Option A: Submit your solutions in writing not much later than February 16.

Option B: Present your solutions orally during my office hours (MWF 3–4 in 19D) not much later than February 16.

What to do when stuck on a problem:

- Solve another problem first;
- Ask your classmate;
- Ask me during my office hours (bring all your work).

1. Which sets of data given below define categories? Warning: some sets of data are incomplete, e.g., do not specify composition. You must reconstruct all missing data (this is a creative process and can have more than one answer). Some items below have negative answers. Some are purposefully nonsensical.

- Topological spaces and proper maps. (A map $f\colon X \to Y$ is *proper* if it is continuous and for any compact subset $T \subset Y$ the subset $f^{-1}(T) \subset X$ is also compact.)
- Sets and relations. (More precisely, given two sets X and Y, morphisms $X \to Y$ are *relations* from X to Y, i.e., subsets of $X \times Y$. Two relations $R \subset X \times Y$ and $S \subset Y \times Z$ are composed as follows: $R \circ S = \{(x,z) \mid \exists y \in Y\colon (x,y) \in R \wedge (y,z) \in Z\}$.
- Sets and surjective functions.
- Sets and partially defined functions. (A partially defined function $X \to Y$ is a function $A \to Y$, where $A \subset X$. If $x \in X$, we say that f is *defined* on x (or: $f(x)$ is defined) if $x \in A$. Partially defined functions are composed as follows: if $f\colon X \to Y$ and $g\colon Y \to Z$ are partially defined function, then the composition gf is defined on $x \in X$ if f is defined on x and g is defined on $f(x)$, in which case $(gf)(x){:}= g(f(x))$.)
- Fix a topological space X and define a category as follows. Objects are continuous functions with codomain X, i.e., $f : Y \to X$ (Y is arbitrary). (Here *morphisms* in Top play the role of *objects* in the category that we are constructing.) Morphisms from an object $f : Y \to X$ to an object $f' : Y' \to X$ are continuous maps $g : Y \to Y'$ such that $f'g = f$.
- The category Open. Objects are open subsets of $\mathbf{R}^n$, where n is arbitrary (not fixed). Morphisms $U \to V$ are infinitely differentiable maps $U \to V$.
- The category Open_*. Objects are pairs (U,x), where $U \subset \mathbf{R}^n$ is an open subset and $x \in U$. Morphisms $(U,x) \to (V,y)$ are infinitely differentiable maps $U \to V$ that map x to y.
- $\mathsf{Mat}_{\mathbf{R}}$: objects are natural numbers $n \geq 0$ and morphisms $m \to n$ are matrices of size $n \times m$. Composition is multiplication of matrices.
- $\mathsf{B}\mathbf{R}$: there is only one object $*$. Morphisms $* \to *$ are real numbers. Composition of morphisms is given by multiplication of real numbers.
- There is only one object $*$. Morphisms $* \to *$ are compactly supported continuous functions $\mathbf{R} \to \mathbf{R}$. Composition of morphisms is given by multiplication of functions.
- Poset: objects are partially ordered sets (i.e., a set X with a relation R that is reflexive ($x \leq x$), transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$), and antisymmetric ($x \leq y$ and $y \leq x$ implies $x = y$). Morphisms are functions that preserve the order: if $x \leq y$, then $f(x) \leq f(y)$.

2. Which sets of data below define functors? (Same warning as above.)

- $\mathsf{Open}_* \to \mathsf{Vect}_{\mathbf{R}}$. Send $U \subset \mathbf{R}^m$ to $\mathbf{R}^m$. Send $f\colon (U,x) \to (V,y)$ to the linear map $\mathbf{R}^m \to \mathbf{R}^n$ given by the Jacobian matrix of f at x, i.e., the entry in ith row and jth column is the value of the ith partial derivative of the jth coordinate of f at point x. In symbols: $a_{i,j} = \frac{\partial f_j}{\partial x_i}(x)$. (The jth coordinate of f is the composition $U \to V \subset \mathbf{R}^n \to \mathbf{R}$, where $\mathbf{R}^n \to \mathbf{R}$ is the projection to the jth component.)
- $\mathsf{Mat}_{\mathbf{R}} \to \mathsf{B}\mathbf{R}$: send any object $n \geq 0$ of $\mathsf{Mat}_{\mathbf{R}}$ to the only object of $\mathsf{B}\mathbf{R}$. Send a matrix of size $m \times n$ to its determinant (a morphism in $\mathsf{B}\mathbf{R}$) if $m = n$. Otherwise send it to zero.
- $\mathsf{OpenSet}\colon \mathsf{Top} \to \mathsf{Poset}$: send any topological space X to the poset $\mathsf{OpenSet}(X)$ whose elements are open subsets of X and the ordering is given by inclusion. Send any continuous map $f\colon X \to Y$ of topological spaces to the map of posets $g\colon \mathsf{OpenSet}(X) \to \mathsf{OpenSet}(Y)$ defined as follows: $g(U) = \bigcup_{V \subset f(U)} V$, where V runs over open subsets of Y.

3. Define a functor $\mathsf{Mat}_{\mathbf{R}}^{\mathsf{op}} \to \mathsf{Mat}_{\mathbf{R}}$. Define a functor $\mathsf{Mat}_{\mathbf{R}} \to \mathsf{Vect}_{\mathbf{R}}$. Define a functor $\mathsf{B}\mathbf{R} \to \mathsf{Mat}_{\mathbf{R}}$. Define a functor $\mathsf{Mat}_{\mathbf{R}} \to \mathsf{Open}_*$. Define a functor $\mathsf{Top}^{\mathsf{op}} \to \mathsf{Poset}$.

4. Fix a category C. A *section* of a morphism $f\colon X \to Y$ in C is a morphism $g\colon Y \to X$ such that $fg = \mathrm{id}_Y$. Give an example of a category C such that all epimorphisms have sections. Give an example of a category C

and an epimorphism f in C that does not have a section. Hint: it suffices to use the examples that we studied in class.

5. Fix a category C. A *bimorphism* in C is a morphism f that is simultaneously a monomorphism and an epimorphism. Is any isomorphism a bimorphism? Give an example of a category C and a bimorphism f in C that is not an isomorphism.

6. Construct two functors $D\colon \mathsf{Set} \to \mathsf{Set}$ and $I\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$ such that for any set X we have $D(X) = I(X) = 2^X$, where 2^X denotes the set of all subsets of X. (In other words, you must define D and I on morphisms and prove that composition and identity maps are respected.)

7. Construct a functor $\mathsf{L}^1\colon \mathsf{Set} \to \mathsf{Ban}_1$ such that for any set S the Banach space $\mathsf{L}^1(S)$ is the space of functions $f : S \to \mathbf{R}$ such that the sum $\sum_{s\in S} f(s)$ exists (and is finite). Construct a functor $\mathsf{L}^\infty\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Ban}_1$ such that $\mathsf{L}^\infty(S)$ is the space of all bounded functions $S \to \mathbf{R}$.

8. Show that the class of epimorphisms in the category of Hausdorff topological spaces coincides with the class of continuous maps whose image is dense.

9. Describe concretely all monomorphisms and epimorphisms in $\mathsf{B}\mathbf{R}$. Same question for Open and Open_*. Same question for Poset.