(f) \colon \mathsf{Spec}_{\mathsf{CC}}(B) \to \mathsf{Spec}_{\mathsf{CC}}(A)$ that sends a point $b \in \mathsf{Spec}_{\mathsf{CC}}(B)$ (i.e., a morphism $B \to \mathbf{C}$) to the composition $A \to B \to \mathbf{C}$, which is a point in $\mathsf{Spec}_{\mathsf{CC}}(A)$. (Again, one must show that this function is a continuous map.)

Remark 8.27. Once again, we will see below that $\mathsf{Spec}_{\mathsf{CC}}$ and O_{CC} form an *equivalence* of categories between $\mathsf{CompHaus}$ and CC^*.

8.28. ***Measure theory***

We recall that the category Meas has triples (X, M, N) as objects and equivalence classes of measurable maps as morphisms.

The following functor exhibits sets as discrete measurable spaces.

Example 8.29. The functor $\mathsf{Disc} \colon \mathsf{Set} \to \mathsf{Meas}$ sends a set S to $(S, 2^S, \emptyset)$ and a function $f \colon S \to T$ to the morphism $(S, 2^S, \emptyset) \to (T, 2^T, \emptyset)$ given by the equivalence class of f.

We establish a connection between measurable spaces and Boolean algebras.

Example 8.30. The functor $\mathsf{MeasAlg} \colon \mathsf{Meas}^{\mathsf{op}} \to \mathsf{BoolAlg}$ sends a measurable space (X, M, N) to the Boolean algebra M/N of equivalences classes of measurable sets modulo negligible sets and a morphism of measurable spaces $f \colon (X, M, N) \to (X', M', N')$ to the induced morphism $M'/N' \to M/N$ of Boolean algebras given by the preimage map. Recall that f is an equivalence class of measurable functions, and equivalent functions induce the same morphism $M'/N' \to M/N$, see Theorem 324A in Fremlin's *Measure Theory* for more details. Used in 7.36.

We now explain how to get measurable spaces from topological spaces.

Definition 8.31. The functor $\mathsf{Borel} \colon \mathsf{Top} \to \mathsf{Meas}$ sends a topological space X to the measurable space $(X, \mathrm{Borel}_X, \{\emptyset\})$, where Borel_X is the σ-algebra of Borel subsets of X (i.e., the σ-algebra generated by open subsets of X). A continuous map $f \colon X \to Y$ is sent to the equivalence class of f. (Any equivalence class consists of a single element, and measurable functions are precisely continuous functions.)

For us, the following *different* construction of a measurable space from a topological space will also be of use. It formalizes the well-known set of analogies between negligible sets and meager sets, see Oxtoby's book *Measure and Category*. (*Meager* sets are defined as countable unions of nowhere dense sets, i.e., sets whose closure has empty interior.)

Definition 8.32. The category $\mathsf{TopOpen}$ has topological spaces as objects and continuous *open* maps as morphisms. A map is *open* if the image of any open set is an open set.

Definition 8.33. The functor $\mathsf{Meager} \colon \mathsf{TopOpen} \to \mathsf{Meas}$ turns a topological space X into a measurable space $(X, \mathrm{BorelMeager}_X, \mathrm{Meager}_X)$, where $\mathrm{BorelMeager}_X$ is the σ-algebra generated by open and meager subsets of X and Meager_X is the σ-ideal of meager subsets of X. A continuous map $X \to Y$ is sent to the equivalence class of the underlying function, which is measurable because preimages of meager subsets are meager, which in its turn follows from the fact that preimages of closed subsets with empty interior again have empty interior because the map is open.

We now establish a connection to functional analysis and operator algebras. In practice, Meas has extremely pathological objects that make most of the familiar theorems of measure theory false. The condition of σ-finiteness is often used to remedy this problem, but we use a less restrictive property.

Definition 8.34. The category $\mathsf{LocMeas}$ of *localizable* measurable spaces is the full subcategory of Meas consisting of measurable spaces (X, M, N) such that the factoralgebra M/N is *complete*: every subset $S \subset M/N$ has a supremum in M/N.

Remark 8.35. Translated in the language of underlying sets, a measurable space is localizable if for any subset $F \subset M$ (not necessarily countable) there is $X \in M$ such that for all $Y \in F$ we have $Y \setminus X \in N$. If F is finite or countable, then X must be equivalent (up to a negligible set) to $\bigcup F$, the union of all elements in F, which is guaranteed to be an element of M. If F is uncountable, there is no relation between X and the union of all elements in F. For example, suppose $X = \mathbf{R}$, the σ-algebra M consists of Lebesgue measurable sets, and the σ-ideal N consists of sets of Lebesgue measure 0. Take as F all singleton subsets of X. Then $X = \emptyset$. Indeed, for any $Y \in F$ we have $Y \setminus X = Y \in N$ because all singleton sets have measure 0.

Example 8.36. The measurable space $\mathtt{Lebesgue}(\mathbf{C}) = (\mathbf{C}, \mathrm{Lebesgue}_{\mathbf{C}}, \mathrm{Null}_{\mathbf{C}})$, where $\mathrm{Lebesgue}_{\mathbf{C}}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbf{C}$ and $\mathrm{Null}_{\mathbf{C}}$ is the σ-ideal of sets of Lebesgue measure 0, is localizable. In fact, any measurable space that admits a finite measure that does not vanish on $M \setminus N$ is localizable.

Example 8.37. The measurable space $\mathtt{Borel}(\mathbf{C}) = (\mathbf{C}, \mathrm{Borel}_{\mathbf{C}}, \{\emptyset\})$ is *not* localizable. (Take as F the uncountable family of singleton subsets a nonmeasurable subset of $\mathbf{C}$.) One can replace it with a certain localizable measurable space $\hat{\mathbf{C}}$, which is *not* $\mathtt{Lebesgue}(\mathbf{C})$. Indeed, there are *no* morphisms $\mathtt{Disc}\{*\} \to \mathbf{C}$. (The preimage of every point in $\mathbf{C}$ would have to be negligible, i.e., empty, a contradiction.) On the other hand, morphisms $\mathtt{Disc}\{*\} \to \hat{\mathbf{C}}$ are in bijection with morphisms $\mathtt{Disc}\{*\} \to \mathtt{Borel}(\mathbf{C})$, i.e., complex numbers. The existence of $\hat{\mathbf{C}}$ can be demonstrated most easily using the tools of category theory developed below: the category $\mathsf{LocMeas}$ is a *reflective subcategory* of Meas and $\hat{\mathbf{C}}$ can be defined as the *reflection* of $\mathtt{Borel}(\mathbf{C})$.

Definition 8.38. The functor $\mathsf{L}^\infty = \mathsf{O}_{\mathtt{Meas}}\colon \mathsf{LocMeas}^{\mathsf{op}} \to \mathsf{CW}^*$ sends a localizable measurable space (X, M, N) to the von Neumann algebra of bounded measurable functions on X. The latter can be defined as the set of all morphisms $(X, M, N) \to (\mathbf{C}, \mathrm{Borel}_{\mathbf{C}}, \{\emptyset\})$, where $\mathrm{Borel}_{\mathbf{C}}$ denotes the σ-algebra of Borel subsets of $\mathbf{C}$. The structure of a complex *-algebra is induced from $\mathbf{C}$. It remains to show (by definition of a von Neumann algebra) that the underlying Banach space is the dual of some other Banach space, the *predual*. In our case we take the Banach space of *complex-valued finite measures* on X, defined as σ-additive functions $M \to \mathbf{C}$ that vanish on N. (This space is also denoted by $\mathsf{L}^1(X, M, N)$ and is an example of L^p-spaces that are important in analysis.)

Definition 8.39. The *von Neumann spectrum functor* $\mathsf{Spec}_{\mathtt{Meas}}\colon (\mathsf{CW}^*)^{\mathsf{op}} \to \mathsf{LocMeas}$ is defined as the composition of three functors: the forgetful functor $(\mathsf{CW}^*)^{\mathsf{op}} \to (\mathsf{CC}^*)^{\mathsf{op}}$, the Gelfand spectrum functor $(\mathsf{CC}^*)^{\mathsf{op}} \to \mathsf{CompHaus}$, and the functor $\mathsf{TopOpen} \to \mathsf{Meas}$. The composition of the first two functors lands in the subcategory of $\mathsf{TopOpen}$ consisting of *hyperstonean* topological spaces and *open* maps, which is why composing with the third functor makes sense. The resulting composition itself lands in localizable measurable spaces. See §III.1, in particular, Theorem III.1.18 in Takesaki's *Theory of Operator Algebras* for the relevant facts about hyperstonean spaces used in this definition.

8.40. ***Algebraic geometry***

Fix an algebraically closed field k.

Example 8.41. The functor $\mathsf{O}_{\mathtt{Sch}}\colon \mathsf{AffVar}_k^{\mathsf{op}} \to \mathsf{CAlg}_k$ sends an affine variety V to the commutative k-algebra of regular maps $V \to k$ and a regular map f of affine varieties $V' \to V$ to the induced homomorphism $\mathsf{O}_{\mathtt{Sch}}(V) \to \mathsf{O}_{\mathtt{Sch}}(V')$ given by the precomposition with f. This functor lands in the full subcategory $\mathsf{CAlgAff}_k$ of finitely generated k-algebras without nilpotent elements.

Example 8.42. The functor $\mathsf{Spec}_{\mathtt{Sch}}\colon \mathsf{CAlgAffGen}_k^{\mathsf{op}} \to \mathsf{AffVar}_k$ has as its source the category $\mathsf{CAlgAffGen}_k^{\mathsf{op}}$ that is defined just like $\mathsf{CAlgAff}_k^{\mathsf{op}}$ except that its objects (i.e., algebras) are equipped with a finite set of generators (but morphisms need not preserve the generators). (This rather awkward kludge is necessary because we defined affine varieties as subsets of k^n. It can be eliminated by passing to *abstract* algebraic varieties, whose category is equivalent to our category.) We define the variety $\mathsf{Spec}_{\mathtt{Sch}}(A, G)$, where G is a finite set of generators of A (which induces a homomorphism of algebras $k[G] \to A$), as the subset of k^G consisting of those points for which the associated evaluation map $k[G] \to k$ factors as the composition $k[G] \to A \to k$ for some homomorphism $A \to k$ (which, if it exists, uniquely determines the corresponding point in k^G). The regular map of varieties $\mathsf{Spec}_{\mathtt{Sch}}(A, G) \to \mathsf{Spec}_{\mathtt{Sch}}(A', G')$ for a homomorphism $A' \to A$ is defined by sending the point corresponding to a homomorphism $A \to k$ to the point corresponding to the homomorphism $A' \to A \to k$.

Below we will see that $\mathsf{O}_{\mathtt{Sch}}$ and $\mathsf{Spec}_{\mathtt{Sch}}$ define an equivalence of categories between AffVar_k and $\mathsf{CAlgAff}_k^{\mathsf{op}}$ (and $\mathsf{CAlgAffGen}_k^{\mathsf{op}}$). In other words, one could *define* the category AffVar_k as $\mathsf{CAlgAff}_k^{\mathsf{op}}$ and dispose of our definition above.

One may ask how we can define morphisms between varieties defined over *different* fields k and k'. Furthermore, how one can perform operations such products and disjoint unions on such varieties? The resulting objects would have to be more general than varieties. The following definition represents a fundamental breakthrough by Grothendieck. (The definition was already used in some form by Wolfgang Krull, but it was Grothendieck who developed the associated theory systematically.)

Definition 8.43. The category AffSch of *affine schemes* is defined as $\mathsf{CRing}^{\mathsf{op}}$.

Below we will see how one can define (nonaffine) schemes (which are in the same relation to affine schemes as varieties are to affine varieties) using a very powerful formalism known as the *functor of points*.

9 Monomorphisms and epimorphisms

Definition 9.1. A *monomorphism* in a category C is a morphism $f\colon X \to Y$ such that for any object W and any pair of morphisms $g_1, g_2\colon W \to X$ such that $fg_1 = fg_2$ we have $g_1 = g_2$.

Definition 9.2. An *epimorphism* in a category C is a morphism $f\colon X \to Y$ that is a monomorphism in the category C^{op}.

Remark 9.3. It is instructive to unfold the above definition: an epimorphism in a category C is a morphism $f\colon X \to Y$ such that for any object Z and any pair of morphisms $h_1, h_2\colon Y \to Z$ such that $h_1 f = h_2 f$ we have $h_1 = h_2$.

Example 9.4. In Set monomorphisms are injective functions (take $W = \{*\}$) and epimorphisms are surjective functions (take $Z = \{*, **\}$).

9.5. ***Algebra***

Example 9.6. In any category of algebraic objects, such as Group, Ab, Ring, Mod_k, Alg_k, etc., monomorphisms are injective homomorphisms. It suffices to take as W the free object on one generator and the remainder of the argument is identical to Set using the fact that morphisms out of such a free object are in bijection with the elements of target. Furthermore, all surjective homomorphisms are epimorphisms, however, some categories may have nonsurjective epimorphisms (see below).

Example 9.7. In Group all epimorphisms are surjective homomorphisms. This can be established most easily using Schreier's theorem: every subgroup $H \subset G$ equals $\{x \mid G \mid g(x) = h(x)\}$ for some homomorphisms $g, h\colon G \to G'$ (observe that $g = h$ if and only if $H = G$). Indeed, the image of any epimorphism $f\colon F \to G$ is a subgroup $H \subset G$, then Schreier's theorem supplies g and h such that $gf = hf$, hence $g = h$ by definition of an epimorphism, and therefore $H = G$, i.e., f is surjective.

Example 9.8. The same argument as for Group also shows that epimorphisms coincide with surjective maps in the categories Ab, Vect_k, and Mod_R. However, the analog of Schreier's theorem is trivial here: given an epimorphism $f\colon X \to Y$ we take $Z = Y/f(X)$, the map h_1 is the canonical quotient map $Y \to Y/f(X) = Z$, and h_2 is the zero map. (For nonabelian groups we can only make sense of $Y/f(X)$ as a set of cosets, not as a group. However, one can use the symmetric group of the set $* \sqcup Y/f(X)$ to a similar effect, as shown by Linderholm.)

Example 9.9. In Ring not all epimorphisms are surjective. For instance, $\mathbf{Z} \to \mathbf{Q}$ is a nonsurjective epimorphism: if two ring homomorphism $g, h \colon \mathbf{Q} \to R$ coincide on $\mathbf{Z}$, then for any integer p and $q \neq 0$ we have $g(p/q) = g(p)g(q)^{-1} = f(p)f(q)^{-1} = f(p/q)$. Here we used the fact that homomorphisms of rings preserve inverses of invertible elements, which we also used to show that $-^{\times} \colon \mathsf{Ring} \to \mathsf{Group}$ is a functor.

One can characterize epimorphisms of rings in more familiar terms. This is a nontrivial result due to Cohn, Isbell, Mazet, and Silver. A morphism of rings $f \colon A \to B$ is an epimorphism if and only if the *dominion* of the subring $f(A)$ in B coincides with B. Here the *dominion* of a subring $S \subset R$ is the set of all elements of R that can be represented as the product of matrices XPY (with coefficients in R), where X, P, and Y have size $1 \times m$, $m \times n$, and $n \times 1$ respectively, and P, XP, and PY have coefficients in S.

9.10. ***Functional analysis***

Example 9.11. In Ban and Ban_1 monomorphisms are injective maps (take $W = \mathbf{R}$ or $W = \mathbf{C}$). Epimorphisms are morphisms with dense image (take $Z = Y/\overline{f(X)}$ and then proceed as for Vect_k; for Ban_1 note that the quotient map is contractive).

9.12. ***General topology***

Example 9.13. In Top mononomorphisms are injective continuous maps (take $W = \{*\}$) and epimorphisms are surjective continuous maps (take $W = \{0, 1\}$ with the antidiscrete topology). In Haus (the full subcategory of Top on Hausdorff spaces) mononomorphisms are injective continuous maps (take $W = \{*\}$) and epimorphisms are continuous maps with dense image (two continuous functions with Hausdorff codomains that coincide on some subset must also coincide on its closure). In $\mathsf{CompHaus}$ (the full subcategory of Haus on compact spaces) monomorphisms are continuous injections for the same reason, whereas epimorphisms are (once again) continuous surjective maps (continuous maps with dense image between compact Hausdorff spaces are automatically surjective).

Example 9.14. In the category $\mathsf{TopGroup}$ of topological groups and continuous group homomorphisms monomorphisms are injections (take $W = \mathbf{Z}$) and epimorphisms are surjections (given an epimorphism $f \colon G \to H$, use Schreier's theorem to construct a homomorphism of discrete groups $H \to K$ whose kernel is precisely the image of f, and equip K with the antidiscrete topology so that $H \to K$ is continuos). In the full subcategory $\mathsf{HausGroup}$ monomorphisms are precisely injections (take $W = \mathbf{Z}$). Any morphism with a dense image is an epimorphism, and judging by what happens for categories Group and Haus one could be led to conjecture that all epimorphisms have dense image, but a counterexample was constructed by Uspenskij. This statement is true, however, for compact topological groups (a theorem of Poguntke) as well as Hausdorff abelian topological groups (the same argument as for Ab).

9.15. ***Measure theory***

Example 9.16. In the category Meas monomorphisms and epimorphisms can be most easily described in terms of the functor $\mathsf{MeasAlg} \colon \mathsf{Meas}^{\mathsf{op}} \to \mathsf{BoolAlg}$: they are precisely those morphisms that are mapped by $\mathsf{MeasAlg}$ to an epimorphism (i.e., surjection) respectively monomorphism (i.e., injection) of Boolean algebras. (The two classes of maps are exchanged because of op.) In more concrete terms, monomorphisms of measurable spaces are morphisms $(X, M, N) \to (X', M', N')$ such that any element of M is equivalent to the preimage of some element of M'. Likewise, epimorphisms are characterized by the property that any element $m' \in M'$ whose preimage belongs to N itself belongs to N', i.e., only negligible sets have negligible preimages.

10 Equivalences of categories

Previously we defined the category Cat of small categories and the "huge" category CAT of categories. Any category has a built-in notion of isomorphism. In particular, we can talk about isomorphisms of categories. These are functors $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ such that there is a functor $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$ and $\mathsf{G} \circ \mathsf{F} = \mathrm{id}_\mathsf{C}$, $\mathsf{F} \circ \mathsf{G} = \mathrm{id}_\mathsf{D}$.

This is a perfectly good definition except that it fails to exhibit many categories as equivalent even though we consider them to be the "same". This is entirely analogous to how "same" groups need not be *equal*, but only *isomorphic*. There are many groups (in fact, a proper class of groups) with one element, but only one isomorphism class of groups with one element.

In the above definition, $\mathsf{G} \circ \mathsf{F} = \mathrm{id}_\mathsf{C}$ means that for any object $X \in \mathsf{C}$ we have $\mathsf{G}(\mathsf{F}(X)) = X$. This is hardly ever true for any of the constructions that we considered above, e.g., $\mathsf{O}_{\mathsf{Ban}}(\mathsf{DUB}(X)) \neq X$ for a Banach space X, even though these two Banach spaces are isomorphic.

We can consider functors $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ for which there is a functor $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$ such that X is *isomorphic* (but not necessarily equal) to $\mathsf{G}(\mathsf{F}(X))$ for any object $X \in \mathsf{C}$ and Y is isomorphic to $\mathsf{F}(\mathsf{G}(Y))$ for any object $Y \in \mathsf{D}$. This can also be formulated by saying that the functions $\pi_0(\mathsf{F})\colon \pi_0(\mathsf{C}) \to \pi_0(\mathsf{D})$ and $\pi_0(\mathsf{G})\colon \pi_0(\mathsf{D}) \to \pi_0(\mathsf{C})$ are mutually inverse to each other.

This modified definition is far too expansive. For instance, recall that any group G gives rise to a category $\mathsf{B}G$ that has a single object $*$ whose endomorphisms are elements of G and composition is given by multiplication. The above definition makes $\mathsf{B}G$ and $\mathsf{B}H$ the "same" for any pair of groups G and H.

Even more so, consider any groupoid C and a discrete category $\pi_0\mathsf{C}$ (all morphisms are identity morphisms) on the set of isomorphism classes of C. We have a canonical functor $\mathsf{C} \to \pi_0\mathsf{C}$. We can also choose an inclusion $\pi_0\mathsf{C} \to \mathsf{C}$ that choose a representative for each equivalence class. The composition $\pi_0\mathsf{C} \to \mathsf{C} \to \pi_0\mathsf{C}$ is the identity functor $\pi_0\mathsf{C} \to \pi_0\mathsf{C}$. The other composition $\mathsf{C} \to \pi_0\mathsf{C} \to \mathsf{C}$ sends any object in C to an isomorphic object.

These examples show us what is wrong with out last attempt: we should take morphisms into account.

Definition 10.1. An *equivalence* of categories is a functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ for which there is a functor $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$ such that the induced maps $\mathsf{F}_{X,X'}\colon \mathsf{Mor}_\mathsf{C}(X, X') \to \mathsf{Mor}_\mathsf{C}(\mathsf{F}(X), \mathsf{F}(X'))$ for any objects $X, X' \in \mathsf{C}$ and $\mathsf{G}_{Y,Y'}\colon \mathsf{Mor}_\mathsf{D}(Y, Y') \to \mathsf{Mor}_\mathsf{D}(\mathsf{G}(Y), \mathsf{G}(Y'))$ for any objects $Y, Y' \in \mathsf{D}$ are isomorphisms and the functions $\pi_0(\mathsf{F})\colon \pi_0(\mathsf{C}) \to \pi_0(\mathsf{D})$ and $\pi_0(\mathsf{G})\colon \pi_0(\mathsf{D}) \to \pi_0(\mathsf{C})$ are mutually inverse to each other.

There is a simple practical criterion for equivalence.

Definition 10.2. Given a functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$, we have an induced function

$$\mathsf{F}_{X,X'}\colon \mathsf{Mor}_\mathsf{C}(X, X') \to \mathsf{Mor}_\mathsf{D}(\mathsf{F}(X), \mathsf{F}(X'))$$

for any pair of objects $X, X' \in \mathsf{C}$. We say that the functor F is

- *faithful* if $\mathsf{F}_{X,X'}$ is injective for any X, X';
- *full* if $\mathsf{F}_{X,X'}$ is surjective for any X, X';
- *fully faithful* if $\mathsf{F}_{X,X'}$ is bijective for any X, X', in which case we denote its inverse by $\mathsf{F}^{-1}_{X,X'}$.

Definition 10.3. An *essentially surjective functor* is a functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ such that the induced function $\pi_0(\mathsf{F})\colon \pi_0(\mathsf{C}) \to \pi_0(\mathsf{D})$ is surjective. (In other words, for any object $Y \in \mathsf{D}$ there is an object $X \in \mathsf{C}$ such that $\mathsf{F}(X)$ is isomorphic to Y.)

Proposition 10.4. A functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is an equivalence if and only if it is fully faithful and essentially surjective.

Proof. Necessity of these two properties follows immediately from the definition of an equivalence. To show sufficiency, we start by constructing the functor $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$. On objects, we invoke the axiom of choice and construct a function $\mathsf{Ob}(\mathsf{G})\colon \mathsf{Ob}(\mathsf{D}) \to \mathsf{Ob}(\mathsf{C})$ by choosing for every object $Y \in \mathsf{D}$ an object $\mathsf{G}(Y) \in \mathsf{C}$ such that there is an isomorpism $\epsilon_Y\colon \mathsf{F}(\mathsf{G}(Y)) \to Y$. We also record a choice of an isomorphism ϵ_Y.

Given a morphism $g\colon Y \to Y'$ in D we define $\mathsf{G}(g) = \mathsf{F}^{-1}_{\mathsf{G}(Y),\mathsf{G}(Y')}(\epsilon^{-1}_{Y'} \circ g \circ \epsilon_Y)$. Here $\epsilon_Y\colon \mathsf{F}(\mathsf{G}(Y)) \to Y$, $g\colon Y \to Y'$, and $\epsilon^{-1}_{Y'}\colon Y' \to \mathsf{F}(\mathsf{G}(Y'))$ compose together into a morphism $\mathsf{F}(\mathsf{G}(Y)) \to \mathsf{F}(\mathsf{G}(Y'))$. By the fully faithfulness of F the function $\mathsf{F}_{\mathsf{G}(Y),\mathsf{G}(Y')}\colon \mathsf{Mor}_\mathsf{C}(\mathsf{G}(Y), \mathsf{G}(Y')) \to \mathsf{Mor}_\mathsf{D}(\mathsf{F}(\mathsf{G}(Y)), \mathsf{F}(\mathsf{G}(Y')))$ is a bijection, so it has an inverse, which we used in the formula for $\mathsf{G}(g)$.

Given morphisms $g: Y \to Y'$ and $g': Y' \to Y''$, we immediately compute

$$\begin{aligned} \mathsf{G}(g') \circ \mathsf{G}(g) &= \mathsf{F}^{-1}_{\mathsf{G}(Y'),\mathsf{G}(Y'')}(\epsilon^{-1}_{Y''} \circ g' \circ \epsilon'_Y) \circ \mathsf{F}^{-1}_{\mathsf{G}(Y),\mathsf{G}(Y')}(\epsilon^{-1}_{Y'} \circ g \circ \epsilon_Y) \\ &= \mathsf{F}^{-1}_{\mathsf{G}(Y),\mathsf{G}(Y'')}(\epsilon^{-1}_{Y''} \circ g' \circ \epsilon'_Y \circ \epsilon^{-1}_{Y'} \circ g \circ \epsilon_Y) \\ &= \mathsf{F}^{-1}_{\mathsf{G}(Y),\mathsf{G}(Y'')}(\epsilon^{-1}_{Y''} \circ g' \circ g \circ \epsilon_Y) = \mathsf{G}(g' \circ g). \end{aligned}$$

(The second equality follows from the preservation of composition by F.) Likewise,

$$\mathsf{G}(\mathrm{id}_Y) = \mathsf{F}^{-1}(\epsilon_Y^{-1} \circ \mathrm{id}_Y \circ \epsilon_Y) = \mathsf{F}^{-1}(\mathrm{id}_{\mathsf{F}(\mathsf{G}(Y))}) = \mathrm{id}_{\mathsf{G}(Y)}.$$

Thus G is a functor.

Finally, the functions $\pi_0(\mathsf{F}): \pi_0(\mathsf{C}) \to \pi_0(\mathsf{D})$ and $\pi_0(\mathsf{G}): \pi_0(\mathsf{D}) \to \pi_0(\mathsf{C})$ are mutually inverse to each other once we show that $\pi_0(\mathsf{F})$ is injective, which by definition of G implies that $\pi_0(\mathsf{G})$ is its inverse. Thus we have to show that for any objects $X, X' \in \mathsf{C}$ such that $\mathsf{F}(X)$ and $\mathsf{F}(X')$ are isomorphic objects in D, the objects X and X' are themselves isomorphic in C. Indeed, if $g: \mathsf{F}(X) \to \mathsf{F}(X')$ is an isomorphism, then so is $\mathsf{F}^{-1}_{X,X'}(g): X \to X'$. ∎

10.5. *Functional analysis*

Example 10.6. The functors $\mathsf{DUB}: \mathsf{Ban}_1^{\mathsf{op}} \to \mathsf{CompBall}$ and $\mathsf{O}_{\mathsf{Ban}}: \mathsf{CompBall} \to \mathsf{Ban}_1^{\mathsf{op}}$ form an equivalence of categories. This is the *Hahn–Banach theorem.* Given $X \in \mathsf{Ban}_1$, the isomorphism $\mathsf{ev}: X \to \mathsf{O}_{\mathsf{Ban}}(\mathsf{DUB}(X))$ sends an element $x \in X$ to the linear function $\mathsf{ev}(x): X^* \to \mathbf{C}$ that sends an element $f \in X^*$ to $f(x) \in \mathbf{C}$, i.e., $\mathsf{ev}(x)(f) = f(x)$. (We have

$$\|\mathsf{ev}(x)\| = \sup_{f \in X^*_{\le 1}} |\mathsf{ev}(x)(f)| = \sup_{f \in X^*_{\le 1}} |f(x)| = \|x\|,$$

so ev is indeed a contractive map.) Given $Y \in \mathsf{CompBall}$, the isomorphism $\mathsf{ev}: Y \to \mathsf{DUB}(\mathsf{O}_{\mathsf{Ban}}(Y))$ is a morphism of compact balls $Y = (V, B) \to ((\mathsf{O}_{\mathsf{Ban}}(V,B))^*, (\mathsf{O}_{\mathsf{Ban}}(V,B))^*_{\le 1})$ that sends $v \in V$ to the continuous linear map $\mathsf{O}_{\mathsf{Ban}}(V,B) \to \mathbf{C}$ that sends an element $f \in \mathsf{O}_{\mathsf{Ban}}(V,B)$ to $f(v) \in \mathbf{C}$, i.e., $\mathsf{ev}(v)(f) = f(v)$. (The norm of $\mathsf{ev}(v)$ equals the norm of v because $|f(v)| \le \|f\| \cdot \|v\|$, so $\mathsf{ev}(v)$ is indeed a morphism of balls.)

Remark 10.7. The traditional formulation of the Hahn–Banach theorem states that any functional $B \to \mathbf{C}$ on a Banach subspace $B \subset A$ can be extended to a functional $A \to \mathbf{C}$ with the same norm. We can deduce it from the above stronger version as follows. The inclusion $B \subset A$ is a monomorphism in Ban_1, equivalently, an epimorphism in $\mathsf{Ban}_1^{\mathsf{op}}$. The above equivalence sends it to an epimorphism in $\mathsf{CompBall}$. The forgetful functor $\mathsf{CompBall} \to \mathsf{CompHaus}$ preserves epimorphisms. The resulting epimorphism in $\mathsf{CompHaus}$ is a map that sends a contractive functional $A \to \mathbf{C}$ to its restriction $B \to \mathbf{C}$. Epimorphisms of compact Hausdorff spaces are precisely surjective maps, which in our case means that any contractive functional $B \to \mathbf{C}$ extends to a contractive functional $A \to \mathbf{C}$, which immediately implies the original statement.

Example 10.8. The functors $\mathsf{Spec}_{\mathsf{CC}}: (\mathsf{CC}^*)^{\mathsf{op}} \to \mathsf{CompHaus}$ and $\mathsf{O}_{\mathsf{CC}}: \mathsf{CompHaus}^{\mathsf{op}} \to \mathsf{CC}^*$ form an equivalence of categories. This is the *Gelfand duality theorem for commutative C*-algebras.* We describe the involved isomorphisms. Given $X \in \mathsf{CompHaus}$, the isomorphism $X \to \mathsf{Spec}_{\mathsf{CC}}(\mathsf{O}_{\mathsf{CC}}(X))$ sends a point $x \in X$ to the morphism $\mathsf{O}_{\mathsf{CC}}(X) \to \mathbf{C}$ that evaluates on X. Given $A \in \mathsf{CC}^*$, the isomorphism $A \to \mathsf{O}_{\mathsf{CC}}(\mathsf{Spec}_{\mathsf{CC}}(A))$ sends an element $a \in A$ to the evaluation map $\mathsf{Spec}_{\mathsf{CC}}(A) \to \mathbf{C}$ that sends an element $f: A \to \mathbf{C}$ of $\mathsf{Spec}_{\mathsf{CC}}(A)$ to $f(a) \in \mathbf{C}$.

Example 10.9. The functor $\mathsf{PD}: \mathsf{LocCompHausAb}^{\mathsf{op}} \to \mathsf{LocCompHausAb}$ constructed in Example 8.14 is an equivalence of categories. This is the *Pontryagin duality for locally compact topological groups.* Its inverse is the *same* functor, now regarded as a functor $\mathsf{LocCompHausAb} \to \mathsf{LocCompHausAb}^{\mathsf{op}}$. Both relevant isomorphisms are also the same: the map $G \to \mathsf{PD}(\mathsf{PD}(G))$ sends $g \in G$ to the continuous homomorphism $\mathsf{PD}(G) \to \mathrm{U}(1)$ that maps $f: G \to \mathrm{U}(1)$ to $f(g) \in \mathrm{U}(1)$.

Example 10.10. The functors $\mathsf{Spec}_{\mathsf{Meas}}: (\mathsf{CW}^*)^{\mathsf{op}} \to \mathsf{LocMeas}$ and $\mathsf{O}_{\mathsf{Meas}}: \mathsf{LocMeas}^{\mathsf{op}} \to \mathsf{CW}^*$ form an equivalence of categories. This is result is due to von Neumann. Given $A \in \mathsf{CW}^*$, the isomorphism $A \to \mathsf{O}_{\mathsf{Meas}}(\mathsf{Spec}_{\mathsf{Meas}}(A))$ is defined by sending an element $a \in A$ to the equivalence class of a bounded measurable function $\mathsf{Spec}_{\mathsf{Meas}}(A) \to \mathbf{C}$ that sends a point $p \in \mathsf{Spec}_{\mathsf{Meas}}(A)$ (i.e., a morphism of C*-algebras $A \to \mathbf{C}$) to $p(a)$. Given $X \in \mathsf{LocMeas}$, the isomorphism $X \to \mathsf{Spec}_{\mathsf{Meas}}(\mathsf{O}_{\mathsf{Meas}}(X))$ is defined as follows.

11 Future topics

Hom is a right adjoint to cartesian product.
The topology on compactly supported functions is a filtered colimit.
Fourier transformation as an isomorphism of functors.