Submit your solutions in writing no later than May 20 or present them orally during my office hours (MWF 3–4 in 19D).

1. Suppose A is a locally compact Hausdorff abelian group (typical cases include $A = \mathbf{R}$, $A = \mathbf{Z}$, and $A = \mathrm{U}(1)$). Recall (Example 7.18) the Pontryagin dual group $\mathsf{PD}(A) = \mathsf{Hom}(A, \mathrm{U}(1))$. Recall also the two functors of the form $\mathsf{LocCompHausAb} \to \mathsf{Ban_1}$, namely, $\mathsf{L^1} \circ \mathsf{HaarMeas}$ and $\mathsf{C_0} \circ \mathsf{PD}$. Here $\mathsf{L^1}(\mathsf{HaarMeas}(A))$ (henceforth $\mathsf{L^1}(A)$) is the Banach space of finite measures on the measurable space $\mathsf{HaarMeas}(A)$ and $\mathsf{C_0}(\mathsf{PD}(A))$ is the Banach space of $\mathbf{C}$-valued continuous functions on the topological group $\mathsf{PD}(A)$ that vanish at infinity. The *Fourier transform* on A is defined as a bounded map $F_A\colon \mathsf{L^1}(A) \to \mathsf{C_0}(\mathsf{PD}(A))$ that sends $\mu \in \mathsf{L^1}(A)$ to the function $\chi \mapsto \int \chi \cdot \mu$, where $\chi \in \mathsf{PD}(A)$, i.e., $\chi\colon A \to \mathrm{U}(1)$, and $\chi \cdot \mu$ denotes the product of a measurable function and a finite measure. Prove that the collection of maps F_A defined above is a natural transformation from $\mathsf{L^1} \circ \mathsf{HaarMeas}$ to $\mathsf{C_0} \circ \mathsf{PD}$.

2. Prove that the category $\mathsf{Ban_1}$ admits arbitrary small coproducts. Prove that the category Ban admits finite coproducts, but does not admit infinite coproducts.

3. Suppose V and W are two Banach spaces in the category $\mathsf{Ban_1}$ of Banach spaces and contractive linear maps. A *contractive bilinear map* is a bilinear map $b\colon V, W \to A$ such that $\|b(v, w)\| \le 1$ for all $v \in V$ and $w \in W$ such that $\|v\| \le 1$ and $\|w\| \le 1$. Prove the existence of a Banach space $V \hat{\otimes}_{\mathbf{C}} W$ with the following universal property: contractive bilinear maps $V, W \to A$ are in natural bijection with contractive linear maps $V \hat{\otimes}_{\mathbf{C}} W \to A$. Prove that the span of elements of the form $v \otimes_{\mathbf{C}} w$ is dense in $V \hat{\otimes}_{\mathbf{C}} W$. Does it always coincide with $V \hat{\otimes}_{\mathbf{C}} W$?

4. Suppose S is a set and $P \subset S \times S$ is a set of pairs of elements in S. Prove that the quotient set $S/\sim$, where $\sim$ is the equivalence relation generated by P, is isomorphic to the coequalizer of $P \underset{p_1}{\overset{p_0}{\rightrightarrows}} S$, where $p_0, p_1\colon P \to S$ are the projection maps.

5. Denote by $\mathsf{U}\colon \mathsf{CAlg}_k \to \mathsf{Vect}_k$ the forgetful functor from the category of commutative k-algebras to the category of vector spaces over k. Fix a vector space $V \in \mathsf{Vect}_k$. Consider the functor $\mathsf{F}\colon \mathsf{CAlg}_k \to \mathsf{Set}$ such that $\mathsf{F}(A)$ is the set of linear maps $V \to \mathsf{U}(A)$ and for a homomorphism $f\colon A \to A'$ the map $F(f)$ sends $g\colon V \to \mathsf{U}(A)$ to $f \circ g\colon V \to \mathsf{U}(A')$. Prove that F is corepresentable. Identify its corepresenting object as a familiar object from your algebra course.

6. Fix some vector spaces $V, W \in \mathsf{Vect}_k$. Consider the functor $\mathsf{F}\colon \mathsf{Vect}_k^{\mathsf{op}} \to \mathsf{Set}$ such that $\mathsf{F}(A)$ is the set of linear maps $A \otimes_k V \to W$ and $F(f)$ for a linear map $f\colon A \to A'$ sends a map $A' \otimes_k V \to W$ to its composition with $A \otimes_k V \to A' \otimes_k V$. Prove that F is representable. Identify its representing object as a familiar object from your algebra course. Same question when Vect_k is replaced by $\mathsf{Ban_1}$ and $\otimes_k$ is replaced by $\hat{\otimes}_{\mathbf{C}}$.