Submit your solutions in writing not much later than March 19 or present them orally during my office hours (MWF 3–4 in 19D).

1. Which of the following assignments of a set $F(U)$ to an open subset $U \subset X$ of a topological space X defines a sheaf on X? (As usual you must fill in all the missing data, e.g., the restriction maps are not specified explicitly if they are obvious.) Below U denotes an arbitrary open subset of X.

- $X = \mathbf{R}$, $F(U)$ is the set of continuous functions $U \to \mathbf{R}$.
- $X = \mathbf{R}$, $F(U)$ is the set of bounded continuous functions $U \to \mathbf{R}$.
- $X = \mathbf{C}$, $F(U)$ is the set of holomorphic functions $U \to \mathbf{C}$.
- $X = \mathbf{R}$, $F(U)$ is the set of Borel measurable functions $U \to \mathbf{R}$.
- $X = \mathbf{R}$, $F(U)$ is the set of constant functions $U \to \mathbf{R}$.
- $X = \mathbf{R}$, $F(U)$ is the set of locally constant functions $U \to \mathbf{R}$.
- $X = \mathbf{R}$, $F(U)$ is the set of increasing functions $U \to \mathbf{R}$ ($x \le y$ implies $f(x) \le f(y)$).
- $X = \mathbf{R}$, $F(U)$ is the set of integrable measurable functions $U \to \mathbf{R}$ (with finite integral).
- $X = \mathbf{R}$, $F(U)$ is the set of convex functions $U \to \mathbf{R}$ in the following sense: for any $x, y \in U$ such that $x \le y$ and $[x, y] \subset U$ and for any $0 \le t \le 1$ we have $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

2. Recall that a *base* B of a topological space X is a set of open subsets of X such that any open subset of X can be represented as a union of sets in the base. Below we consider B to be a category whose objects are elements of B and morphisms are inclusions (i.e., the set of morphisms between any two objects is either empty or consists of a single element). A *sheaf on a topological space X with respect to a base B of X* (or simply a *sheaf on B*) is a functor $B^{\mathsf{op}} \to \mathsf{Set}$ that satisfies the gluing property, which is formulated in exactly the same way as for sheaves on X, but using elements of B everywhere instead of arbitrary open sets. Morphisms of sheaves on B are natural transformations of functors.

- Prove that any sheaf on X restricts to a sheaf of B.
- Prove that any sheaf on X can be reconstructed (up to an isomorphism) from its restriction to B.
- Prove that the category of sheaves on X is equivalent to the category of sheaves on B.

3. A *idempotent ring* is a ring R such that $x^2 = x$ for any $x \in R$. (Rings are assumed to be associative and unital, homomorphisms of rings preserve units.)

- Show that any idempotent ring is commutative: $xy = yx$ for all x and y.
- Show that the relation $x \le y := (x = xy)$ defines a partial order on R.
- Show that given a set X, equipping 2^X (the set of subsets of X) with the following operations: $0 := \emptyset$, $x + y := (x \setminus y) \cup (y \setminus x)$, $-x := X \setminus x$, $1 := X$, $xy := x \cap y$ produces an idempotent ring.
- Recall that the supremum of a subset $A \subset R$, if it exists, is the unique element $s \in R$ such that for all $a \in A$ we have $a \le s$ and if s' is another element with the same property, then $s \le s'$. Show that in the idempotent ring 2^X every subset has a supremum.
- An *atom* in an idempotent ring is an element $a \in R$ such that $a \ne 0$ and if $0 \le b \le a$ for some $b \in R$, then $b = 0$ or $b = a$. Show that in the idempotent ring 2^X every element can be represented as the supremum of a set of atoms.
- Show that the assignment $X \mapsto 2^X$ can be extended to a contravariant functor $2^{(-)}$ from the category of sets to the category whose objects are idempotent rings in which every subset has a supremum and every element is the supremum of a set of atoms, and morphisms are homomorphisms of rings that preserve suprema ($f : R \to R'$ preserves suprema if for any $S \subset R$ we have $\sup f(S) = f(\sup S)$).
- Construct a contravariant functor going in the opposite direction. (Hint: it is useful to keep the example of the idempotent ring 2^X when constructing this functor.)
- Prove that the two functors together form an equivalence of categories.

4. Recall that topological space X is *connected* if it is nonempty and the only subsets of X that are both closed and open are $\emptyset$ and X. A topological space is *locally connected* if it admits a base consisting of connected open subsets. The purpose of this exercise is to demonstrate an alternative theory of the fundamental group, which is applicable to all locally connected topological spaces, as opposed to the much more narrow class of locally path connected semilocally simply connected topological spaces.

A *connected base* for a locally connected topological space X is a base B such that for any $P \in B$ the space P is a connected open subset of X and if $Q \subset P$ is a connected open subset of P, then $Q \in B$. Given a connected base B, we define a category as follows. Objects are elements of B. Morphisms $X \to Y$ (for some $X, Y \in B$) are finite sequences $C_0, C_1, \ldots, C_n \in B$ of elements of B, where $X = C_0$, $Y = C_n$, $n \geq 0$ is arbitrary, and $C_i \cap C_{i+1}$ is connected for any $0 \leq i < n$ (recall that C_i are also connected by definition of a connected base). The composition of $B_0, \ldots, B_m$ (a morphism $B_0 \to B_m$) and $C_0, \ldots, C_n$ (a morphism $C_0 \to C_n$) is the concatenation with the middle duplicate element removed: $B_0, \ldots, B_m = C_0, C_1, \ldots, C_n$ (recall that we must have $B_m = C_0$ for the composition to be defined). For $n = 0$ we get the identity morphism on $X = Y$.

- Prove that the above data defines a category Chain_B. Prove that any morphism in Chain_B can be presented as a composition of chains with $n = 1$.

Consider the equivalence relation R on morphisms $X \to Y$ of Chain_B generated by the following two types of equivalences. First, we postulate that a sequence $X = B_0, \ldots, B_n = Y$ is equivalent to the sequence $X = C_0, \ldots, C_n = Y$ if for all i we have $B_i \subset C_i$. Secondly, we postulate that any sequence $B_0, \ldots, B_n$ is equivalent to the same sequence, but with the element B_i for some i repeated twice: $B_0, \ldots, B_i, B_i, \ldots, B_n$.

- Prove that this relation defines a congruence on Chain_B (see Definition 4.28 in the lecture notes) and therefore we have the quotient category $\Pi_B := \mathsf{Chain}_B/R$.
- Prove that nonidentity morphisms in Chain_B are noninvertible. Prove that every morphism in Π_B is invertible, i.e., Π_B is a groupoid.

As it turns out, fixing some connected base B is not enough because the answer may turn out to be trivial.

- Prove that if $X \in B$ (in particular, X is connected), then Π_B is equivalent to the trivial group (a single object with a single identity morphism).

We resolve this defect by allowing arbitrary refinements of bases. We define a category $\mathsf{ConBase}_X$ whose objects are connected bases of X as defined above, and morphism $B_1 \to B_2$ between two bases B_1 and B_2 are functions $b: B_1 \to B_2$ such that for any $P \in B_1$ we have $P \subset b(P)$ (here both P and $b(P)$ are some connected open subsets of X).

- Given a morphism $b: B_1 \to B_2$, define a functor $\Pi_b: \Pi_{B_1} \to \Pi_{B_2}$.
- Prove that the previous construction extends to a functor $\Pi: \mathsf{ConBase} \to \mathsf{Cat}$ (recall that Cat is the category of small categories).

We now define a new category Π_X that does not depend on the choice of B. Its objects p are given by the following data: for every base B we specify an object $p(B)$ of Π_B (i.e., an element of B). This data must satisfy the following condition: for any morphism $b: B_1 \to B_2$ of bases we have $\Pi_b(p(B_1)) = p(B_2)$. Its morphisms $f: p \to q$ are given by the following data: for every base B we specify a morphism $f(B): p(B) \to q(B)$ in Π_B. This data must satisfy the following condition: for any morphism $b: B_1 \to B_2$ of bases we have $\Pi_b(f(B_1)) = f(B_2)$.

- Prove that the above construction defines a category Π_X.

It may not be obvious why such a category should at all be related to $\pi_{\leq 1}(X)$ defined in the traditional way using homotopy equivalence classes of paths.

- Prove that if X is locally compact, then any object of Π_X canonically defines a point in X. Show that any point in X arises from some object of Π_X (not necessarily unique). Show that any path in X arises from some morphism in Π_X (not necessarily unique).
- Look up the Hahn–Mazurkiewicz theorem and prove that any morphism in Π_X defines a (noncanonical) path in X. Assume X to be second countable and locally compact.

This construction allows one to develop the theory of covering spaces for locally connected spaces, a much bigger class than locally path-connected semilocally simply connected spaces. Such a generality is necessary for Grothendieck's Galois theory. We will return to this later in the course.

5. Prove that any continuous function defined on a closed subset of a compact Hausdorff space can be extended to a continuous function defined on the entire space. (This is meant to be proved using what we have learned about various equivalences of categories in functional analysis, not by referencing the Urysohn lemma or the Tietze extension theorem.)

6. Recall that any Hilbert space H admits an isomorphism $H \cong H^*$, which equips H with a weak-* topology (alias weak topology). Suppose we are given an operator $P: H \to H$ that becomes continuous if we equip its source with the weak topology and its target with the norm topology. Prove that H decomposes as a direct sum of eigenspaces of P.