**Option A:** Submit your solutions in writing not much later than February 16.

**Option B:** Present your solutions orally during my office hours (MWF 3–4 in 19D) not much later than February 16.

**What to do when stuck on a problem:**

- Solve another problem first;
- Ask your classmate;
- Ask me during my office hours (bring all your work).

**1.** Which sets of data given below define categories? Warning: some sets of data are incomplete, e.g., do not specify composition. You must reconstruct all missing data (this is a creative process and can have more than one answer). Some items below have negative answers. Some are purposefully nonsensical.

- Topological spaces and proper maps. (A map $f\colon X \to Y$ is *proper* if it is continuous and for any compact subset $T \subset Y$ the subset $f^{-1}(T) \subset X$ is also compact.)
- Sets and relations. (More precisely, given two sets $X$ and $Y$, morphisms $X \to Y$ are *relations* from $X$ to $Y$, i.e., subsets of $X \times Y$. Two relations $R \subset X \times Y$ and $S \subset Y \times Z$ are composed as follows: $R \circ S = \{(x, z) \mid \exists y \in Y\colon (x, y) \in R \land (y, z) \in Z\}$.
- Sets and surjective functions.
- Sets and partially defined functions. (A partially defined function $X \to Y$ is a function $A \to Y$, where $A \subset X$. If $x \in X$, we say that $f$ is *defined* on $x$ (or: $f(x)$ is defined) if $x \in A$. Partially defined functions are composed as follows: if $f\colon X \to Y$ and $g\colon Y \to Z$ are partially defined function, then the composition $gf$ is defined on $x \in X$ if $f$ is defined on $x$ and $g$ is defined on $f(x)$, in which case $(gf)(x) := g(f(x))$.)
- Fix a topological space $X$ and define a category as follows. Objects are continuous functions with codomain $X$, i.e., $f\colon Y \to X$ ($Y$ is arbitrary). (Here *morphisms* in $\mathsf{Top}$ play the role of *objects* in the category that we are constructing.) Morphisms from an object $f\colon Y \to X$ to an object $f'\colon Y' \to X$ are continuous maps $g\colon Y \to Y'$ such that $f'g = f$.
- The category $\mathsf{Open}$. Objects are open subsets of $\mathbf{R}^n$, where $n$ is arbitrary (not fixed). Morphisms $U \to V$ are infinitely differentiable maps $U \to V$.
- The category $\mathsf{Open}_*$. Objects are pairs $(U, x)$, where $U \subset \mathbf{R}^n$ is an open subset and $x \in U$. Morphisms $(U, x) \to (V, y)$ are infinitely differentiable maps $U \to V$ that map $x$ to $y$.
- $\mathsf{Mat}_\mathbf{R}$: objects are natural numbers $n \geq 0$ and morphisms $m \to n$ are matrices of size $n \times m$. Composition is multiplication of matrices.
- $\mathsf{B}\mathbf{R}$: there is only one object $*$. Morphisms $* \to *$ are real numbers. Composition of morphisms is given by multiplication of real numbers.
- There is only one object $*$. Morphisms $* \to *$ are compactly supported continuous functions $\mathbf{R} \to \mathbf{R}$. Composition of morphisms is given by multiplication of functions.
- $\mathsf{Poset}$: objects are partially ordered sets (i.e., a set $X$ with a relation $R$ that is reflexive ($x \leq x$), transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$), and antisymmetric ($x \leq y$ and $y \leq x$ implies $x = y$). Morphisms are functions that preserve the order: if $x \leq y$, then $f(x) \leq f(y)$.

**2.** Which sets of data below define functors? (Same warning as above.)

- $\mathsf{Open}_* \to \mathsf{Vect}_\mathbf{R}$. Send $U \subset \mathbf{R}^m$ to $\mathbf{R}^m$. Send $f\colon (U, x) \to (V, y)$ to the linear map $\mathbf{R}^m \to \mathbf{R}^n$ given by the Jacobian matrix of $f$ at $x$, i.e., the entry in $i$th row and $j$th column is the value of the $i$th partial derivative of the $j$th coordinate of $f$ at point $x$. In symbols: $a_{i,j} = \frac{\partial f_j}{\partial x_i}(x)$. (The $j$th coordinate of $f$ is the composition $U \to V \subset \mathbf{R}^n \to \mathbf{R}$, where $\mathbf{R}^n \to \mathbf{R}$ is the projection to the $j$th component.)
- $\mathsf{Mat}_\mathbf{R} \to \mathsf{B}\mathbf{R}$: send any object $n \geq 0$ of $\mathsf{Mat}_\mathbf{R}$ to the only object of $\mathsf{B}\mathbf{R}$. Send a matrix of size $m \times n$ to its determinant (a morphism in $\mathsf{B}\mathbf{R}$) if $m = n$. Otherwise send it to zero.
- $\mathsf{OpenSet}\colon \mathsf{Top} \to \mathsf{Poset}$: send any topological space $X$ to the poset $\mathsf{OpenSet}(X)$ whose elements are open subsets of $X$ and the ordering is given by inclusion. Send any continuous map $f\colon X \to Y$ of topological spaces to the map of posets $g\colon \mathsf{OpenSet}(X) \to \mathsf{OpenSet}(Y)$ defined as follows: $g(U) = \bigcup_{V \subset f(U)} V$, where $V$ runs over open subsets of $Y$.

**3.** Define a functor $\mathsf{Mat}_\mathbf{R}^{\mathsf{op}} \to \mathsf{Mat}_\mathbf{R}$. Define a functor $\mathsf{Mat}_\mathbf{R} \to \mathsf{Vect}_\mathbf{R}$. Define a functor $\mathsf{B}\mathbf{R} \to \mathsf{Mat}_\mathbf{R}$. Define a functor $\mathsf{Mat}_\mathbf{R} \to \mathsf{Open}_*$. Define a functor $\mathsf{Top}^{\mathsf{op}} \to \mathsf{Poset}$.

**4.** Fix a category $\mathsf{C}$. A *section* of a morphism $f\colon X \to Y$ in $\mathsf{C}$ is a morphism $g\colon Y \to X$ such that $fg = \mathrm{id}_Y$. Give an example of a category $\mathsf{C}$ such that all epimorphisms have sections. Give an example of a category $\mathsf{C}$

and an epimorphism $f$ in $\mathsf{C}$ that does not have a section. Hint: it suffices to use the examples that we studied in class.

**5.** Fix a category $\mathsf{C}$. A *bimorphism* in $\mathsf{C}$ is a morphism $f$ that is simultaneously a monomorphism and an epimorphism. Is any isomorphism a bimorphism? Give an example of a category $\mathsf{C}$ and a bimorphism $f$ in $\mathsf{C}$ that is not an isomorphism.

**6.** Construct two functors $D\colon \mathsf{Set} \to \mathsf{Set}$ and $I\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$ such that for any set $X$ we have $D(X) = I(X) = 2^X$, where $2^X$ denotes the set of all subsets of $X$. (In other words, you must define $D$ and $I$ on morphisms and prove that composition and identity maps are respected.)

**7.** Construct a functor $\mathsf{L}^1\colon \mathsf{Set} \to \mathsf{Ban}_1$ such that for any set $S$ the Banach space $\mathsf{L}^1(S)$ is the space of functions $f\colon S \to \mathbf{R}$ such that the sum $\sum_{s\in S} f(s)$ exists (and is finite). Construct a functor $\mathsf{L}^\infty\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Ban}_1$ such that $\mathsf{L}^\infty(S)$ is the space of all bounded functions $S \to \mathbf{R}$.

**8.** Show that the class of epimorphisms in the category of Hausdorff topological spaces coincides with the class of continuous maps whose image is dense.

**9.** Describe concretely all monomorphisms and epimorphisms in $\mathsf{B}\mathbf{R}$. Same question for $\mathsf{Open}$ and $\mathsf{Open}_*$. Same question for $\mathsf{Poset}$.