Category Theory

Def^{n}

A category $\mathcal{C}$ is given by the following data

- a class $ob(\mathcal{C})$ of objects
- a collection of morphisms $Mor_{\mathcal{C}}(X, Y)$

  For any $X, Y \in ob(\mathcal{C})$ we have a set of morphisms from X to Y, but instead of $f \in Mor_{\mathcal{C}}(X, Y)$ we write $f: X \to Y$ or $X \xrightarrow{f} Y$.

  domain object — codomain object.

- Eeach object has a designated identity morphism.

  $1_X : X \to X$

- For any pair of morphisms f, g with the codomain of f equal to the domain of g, there exists a specified "composite morphism" gf whose domain is the domain of f and codomain is the codomain of g.

  i.e $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z \rightsquigarrow X \xrightarrow{gf} Z$

such that

(i) Associative

$X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$, $Z \xrightarrow{h} W$

$h(gf) = (hg)f = hgf$

(ii) Unital

$X \xrightarrow{f} Y$, then $f \cdot 1_X = f$

$1_Y \cdot f = f$

$\mathcal{C} = (Ob, Mor(-,-), \circ, id)$

Examples

① Set category

$Ob(Set)$: class of all sets

$Mor(X,Y)$: functions $X \longrightarrow Y$

$\circ$: composition of functions

1_X : identity function of X

② BG (group or a monoid)

$Ob(BG) : \{G\} = m$

$Mor(X,Y)$: $m \times m \longrightarrow m$

$1 \in m$ $\quad h(gf) = (hg)f$

$1 \cdot f = f \cdot 1 = f$

A one point category is precisely a monoid.

(The set of morphisms of) (a group without inverses).

Def^2

A full subcategory of a category $\mathcal{C}$ on a class of objects $X \subset \mathcal{C}$ is a category that has X as its class of objects, whereas the set of morphisms as well as identities and composition are inherited from $\mathcal{C}$.

objects of a category ⟷ bijective correspondance identity morphisms

Example

(i) the category FinSet of finite sets is the full subcategory of Set on the class of finite sets.

(ii) Set_* has sets with specific base point as objects and basepoint preserving functions as morphisms.

We now give more examples of categories from various areas of mathematics.

(3) Group

the category of Group has groups as objects and $Mor(X, Y)$ is the set of group homomorphisms from $X \to Y$. Composition is given by the composition of group homomorphisms (which is again a gp homo^{m}) and id_X is the identity homomorphism on a group X.

The category Ab of abelian groups is the full subcategory of Group on the class of abelian groups.

(4) Ring : rings and ring homomorphisms.

and the category of commutative rings is a full subcategory.

we require ring to be associative, have a unit and their homo^{m} to preserve units.

CRings and this has a full subcategory Feild of feilds.

(5) Modules over a fixed ring R and module homomorphisms.

(6) Top : topological spaces and continuous maps
Top_* has spaces with a specified basepoint as objects and base-point preserving continuous function as morphisms.

Top. Group : Topological groups and continuous homomorphism of groups.

(7) Met : metric spaces and continuous maps
Met_1 : metric spaces and contractive maps

$$(X, d) \xrightarrow{f} (X', d')$$
$$d'(f(x), f(y)) \leqslant d(x, y)$$

↖ This is a full subcategory of TopVect but Ban_1 is not

(8) Ban : Banach spaces and continuous linear maps
Ban_1 : Banach spaces and contractive linear maps
(Hahn-Banach theorem is a theorem about the second category, but not Ban).

Ban. Alg. : Banach algebras and continuous algebra homomorphism.

(9) C^* - algebras and $*$-homomorphisms.
Von Neumann algebras and (ultra weakly) continuous $*$-homomorphisms.

(10) Meas : measurable spaces $(X, \mathfrak{m})$
X : set
$\mathfrak{m}$: σ-algebra of subsets of X
(measurable subsets)
measurable maps
$$(X, \mathfrak{m}) \xrightarrow{f} (X', \mathfrak{m}')$$
$\forall m' \in \mathfrak{m}'$ $f^{-1}(m') \in \mathfrak{m}$.
problem : need to know which sets have measure "0".
Thus we need additional data!
(i) negligible sets
$\mathfrak{n}$: σ-algebra
A collection of subsets of X closed under intersections and unions (and complements) with $\phi \in \mathfrak{m}$. - finite or countable
$\mathfrak{n} \subseteq \mathfrak{m}$: a σ-ideal in $\mathfrak{m}$
closed under same above operations.
(ii) measure class
$\mu : \nu$: $\mu \ll \nu$ and $\nu \ll \mu$
i.e. there are some measurable f, g functions on X with $\mu = f\nu$ and $\nu = g\mu$.

and $\mathcal{N} = \{ m \in \mathcal{M} : \mu(m) = 0 \}$

(Localizable measurable spaces) $\longrightarrow$ $CW^{*\,op}$ commutative von neumann algebras.

Objects : $(X, \mathcal{M}, \mathcal{N})$

- X: set
- $\mathcal{M}$: σ-algebra on X
- $\mathcal{N}$: negligible sets of $\mathcal{M}$

morphisms : $(X, \mathcal{M}, \mathcal{N}) \xrightarrow{f} (X', \mathcal{M}', \mathcal{N}')$

equivalance classes of functions $X \to X'$ such that

$\forall\, m' \in \mathcal{M}'$, $f^{-1}(m') \in \mathcal{M}$

$\forall\, n' \in \mathcal{N}$, $f^{-1}(n') \in \mathcal{N}$.

and $f \sim g$ if $\{ x \in X : f(x) \neq g(x) \} \in \mathcal{N}$.

composition : composition of equivalance classes is defined by the compositions of functions.

well defineness : $f \sim g \Rightarrow f \circ h \sim g \circ h$

$\Rightarrow h \circ f \sim h \circ g$

proof

$$W \xrightarrow{h} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Consider $\{ w \in W : f(h(w)) \neq g(h(w)) \}$

$= \{ w \in W : h(w) \in \{ x \in X : f(x) \neq g(x) \} \}$

$= h^{-1}\{ x \in X : f(x) \neq g(x) \}$

$\in \mathcal{N}_W$ as $\{ x \in X : f(x) \neq g(x) \} \in \mathcal{N}_X$.

(11) Diff op_m : category

ob = Vector bundles over a smooth manifold m

morphisms = $V \longrightarrow V'$ differential operator $T: C^\infty(V) \longrightarrow C^\infty(V')$ that is $\mathbb{R}$-linear and (not increasing) preserve the support

$f \in C^\infty(V)$ $\operatorname{supp} f = \{x \in m : f(x) \neq 0\}$

i.e $\operatorname{supp} T(f) \subseteq \operatorname{supp} f$.

Δ - the Laplace operator is a morphism in Diff Op

(12) Man has smooth (infinitely differentiable) manifolds as objects and smooth maps as morphisms.

(13) Poset has partially ordered sets as objects and order preserving functions as morphisms. Also, a partially ordered set $(P, \leqslant)$ may be regarded as a category. The elements of P are objects and there is a unique morphism $x \longrightarrow y$ if and only if $x \leqslant y$. Transitivity implies that the composite morphisms exists and reflexivity implies identity morphisms exists.

(14) Graph has graphs as objects and graph morphisms as morphisms (functions carring vertices to vertices, edges to edges and incidence relation). In the variant DirGraph, objects are directed graphs, whose edges are now depicted as arrows, and morphisms are directed graph morphisms, which must preserve source and target.

(15) For a unital ring R, Mat_R is the category whose objects are positive integers and in which the set of morphisms from n to m is the set of mxn matrices with values in R. composition is by matrix multiplication

$$n \xrightarrow{A} m \;,\; m \xrightarrow{B} k \quad \rightsquigarrow \quad n \xrightarrow{BA} k$$

with identity matrices serving as 'identity morphisms.

GSet : sets with a G-action

BoolAlg , ComplBoolAlg $\rightsquigarrow$ stone duality

Species : category of functors $\mathrm{FinSet}^{\times} \longrightarrow$ Set

VBun_m : m - a fixed smooth manifold

vb - vector bundels over m

mor - smooth linear maps

$\lesssim \mathrm{mod}_{C^\infty(m)}$

Def^n

e: category and $x, y \in e$.

$f: x \to y$ is said to be an isomorphism if there is $g: y \to x$ such that

$gf = 1_x$ and $fg = 1_y$.

Here g is unique and denoted by f^{-1}.

Examples

Set : iso^s = bijections

Group : group isomorphisms

Diff Op. : Δ is not an isomorphism $D(f) = gf$ for some $g \neq 0$.

ΨDO/smoothig operator : Δ is an isomorphism and Δ^{-1} is called parametrix.

Top : homeomorphisms. $\mathbb{R}^m \sim \mathbb{R}^n$ iff $m=n$ Elliptic $\Rightarrow$ invertible. Δ is elliptic

Ban : linear homeomorphisms $T(x) = 2x$.

Ban_1 : linear isometries

Homotopy category

HoTop is the quotient of Top wrt equivalance on the morphisms $f, g: x \to y$

$f \sim g$ if the are homotopic.

i.e There is continuous $h: x \times [0,1] \to y$

such that $h|_{x \times \{0\}} = f$

$h|_{x \times \{1\}} = g$.

Category with a single object is a monoid
groupoid with a single object is a group

HoTop : Iso^m = homotopy equivalance
($\mathbb{R}^m$ isomorphic to $\mathbb{R}^n$).

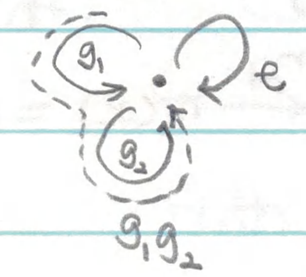
Ob = topological spaces
mor = homotopy classes of continuous maps

Def^n

A groupoid is a category in which all morphisms are invertible.

Observation

BG: groupoid
Ob(BG) = {G}



group elements are morphisms as each group element representing a distinct endomorphism of the single object G.

composition given by multiplication and the identity element acts as the identity morphism.

i.e. mor (G, G) is a group.

Quotient category $\begin{cases} Obj(\mathcal{C}/\sim) = Obj(\mathcal{C}) \\ Mor_{\mathcal{C}/\sim}(X,Y) = Mor_{\mathcal{C}}(X,Y)/\sim \end{cases}$

Def^{n}

$X \in Top$

this is a groupoid $\longrightarrow$ $\pi_{\leq 1}(X)$: The fundamental groupoid of X.

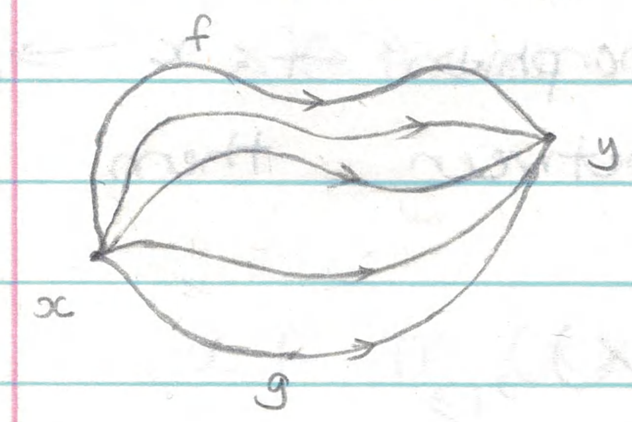
Ob: points of X

$x, y \in Obj$

$Mor(x,y)$: homotopy equivalance classes of continuous maps with

$f: [0,1] \longrightarrow X$

$f(0) = x \qquad f(1) = y$



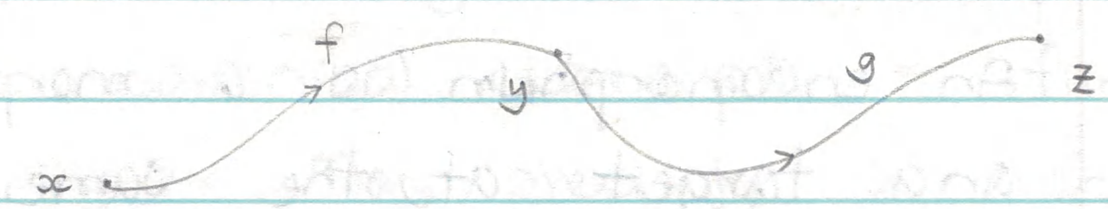
(end point preserving homotopy equivalance classes of paths).

$f \sim g$ if there is continuous $h: [0,1] \times [0,1] \longrightarrow X$

such that $h|_{\{0\}\times[0,1]} = f \qquad h|_{[0,1]\times\{0\}} = x$

$h|_{\{1\}\times[0,1]} = g \qquad h|_{[0,1]\times\{1\}} = y.$

Composition :



$f: x \longrightarrow y \qquad g: y \longrightarrow z$

$$gf(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and

$id_x = const_x$

Inverse of $f: [0,1] \longrightarrow X$

is $f^{-1}(t) = f(1-t)$

Def^{n}

G is a groupoid

$\pi_0 G$:= The set of equivalance classes of objects in G such that

$x, y \in G$ $\quad x \sim y$ if there is a morphism $f: x \to y$ between them.

Example

Suppose $G = \pi_{\leq 1}(X)$,

then

$\pi_0 G = \pi_0(\pi_{\leq 1}(X))$

= set of path components of X.

Def^{n}

An endomorphism is a morphism whose source and target are the same. An automorphism is an endomorphism that is also an isomorphism.

Given an object X in a category $\mathcal{C}$, the group of automorphisms of X is the set of all automorphisms of X equipped with the operation of composition and denoted by $Aut_{\mathcal{C}}(X)$.

$\mathcal{C}$: a category

$x \in \mathcal{C}$ $\quad Aut_{\mathcal{C}}(x) = Aut(x)$

$=$ the group of isomorphisms $x \to x$.

Example

$X \in$ FinSet, $Aut(X)$ is the symmtric group on X

$\mathbb{K}^n \in Vect_{\mathbb{K}}$, $Aut(\mathbb{K}^n) = GL(n, \mathbb{K})$ is the general linear group of order n over the feild $\mathbb{K}$

$X \in$ Meas, the group of measurable automorphisms play an important role in ergodic theory.

$x \in \Pi_{\leq 1}(X)$, the group $Aut_{\Pi_{\leq 1}}(x)$ denoted by $\pi_1(X, x)$ and is referred as the fundamental group of the pointed space (X, x)

note

A different choice y of a base point yeild a noncanonical isomorphic fundamental group $\pi_1(X, y)$. More precisely a morphism $f: x \to y$ (i.e. a homotopy class of paths) in $\Pi_{\leq 1}(X)$ gives rise to an isomorphism $\pi_1(X, x) \to \pi_1(y)$ (namely $p \mapsto fpf^{-1}$), and different morphism can gives a different isomorphism of groups.

Def^{n} $\mathcal{G}al(F))$ ← a groupoid

Let $F \in$ Field. The absolute Galois groupoid of F is a category with

ob = algebraic closures (or separably closed if $ch(F) \neq 0$) of F

i.e. $F \hookrightarrow L$ $\quad L/F$

(i) L is algebraically closed.

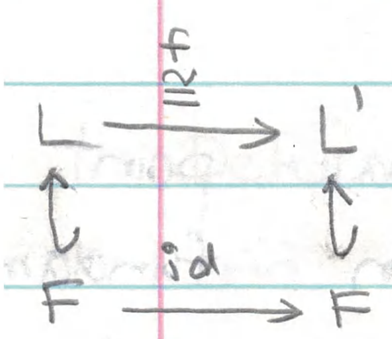
Every polynomial with coefficients in L has a root in L.

(ii) L is an algebraic extension of F

Every element of L is a root of some polynomial with coefficients in F.

$Mor(L, L')$ = isomorphisms $L \to L'$ of extensions over F

(The action over F is identity)



$f|_F = id$

Example: $\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}$ (algebraic numbers)

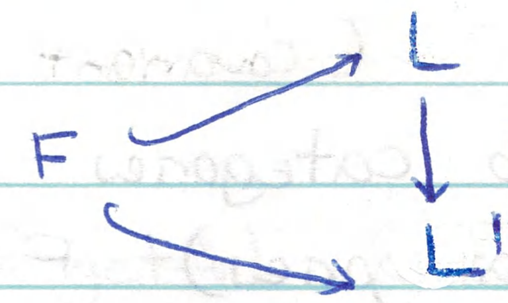
$\mathbb{R} \hookrightarrow \mathbb{C}$ conjugation $\in Aut(\mathbb{Q} \to \bar{\mathbb{Q}})$

Recall

$\pi_0(Gal(F)) = \{*\}$

algebraic closure exist and a unique up to an isomorphism.

Gothendick Galois theory, Borceux Janelidze (Galois theories)



$\cong$ (no canonical way to choose an isomorphism)

The automorphism group $Aut_{Gal(F)}(L/F)$ of some algebraic closure L is called the (absolute) Galois group of F (A different choice of L/F yields a noncanonically isomorphic Galois group)

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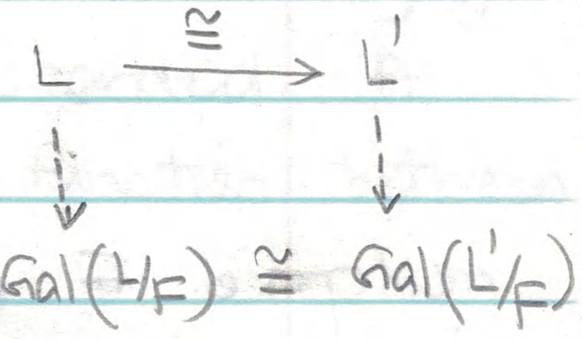


Remark

The above examples of $\pi_1(X)$ and $Gal(F)$ seems to be analogous. Indeed, both groupoids can be defined using the same construction: The fundamental groupoid of a topol. ← another category

Top ⇝ topol of sheaves

Field ⇝ étale topol Spec(F) étale topology

Note

G - groupoid and $x, y \in G$. Then $x \xrightarrow{f} y$ induces a homomorphism of groups $Aut_G(x) \xrightarrow{f_*} Aut_G(y)$.

Functors (covariant functor)

Let $\mathcal{C}, \mathcal{D}$ be two categories. A functor (morphism between categories) $F: \mathcal{C} \to \mathcal{D}$ given by following data:

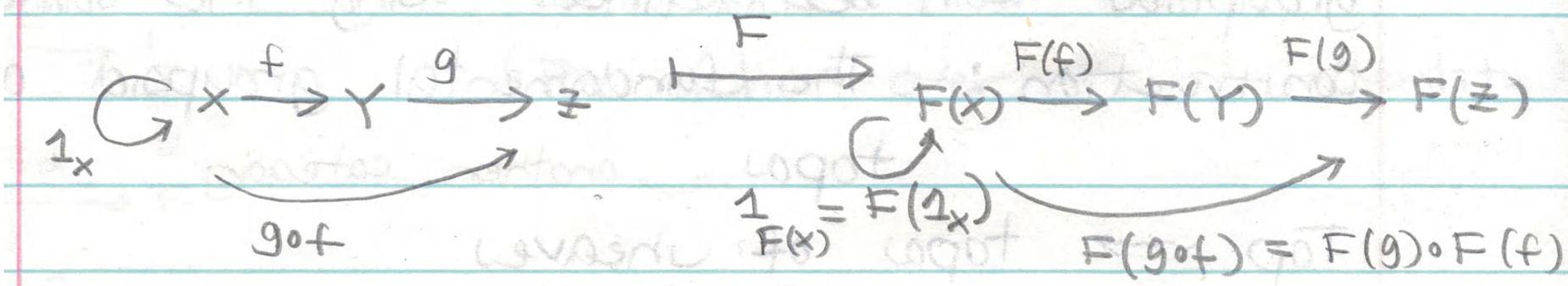
- ⊙ A function $ob(F): ob(\mathcal{C}) \longrightarrow ob(\mathcal{D})$ that sends objects of $\mathcal{C}$ to objects of $\mathcal{D}$.
- ⊙ For any objects $X, Y \in \mathcal{C}$ we have a function $Mor_F(X,Y): Mor_{\mathcal{C}}(X,Y) \longrightarrow Mor_{\mathcal{D}}(F(X), F(Y))$

Such that

(i) $F(g \circ f) = F(g) \circ F(f)$

(ii) $F(1_X) = 1_{F(X)}$

i.e. F preserves composition and identity morphisms.



Remark

To specify a functor $F: \mathcal{C} \to \mathcal{D}$ one must specify

(i) $\forall X \in \mathcal{C}$, $F(X) \in \mathcal{D}$

(ii) $X \xrightarrow{f} Y$ in $\mathcal{C}$, $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathcal{D}$

such that the composition and identity morphism are preserved.

Examples

① The Forgetful functor

(i)

$$Ab \xrightarrow{u} Set$$

$$X \longmapsto u(X) \text{ - the underlying set of } X$$

$$X \xrightarrow{f} Y \longmapsto u(X) \xrightarrow{u(f)} u(Y)$$

- the underlying function of f

We send an abelian group X to its underlying set with the algebraic operations discarded. A homo^{m} f of abelian groups is a function between the underlying sets, hence already a morphism of sets. The forgetful functor preserves compositions morphisms of abelian groups are composed by composing their underlying functions. The identity morphism preserved for the same reason.

(ii)

Group $\longrightarrow$ Set

Ring $\longrightarrow$ Ab

$Ban_1 \longrightarrow Met_1$

(but not $Ban \to Met_1$)

$Ban \longrightarrow Top$

② Inclusion functors

$$\mathbf{Met}_1 \longrightarrow \mathbf{Met}$$

$$\mathbf{Ban}_1 \longrightarrow \mathbf{Ban}$$

$$\mathbf{Ab} \longrightarrow \mathbf{Group}$$

abelianization is left adjoint to this functor

③ Free group functor

$$\mathbf{Set} \xrightarrow{F} \mathbf{Group}$$

$$X \longmapsto FX = \text{the free group on } X$$

$$X \xrightarrow{f} Y \longmapsto FX \xrightarrow{F(f)} FY$$

$$x \longmapsto f(x) \qquad x \longmapsto f(x)$$

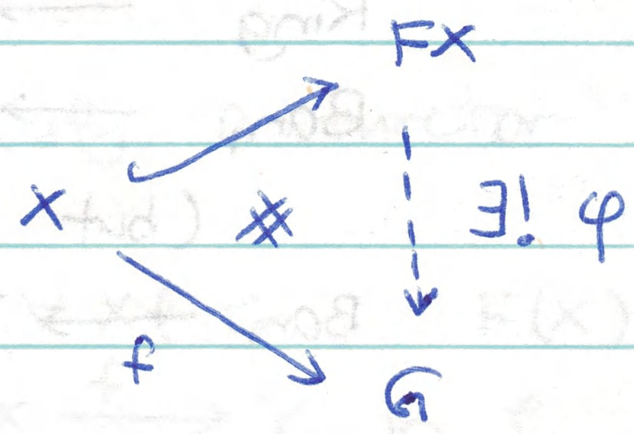
$FX = \{$ words of elements of $X\}$

- ⊙ empty word is the identity element
- ⊙ Concatanation of words is the group operation.

If $X = \{x_1, x_2\}$, then

$x_1 x_2^{-3} x_1^4 x_2 \in F(X)$

Universal property



This functor is left adjoint to the forgetful functor from Group to Sets.

④ Free vector space Functor

$$\text{Set} \xrightarrow{F} \text{Vect}_k$$

$$S \longmapsto F(S) = \{\text{formal } k\text{-linear combinations of elements of } S\}$$

Vector space that has S as a basis.

or vector space of linearly support functions $S \xrightarrow{f} k$

$$S \xrightarrow{h} T \longmapsto F(S) \xrightarrow{F(h)} F(T)$$

$$s \longmapsto h(s) \qquad \sum k_j s_j \longmapsto \sum k_j h(s_j)$$

Check the composition and image of identity morphism.

"Free" (free of relations between elements of S)

$\forall S \in \text{Set} \quad \forall V \in \text{Vect}_k$

$$\{F(S) \xrightarrow{\text{linear}} V\} \equiv \{S \xrightarrow{\text{function}} U(V)\}$$

adjoint to forgetful functor

$$\begin{array}{ccc} S & \xhookrightarrow{F} & F(S) \\ \exists!\ f_\varphi \big\downarrow & & \big\downarrow \varphi \\ U(V) & \xleftarrow{U} & V \end{array}$$

⑤ Group of units functor

$$(_)^\times : \text{Ring} \longrightarrow \text{Group}$$

This functor sends a ring R to its group $R^\times$ of invertible elements.

i.e. $x \in R$ for which there is $y \in R$ such that

$$xy = yx = 1.$$

$$\mathrm{Top}_* \xrightarrow{\pi_1} \mathrm{Grp}$$

is a functor.

Any homomorphism of rings $R \to S$ preserves invertible elements and therefore induces a homomorphism of groups $R^\times \to S^\times$.

⑥ Group ring functor (Group algebra)

$$\mathrm{Group} \xrightarrow{R} \mathrm{Ring}$$

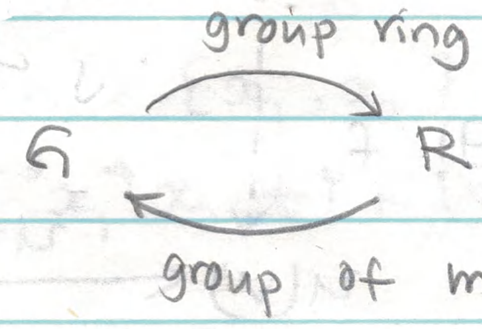
$$G \longmapsto R(G) = \{\text{set of finitely supported mappings } f: G \to R\}$$

$$(\alpha, f) \xmapsto{\cdot} \alpha f(x)$$

$$(f, g) \xmapsto{+} f(x) + g(x)$$

$$(f, g) \xmapsto{\times} \sum_{uv=x} f(u)g(v) = \sum_{u \in G} f(u)\, g(u^{-1}x)$$

When $R = \mathbb{Z}$ this gives an adjunction between Group and Ring



⑦ Polynomial ring functor

$$_[x] : \mathrm{Ring} \to \mathrm{Ring}$$

$$R \longmapsto R[x] : \text{polynomials in a single variable } x \text{ with coefficients in } R$$

$$R \xrightarrow{f} S \longmapsto R[x] \to S[x]$$

$$\sum r_j x^j \mapsto \sum f(r_j) x^j$$

$$\begin{aligned} \mathrm{mor}_{\mathcal{C}^{op}}(X,Y) \times \mathrm{mor}_{\mathcal{C}^{op}}(Y,Z) &= \mathrm{mor}_{\mathcal{C}}(Y,X) \times \mathrm{mor}_{\mathcal{C}}(Z,Y) \\ &\cong \mathrm{mor}_{\mathcal{C}}(Z,Y) \times \mathrm{mor}_{\mathcal{C}}(Y,X) \\ &\longrightarrow \mathrm{mor}_{\mathcal{C}}(Z,X) \\ &= \mathrm{mor}_{\mathcal{C}^{op}}(X,Z) \end{aligned}$$

Def^{n}

Let $\mathcal{C}$ be a category. The opposite category of $\mathcal{C}$ denoted by $\mathcal{C}^{op}$ has

$$ob(\mathcal{C}^{op}) = ob(\mathcal{C})$$

$$\mathrm{mor}_{\mathcal{C}^{op}}(X,Y) = \mathrm{mor}_{\mathcal{C}}(Y,X)$$

i.e $f^{op}: X \longrightarrow Y \in \mathcal{C}^{op} \longleftrightarrow f: Y \longrightarrow X \in \mathcal{C}$

(each morphism is pointing in the opposite direction)

The remaining structure of $\mathcal{C}^{op}$ is given as follows:

① For each object X, the arrow $1_X^{op} = 1_X$ serves as its identity in $\mathcal{C}^{op}$.

②

$$X \xrightarrow{f^{op}} Y \xrightarrow{g^{op}} Z \longleftrightarrow Z \xrightarrow{g} Y \xrightarrow{f} X$$

$$g^{op} \circ f^{op} \in \mathcal{C}^{op} \qquad f \circ g \in \mathcal{C}$$

The composition is associative and unital if and only if $\mathcal{C}$ defines a category.

Def^{n}

A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F: \mathcal{C}^{op} \longrightarrow \mathcal{D}$. Explicitely this consists of the following data: or equivalently $\mathcal{C} \xrightarrow{F} \mathcal{D}^{op}$

(i) $X \in \mathcal{C} \longmapsto F(X) \in \mathcal{D}$

(ii) $X \xrightarrow{f} Y \in \mathcal{C} \longmapsto F(Y) \xrightarrow{F(f)} F(X) \in \mathcal{D}$

satisfying the following two functoriality axioms

① $F(f \circ g) = F(g) \circ F(f)$ for any compatible pair

② $F(1_x) = 1_{F(x)}$ for any object of e.

Examples

① $\Theta : Jet^{op} \longrightarrow Alg_k$

$S \longmapsto \{$ Algebra of functions $S \to k\}$

$S \xrightarrow{f} T \longmapsto \Theta(T) \xrightarrow{\text{linear}} \Theta(S)$

$T \xrightarrow{\varphi} k \longmapsto S \xrightarrow{f} T$, φ : $T \to k$, $\varphi \circ f$: $S \to k$

$\Theta(f)(\varphi) = \varphi \circ f$

precomposition

② Composition is preserved

$S \xrightarrow{f} T \xrightarrow{g} U$, $\varphi : U \to k$, $\varphi \circ g$: $T \to k$, $\varphi \circ (g \circ f)$: $S \to k$

$\Theta(U) \xrightarrow{\Theta(g)} \Theta(T) \xrightarrow{\Theta(f)} \Theta(S)$

$(\Theta(f) \circ \Theta(g))(\varphi) = \Theta(f)(\Theta(g)(\varphi))$

$= \Theta(f)(\varphi \circ g)$

$= \varphi \circ (g \circ f)$

$= \Theta(g \circ f)(\varphi)$

$\forall \varphi \in \Theta(U)$

and similarly identity preserved.

This construction does not provide a functor if we use the algebra of finitely supported functions on S to k.

$V \in Vect_k \quad S \in Set \Rightarrow Hom(S,V) \in Vect_k$ and

$\dim(Hom(S,V)) = (\dim V)^{card(S)}$

For: Let $T = \{*\}$ and S = infinite set.

$g: T \to k$, $h: S \to T$

$g(*) = 1$ is finitely supported, but

$h(s) = * \quad \forall s \in S$ is not. Then

$g \circ h: S \to k$ is $g \circ h(s) = 1 \quad \forall s \in S$

is not finitely supported.

② Dual vector space Functor

$DV: Vect_k^{op} \longrightarrow Vect_k$

$V \longmapsto V^* = Hom(V,k)$

$V \xrightarrow{T} V' \longmapsto Hom(V',k) \xrightarrow{T^*} Hom(V,k)$

$\varphi \longmapsto \varphi \circ T$

fullback

Def^n (Ball category)

Ob: (V,B) (unit ball)

V - Hausdorff locally convex topological vector space

B - Hausdorff $\wedge$ topological subspace of V

Balanced (if $0 \in B$ and for any $x \in B$ and $|t| \leq 1$ we have $tx \in B$)

Convex (for any $x, y \in B$ and $\lambda \in (0,1)$ we have $\lambda x + (1-\lambda) y \in B$)

$Mor((V,B),(V',B')): V \xrightarrow{f} V'$ continuous linear map that sends B to B'

i.e. $f(B) \subseteq B'$

The category of Ball is also known as the category of Saks spaces.
CompBall is the full subcategory of Ball consisting of balls (V, B) such that B is compact. It is also known as the category of Waelbroeck spaces.

Example

① unit ball functor

$$\mathrm{Ban}_1 \longrightarrow \mathrm{Ball}$$
$$X \longmapsto (X, X_{\leq 1})$$

typically noncompact

$$X_{\leq 1} = \{ x \in X : \|x\| \leq 1 \}$$

contractive map $X \xrightarrow{f} Y$ $\longmapsto$ $(X, X_{\leq 1}) \xrightarrow{\text{induced map}} (Y, Y_{\leq 1})$

$$(x, X_{\leq 1}) \longmapsto (f(x), f(X_{\leq 1}))$$

② Dual unit ball functor

$$\mathrm{Ban}_1^{op} \xrightarrow{\mathrm{DuB}} \mathrm{CompBall}$$
$$X \longmapsto (X^*, X^*_{\leq 1})$$

X^* - vector space of continuous linear functionals on X equiped with the weak $*$-topology.

One of the most useful procedures for constructing topologies on vector spaces is by requiring the continuity of certain linear maps. Let $X \in \mathrm{ob}\,\mathbb{V}\mathrm{ect}$.

weak* topology – The coarsest topology in which every function on X^* given by evaluation on some fixed element of X is continuous.

$\forall x \in X$, $X^* \xrightarrow{ev_x} \mathbb{C}$, $f \longmapsto f(x)$ $\quad ev_x \in X^{**}$ is continuous (such smallest topology on X^*)

The weak topology generated by X^* is known simply as the weak topology on X, is the unique weakest topology on X for which all linear functionals $X \longrightarrow \mathbb{C}$ are continuous. Convergence with respect to this topology is known as weak convergence. When X is infinite dimensional, the weak topology is always weaker than the norm topology.

Next, the weak topology on X^* as defined above is the topology generated by X^{**}; of more interest is the topology generated by X (as a subspace of X^{**}) which is called the weak* topology.

If $x \in X$, define $\hat{x}: X^* \longrightarrow \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then the map $x \longmapsto \hat{x}$ is a linear isometry from X to X^{**}.

By the Alaoglu theorem, the closed unit ball $X^*_{\leq 1} = \{f \in X^* : \|f\| \leq 1\}$ is compact in weak* topology

$$X^*_{\leq 1} = \{f \in X^* : \sup_{\|x\| \leq 1} |f(x)| \leq 1\}$$

$$X \xrightarrow{f} Y \longmapsto (Y^*, Y^*_{\leq 1}) \xrightarrow{DUB(f)(g) = g \circ f} (X^*, X^*_{\leq 1})$$

$$Y \xrightarrow{g} \mathbb{C} \longmapsto \quad X \xrightarrow{f} Y \xrightarrow{g} \mathbb{C}, \quad g \circ f : X \to \mathbb{C}$$

(continuous and linear)

$\|g\circ f\| = \sup_{\|x\|\leq 1} |g(f(x))| \leq 1$

since we are precomposing f.

Thus $g\circ f \in X^*_{\leq 1}$. Therefore any contractive linear map $X \to Y$ of Banach spaces induces a continuous linear map $Y^* \to X^*$ (contravariant) which restricts to $Y^*_{\leq 1} \longrightarrow X^*_{\leq 1}$.

(3) Θ: $\mathrm{CompBall}^{op} \longrightarrow \mathrm{Ban}_1$

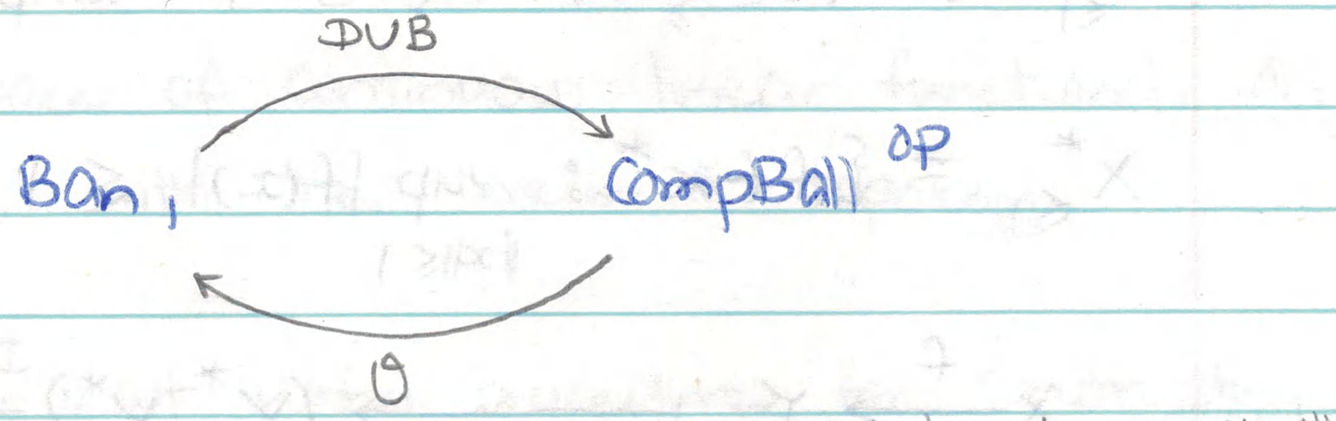
$(V,B) \longmapsto V^* = \{ V \xrightarrow[\text{continuous}]{f\text{-linear}} \mathbb{C} : \|f\| = \sup_{x\in B} |f(x)| \}$

$(V,B) \xrightarrow{f} (V',B') \longmapsto \Theta(V',B') \longrightarrow \Theta(V,B)$

$V' \xrightarrow{g} \mathbb{C} \longmapsto V \xrightarrow{g\circ f} \mathbb{C}$

$\|g\circ f\| = \sup_{x\in B} |g(f(x))| \leq \sup_{x\in B'} |g(x)| \leq 1$

Infact DUB and Θ are mutually inverse to each other in some sense and identify Ban_1 and CompBall. (equivalance of categories)



Claim:

$\forall X \in \mathrm{Ban}_1 \quad X \xrightarrow{\cong} \Theta(\mathrm{DUB}(X))$ (but not necessarily equal)

$\forall (V,B) \in \mathrm{CompBall} \quad (V,B) \xrightarrow{\cong} \mathrm{DUB}(\Theta(V,B))$

this is a strengthening of the traditional Hahn-Banach theorem. Later we will prove this rigorously.

Traditional form

Let $X \in \mathrm{Ban}$ and $Y \subseteq X$ be a closed subspace. If $Y \xrightarrow{f} \mathbb{C}$ s.t. $\|f\| \leq 1$, then $\exists\ X \xrightarrow{F} \mathbb{C}$ s.t. $\|F\| \leq 1$ and $Y \xrightarrow{i} X \xrightarrow{F} \mathbb{C} = f$.

Our formulation

$\forall\ f \in \mathbb{DUB}(Y) \in Y^*_{\leq 1} \quad \exists\ F \in \mathbb{DUB}(X)$ such that $\mathbb{DUB}(i)(F) = f$

equivalently,

$\mathbb{DUB}(i)$ is a surjective map or $\mathbb{DUB}(i)$ is an epimorphism in CompBall.

i.e. $Y \overset{i}{\subseteq} X$, $\mathbb{DUB}(i)$ is an monomorphism in $\mathrm{CompBall}^{op}$.

Def^n

A category $\mathcal{C}$ is small if the class of its objects is a set.

Example: The category Set is not a small category. The full subcategory of Set

whose objects are subsets of some fixed set in a small category.

Def^n

The category Cat of small categories has small categories as objects and functors as morphisms. Composition of morphisms $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to E$ is given by the composition of functors.

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} E$$

$$X \longmapsto F(X) \longmapsto G(F(X)) \qquad (G \circ F(X))$$

$$X \xrightarrow{f} Y \longmapsto F(X) \xrightarrow{F(f)} F(Y) \longmapsto G(F(X)) \xrightarrow{G(F(f))} G(F(Y)) \qquad (G \circ F(f))$$

The identity morphism on $\mathcal{C}$ is the identity functor $1_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ such that $1_{\mathcal{C}}(X) = X$ and $1_{\mathcal{C}}(f) = f$ for all $X \in Ob(\mathcal{C})$ and $f \in Mor_{\mathcal{C}}$.

① The functor $Ob: Cat \to Set$ sends a category $\mathcal{C}$ to its set of objects and a functor to its underlying function on objects.

$$\mathcal{C} \longmapsto Ob(\mathcal{C})$$

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longmapsto Ob(\mathcal{C}) \xrightarrow{Ob(F)} Ob(\mathcal{D})$$

$$X \longmapsto F(X) \qquad X \longmapsto F(X)$$

$$X \xrightarrow{f} Y \longmapsto F(X) \xrightarrow{F(f)} F(Y)$$

(2) The functor $\pi_0 : Cat \longrightarrow Set$ sends a category $\mathcal{C}$ to the set $Ob(\mathcal{C})/\sim$, where $\sim$ denotes the equivalance of isomorphism of objects.

$$\mathcal{C} \longmapsto Ob(\mathcal{C})/\sim$$

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longmapsto Ob(\mathcal{C})/\sim \xrightarrow{\pi_0(F)} Ob(\mathcal{D})/\sim$$

$$x \longmapsto F(x) \qquad [x] \longmapsto [F(x)]$$

(3) For a given category $\mathcal{C}$, the functor $Mor_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow Set$ sends a pair of objects $(X, Y) \in \mathcal{C}$ to the set $Mor_{\mathcal{C}}(X, Y)$ and a morphism $(X, Y) \xrightarrow{(f,g)} (X', Y')$ to the induced function $Mor_{\mathcal{C}}(X, Y) \longrightarrow Mor_{\mathcal{C}}(X', Y')$

$$(X, Y) \longmapsto Mor_{\mathcal{C}}(X, Y)$$

$$(X, Y) \xrightarrow{(f,g)} (X', Y') \longmapsto Mor_{\mathcal{C}}(X, Y) \longrightarrow Mor_{\mathcal{C}}(X', Y')$$

$$X' \xrightarrow{f} X \qquad X \xrightarrow{\varphi} Y \longmapsto X' \xrightarrow{g \varphi f} Y'$$

$$Y \xrightarrow{g} Y'$$

Def^n

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is faithful if for any pair of objects $X, Y \in \mathcal{C}$ the induced function $Mor_{\mathcal{C}}(X, Y) \longrightarrow Mor_{\mathcal{D}}(F(X), F(Y))$ is injective.

i.e. $F(f) = F(g) \Rightarrow f = g \quad \forall f, g : X \longrightarrow Y$

$\forall X, Y \in \mathcal{C}$.

$F: \mathcal{C} \to \mathcal{D}$ is faithful

if $\forall X, Y \in \mathcal{C}$

$$\mathrm{mor}_{\mathcal{C}}(X,Y) \xrightarrow{\text{injective}} \mathrm{mor}_{\mathcal{D}}(FX, FY)$$

Def^{n}

A concrete category is a pair $(\mathcal{C}, \mathcal{U})$, where $\mathcal{C}$ is a category and $\mathcal{U}: \mathcal{C} \to \mathrm{Set}$ is a faithful functor. A category $\mathcal{C}$ is Concretizable if there is $\mathcal{U}$ such that $(\mathcal{C}, \mathcal{U})$ is a concrete category.

Examples (concrete categories)

① (Set, id)

② (Group, $\mathcal{U}$), $\mathcal{U}$ - underlying set functor

$\mathcal{U}: \mathrm{Group} \longrightarrow \mathrm{Set}$

$$G \xrightarrow{\varphi} H \longmapsto \mathcal{U}(G) \xrightarrow{\mathcal{U}(\varphi)} \mathcal{U}(H)$$

function between underlying sets.

If $\mathcal{U}(\varphi) = \mathcal{U}(\psi)$ then $\varphi(x) = \psi(x)$ for all $x \in G$. i.e. $\varphi = \psi$. thus $\mathcal{U}$ is faithful.

③ Ring, Vect_k, Alg_k

④ Top, TopGroup, $\mathrm{TopVect}_k$

⑤ Ban, Ban_1, C^*, W^*

⑥ Man, LieGroup, VBun for the forgetful functor $\mathcal{U}$.

⑦ Set^{OP}

$\mathcal{U}: \mathrm{Set}^{OP} \longrightarrow \mathrm{Set}$ ~~complAtomBoolAlg~~

$$X \longmapsto 2^X = \{A : A \subset X\}$$

f^{OP} - morphism in Set^{OP}

$$X \xrightarrow{f} Y \longmapsto 2^X \longrightarrow 2^Y$$

f - function

$Y \longrightarrow X \in \mathrm{Set}$

$A \longmapsto f^{-1}(A)$ - function

If $f^{-1}(A) = g^{-1}(A) \quad \forall A \in 2^X$, for two functions $f, g : Y \to X$, then $f = g$. So the functor $\mathcal{U}$ is faithful and therefore Set^{op} is concrete.

Proposition

If $\mathcal{C}$ is concrete, then so is $\mathcal{C}^{op}$.

proof:

$$\mathcal{C} \xrightarrow{\mathcal{U}\text{-faithful}} Set$$

$$\mathcal{C}^{op} \xrightarrow{\mathcal{U}^{op}\text{-faithful}} Set^{op} \xrightarrow{2^-\text{-faithful}} Set$$

$$\mathcal{C}^{op} \xrightarrow{\text{faithful}} Set$$

Remark

A given category $\mathcal{C}$ can admit many different functors $\mathcal{U}$ that make it concrete. For instance, for Ban, apart from the underlying set functor we can take the functor

$$U(X) = \{ x \in X : \|x\| \leq 1 \}.$$

Theorem (Freyd)

HoTop is not concretizable.

Def²

A monomorphism in a category $\mathcal{C}$ is a morphism $f: X \to Y$ such that for any object W and any pair of morphisms $g_1, g_2: W \to X$ such that $fg_1 = fg_2$ we have $g_1 = g_2$.

i.e. $\forall W \in \mathcal{C}$ with

$$W \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

s.t. $fg_1 = fg_2 \Rightarrow g_1 = g_2$.

left cancellation

An epimorphism in a category $\mathcal{C}$ is a monomorphism in $\mathcal{C}^{op}$.

i.e. $\forall W \in \mathcal{C}$ with

$$W \underset{g_2}{\overset{g_1}{\leftleftarrows}} X \xleftarrow{f} Y \quad \text{in } \mathcal{C}$$

s.t. $g_1 f = g_2 f$ we have $g_1 = g_2$.

right cancellation

Examples

(1) In Set monomorphisms are injective functions and epimorphisms are surjective functions.

proof

(i) f is injective $\Rightarrow$ f is a monomorphism

$W \underset{g_2}{\overset{g_1}{\rightrightarrows}} X$: $fg_1 = fg_2$

$\Rightarrow fg_1(w) = fg_2(w) \quad \forall w \in W$

$\Rightarrow g_1(w) = g_2(w) \quad \forall w \in W$

$\Rightarrow g_1 = g_2$.

f is a monomorphism $\Rightarrow$ f is injective

Let $W = \{*\}$ and $W \overset{g_1}{\underset{g_2}{\rightrightarrows}} X \xrightarrow{f} Y$,

where $g_1(*), g_2(*) \in X$ are arbitrary such that $fg_1 = fg_2$. Then $g_1 = g_2$

$g_1(*) = g_2(*)$.

i.e. f is injective.

(ii) f is an epimorphism $\Rightarrow$ f is a surjection.

$\forall Z \quad X \xrightarrow{f} Y \overset{g_1}{\underset{g_2}{\rightrightarrows}} Z \; : \; g_1 f = g_2 f \Rightarrow g_1 = g_2$

$\forall y \in Y \quad \exists x \in X : y = f(x)$.

If $\exists y_0 \notin \mathrm{Im} f$, define $Y \overset{g_1}{\underset{g_2}{\rightrightarrows}} Z = Y \sqcup \{*\}$

by $g_1(y) = y \quad \forall y \in Y$

and $g_2(y) = \begin{cases} *, & y = y_0 \\ y, & \text{otherwise} \end{cases}$

In fact $Z = \{0,1\}$ would be enough.

Then $g_1 f = g_2 f$, but $g_1 \neq g_2$ a contradiction.

f is surjective $\Rightarrow$ f is an epimorphism.

For any $y \in Y$ there is $x \in X$ such that $y = f(x)$. AND if $g_1(f(x)) = g_2(f(x)) \quad \forall x \in X$

$\Rightarrow g_1(y) = g_2(y) \quad \forall y \in Y$

$\Rightarrow g_1 = g_2$

(2) In any category of algebraic objects such as Group, Ab, Ring, Mod_R, $Vect_K$, Alg_K, ect

monomorphisms $\Leftrightarrow$ injective homomorphisms

(It suffices to take $\mathbb{N}$ as the free object on one generator and the remainder of the argument is identical to set using the fact that morphisms out of such a free object are in bijection with the elements of target)

Further

surjective homomorphism $\Rightarrow$ epimorphism.

However in some categories there are nonsurjective epimorphisms.

For: In Ring not all epimorphisms are surjective. For instance, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a nonsurjective epimorphism

$$\mathbb{Z} \xhookrightarrow{i} \mathbb{Q} \underset{g_2}{\overset{g_1}{\rightrightarrows}} R$$

If $g_1 i = g_2 i$, then $g_1(i(1)) = g_2(i(1))$

$$\Rightarrow g_1(1) = g_2(1)$$

$$\Rightarrow g_1 = g_2.$$

Or

$$g_1\left(\frac{p}{q}\right) = g_1(p)(g_1(q))^{-1} = g_2(p)(g_2(q))^{-1} = g_2\left(\frac{p}{q}\right)$$

$\forall\ p, q \in \mathbb{Z},\ q \neq 0.$

(3) In Group all epimorphisms are surjective homomorphisms.

This can be easily establish using Schreier's theorem: Every subgroup $H \leq G$ equals $\{x \in G : g_1(x) = g_2(x)\}$ for some homomorphism $g_1, g_2 : G \to G'$ (observe that $g_1 = g_2$ iff $G = H$). Indeed the image of any epimorphism $f : F \to G$ is a subgroup $H \leq G$, then Schreier's supplies g_1, g_2 such that $g_1 f = g_2 f$, hence $g_1 = g_2$ by definition of an epimorphism, and therefore $H = G$. i.e. f is surjective.

The same argument as for Group also shows that epimorphisms coincide with surjective maps in the categories Ab, Vect_k and Mod_R. However the analog of Schreier's theorem is trivial here: Given an epimorphism $f : X \to Y$ consider $Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z = Y/f(X)$, where g_1 is the canonical quotient map and g_2 is the zero map (For nonabelian groups we can only make sense of $Y/f(X)$ as a set of cosets, not as a group).

④ $\mathrm{Vect}_{\mathbb{K}}$

monomorphism $\Leftrightarrow$ linear injection

proof

monomorphism $\Rightarrow$ injective

Let $W = \mathbb{K}$ (one dimensional vector space) and $f: X \to Y$ be a monomorphism. If $f(x) = f(y)$ for some $x, y \in X$, define $g_1, g_2 : W \to X$ by $g_1(t) = xt$ and $g_2(t) = yt$. Then $fg_1(t) = tf(x) = tf(y) = fg_2(t)$ (note that all these maps are linear over $\mathbb{K}$). Then by the def^n of monomorphism $g_1 = g_2$

$\Rightarrow x = y$.

$\Rightarrow f$ is injective.

epimorphism $\Leftrightarrow$ linear surjection.

proof:

epimorphism $\Rightarrow$ surjective

$$X \xrightarrow{f} Y \underset{g_2 = 0}{\overset{g_1 = \pi}{\rightrightarrows}} Y/\mathrm{Im}f$$

$\mathrm{Im}f$ is a vector subspace of Y

Clearly $\pi f = g_1 f = 0$ (canonical quotient map)

$\Rightarrow \pi = 0$, by the def^n of epimorphism

$\Rightarrow Y = \mathrm{Im}f$

$\Rightarrow f$ is surjective.

(5) In Top monomorphisms are injective continuous maps and epimorphisms are surjective continuous maps.

monomorphism $\Rightarrow$ injective

proof

Let $W = \{*\}$ be equip with trivial topology.

$$W \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

If $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$, define $g_1(*) = x_1$ and $g_2(*) = x_2$. Since these are constant functions, they are continuous.

And the fact that $fg_1 = fg_2$ implies $x_1 = x_2$.

Epimorphism $\Rightarrow$ surjective

Let $I = \{0, 1\}$ equip with antidiscrete topology.

If $\exists\, y_0 \notin \mathrm{Im} f$ define $g_1(y) = 0 \quad \forall y \in Y$ and

$$g_2(y) = \begin{cases} 1, & y = y_0 \\ 0, & \text{otherwise.} \end{cases}$$

Both functions are continuous and $g_1 f = g_2 f$, but $g_1 \neq g_2$.

(6) In Top. Haus (the full subcategory of Hausdorff topological spaces) monomorphisms are injective continuous maps and epimorphisms are continuous maps with dense image.

(Two continuous functions with Hausdorff

codomains that coincide on some subset must also coincide on its closure.)

Continuous function $\Rightarrow$ epimorphism with a dense image

$$X \xrightarrow{f} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z$$

and $\overline{\text{Im} f} = Y$. Let $g_1 f = g_2 f$ on X

$$\Rightarrow g_1|_{\text{Im} f} = g_2|_{\text{Im} f}$$

$$\Rightarrow g_1|_{\overline{\text{Im} f}} = g_2|_{\overline{\text{Im} f}}$$

$$\Rightarrow g_1 = g_2$$

$\Rightarrow f$ is an epimorphism

$f, g : X \xrightarrow{\text{Continuous}} Y$

$$f|_A = g|_A \Rightarrow f|_{\overline{A}} = g|_{\overline{A}}$$

$f \times g : X \longrightarrow Y \times Y$

$x \longmapsto (f(x), g(x))$ is continuous.

$$\{x \in X : f(x) = g(x)\} = (f \times g)^{-1}(\{(y,y) : y \in Y\})$$

$$= (f \times g)^{-1}(\text{diag}(Y))$$

Hausdorff $\Leftrightarrow$ diagonal is closed

Thus $\{x \in X : f(x) = g(x)\}$ is closed and $f|_{\overline{A}} = g|_{\overline{A}}$.

The same results are true for CompHaus (the full subcategory of Haus on compact spaces)

Recall

A category $\mathcal{C}$ is small if $Ob(\mathcal{C})$ is a set (as opposed to a class). In any category $\mathcal{C}$ for any $X, Y \in Ob(\mathcal{C})$, $Mor_{\mathcal{C}}(X,Y)$ is a set.

Example: Set is not small (class of all sets is not a set). Consider the functor

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\text{const}_A} & \mathbf{Set} \\ X & \longmapsto & A \\ f & \longmapsto & 1_A \end{array}$$

The class of all functors $\mathbf{Set} \longrightarrow \mathbf{Set}$ contain the class of all sets, therefore not a set.

Cat := the category of small categories and functors

Equivalance of Categories

not necessarily

An isomorphism of categories (in Cat) is a functor $F: \mathcal{C} \to \mathcal{D}$ such that there is $G: \mathcal{D} \to \mathcal{C}$ satisfying $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$

$G \circ F = 1_{\mathcal{C}} \Rightarrow$ for any object $X \in \mathcal{C}$

$$X = G(F(X))$$

If $\mathcal{C}$ is a group, is an equality of groups?

Thesis

One should not talk about equalities of objects, only isomorphisms of objects.

e.g. Group

Above is a prefectly good definition except that it fails to exhibit many categories as equivalent even though we need them to be the "same".

naive Definition

An equivalance of categoire is a pair of functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ such that $\forall X \in \mathcal{C}$ $X \cong G(F(X))$ and $\forall Y \in \mathcal{D}$ $Y \cong F(G(Y))$.

(X, R) (thin groupoid)

set ↑ ↑ equivalence relation

$$\text{obj} : X$$
$$\text{mor}(x, x') = \begin{cases} *, & xRx' \\ \phi, & \text{o/w} \end{cases}$$

$(X, R) \xrightarrow{\text{d-Function}} (Y, \sim)$

$x \longmapsto d(x)$ ← function

$xRx' \longmapsto d(x) \sim d(x')$

i.e. $x \sim x' \Rightarrow d(x) \sim d(x')$ (*)

This modified definition is far too expensive. For instance, recall that any group G gives rise to a category BG that has a single object whose endomorphisms are elements of G and composition is given by multiplication. The above definition makes BG and BH the "same" for any two groups G and H.

Def$^{\underline{n}}$ (Discrete category)

$$\text{Disc} : \text{Set} \longrightarrow \text{Cat}$$
$$A \longmapsto \text{Disc } A$$
$$\text{ob} := A$$
$$\text{mor}(x, y) := \begin{cases} *, & \text{if } x = y \\ \phi, & \text{if } x \neq y \end{cases}$$

$$A \xrightarrow{f} B \longmapsto \text{Disc } A \xrightarrow{\text{Disc } f} \text{Disc } B$$

Recall

$$\pi_0 : \text{Cat} \longrightarrow \text{Set}$$
$$\mathcal{C} \longmapsto \pi_0 \mathcal{C} = \{\text{isomorphism classes of objects of } \mathcal{C}\}$$

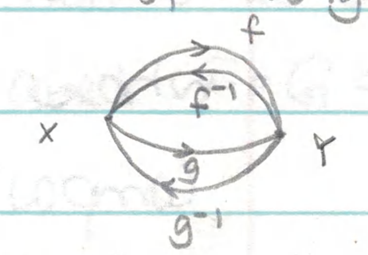
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longmapsto \pi_0 \mathcal{C} \xrightarrow{\pi_0 F} \pi_0 \mathcal{D}$$

(Functors preserve isomorphisms)

Equivalance between (X,R) and (Y,S) is a pair of fmctions $X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y$ such that (*) and $x R \beta(\alpha(x))$, $y S \alpha(\beta(y))$ $\forall x \in X,\ y \in Y$

Claim: The naive definition makes $\mathcal{C}$ and $Disc(\pi_0 \mathcal{C})$ equivalent ~~for a groupoid $\mathcal{C}$.~~

not wrong!



but redundant.

$\mathcal{C} \longleftrightarrow Disc(\pi_0 \mathcal{C})$

$x \longmapsto [x]$

a representative of class $y \longleftarrow\!\shortmid\ y$

$1_{Disc(\pi_0 \mathcal{C})}$

$Disc(\pi_0 \mathcal{C}) \longrightarrow \mathcal{C} \longrightarrow Disc(\pi_0 \mathcal{C})$

$\mathcal{C} \longrightarrow Disc(\pi_0 \mathcal{C}) \longrightarrow \mathcal{C}$

$x \longmapsto [x] \longmapsto Y \cong X$

But the morphism between non-isomorphic elements of ~~groupoid~~ wrong! $\mathcal{C}$ contains important information, invariants (like automorphism groups) and $Disc(\pi_0 \mathcal{C})$ is merely a set.

These examples show us what is wrong with our last attempt: we should take morphisms in to account and need to recover morphisms.

$X = C^1(\mathbb{R}, \mathbb{R})$ $\quad$ $Y = C^0(\mathbb{R}, \mathbb{R})$

$f R g \Leftrightarrow f - g$ is constant $\quad$ $f S g \Leftrightarrow f = g$

$X \underset{\int}{\overset{\frac{d}{dx}}{\rightleftarrows}} Y$ is an equivalance of two categories.

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}$$

$$X \longmapsto F(X) \longmapsto G(F(X)) \cong X$$

$$\mathrm{Mor}_{\mathcal{C}}(X,Y) \xrightarrow{\text{injective}} \mathrm{Mor}_{\mathcal{D}}(FX, FY) \xrightarrow{\text{surjective}} \mathrm{Mor}_{\mathcal{C}}(GFX, GFY)$$

need to be isomorphic as sets. $\quad \forall X, Y \in \mathcal{C}$

similarly

$F(G(Y)) \cong Y$ and

$\mathrm{Mor}_{\mathcal{D}}(X,Y) \hookrightarrow \mathrm{Mor}_{\mathcal{C}}(GX, GY) \hookrightarrow \mathrm{Mor}_{\mathcal{D}}(FGX, FGY)$

Def^n (Equivalance of Categories)

A functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalance of categories $\mathcal{C}$ and $\mathcal{D}$ if there is $G: \mathcal{D} \to \mathcal{C}$ such that

(i) $X \cong G(F(X)) \quad \forall X \in \mathcal{C}$

(ii) $Y \cong F(G(Y)) \quad \forall Y \in \mathcal{D}$

(iii) $\mathrm{Mor}_{\mathcal{C}}(X, X') \cong \mathrm{Mor}_{\mathcal{D}}(FX, FX') \quad \forall X, X' \in \mathcal{C}$

(iv) $\mathrm{Mor}_{\mathcal{D}}(Y, Y') \cong \mathrm{Mor}_{\mathcal{C}}(GY, GY') \quad \forall Y, Y' \in \mathcal{D}$

$F_{X,X'}$ and $G_{Y,Y'}$ are isomorphisms and $\pi_0(F), \pi_0(G)$ are mutually inverse to each other.

- inverse up to isomorphism.

Def 2

Given a functor $F: \mathcal{C} \to \mathcal{D}$, we have an induced function

$$F_{x,x'}: \mathrm{Mor}_{\mathcal{C}}(x, x') \longrightarrow \mathrm{Mor}_{\mathcal{D}}(Fx, Fx')$$

for any pair of objects $x, x' \in \mathcal{C}$. We say that F is

1) faithful if $F_{x,x'}$ is injective $\forall x, x'$
2) full if $F_{x,x'}$ is surjective $\forall x, x'$
3) fully faithful if $F_{x,x'}$ is bijective $\forall x, x'$
4) essentially surjective (isomorphism - dense) if $\pi_0 F$ is surjective
   i.e. $\forall Y \in \mathcal{D}\ \exists x \in \mathcal{C} : Y \cong F(x)$

Proposition

A functor is an isomorphism if and only if it is fully faithful and bijective on objects.

Proposition

A functor is an equivalance if and only if it is fully faithful and essentially surjective.

proof:

necessity of these two properties follows immediately from the definition of an equivalance. To show the sufficiency, we start by construct

the functor $G: \mathcal{D} \to \mathcal{C}$. On objects, we invoke the axiom of choice and construct a function $ob(G): ob(\mathcal{D}) \to ob(\mathcal{C})$ by choosing for every for every object $Y \in \mathcal{D}$ an object $G(Y) \in \mathcal{C}$ such that there is an isomorphism $\varepsilon_Y : F(G(Y)) \to Y$ (we also record a choice of an isomorphism ε_Y). Given a morphism $g: Y \to Y'$ in $\mathcal{D}$ we define $G(g) = F^{-1}_{G(Y), G(Y')}(\varepsilon_{Y'}^{-1} g \varepsilon_Y)$

$$G: \mathcal{D} \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$Y \longmapsto G(Y) : F(G(Y)) \overset{\varepsilon_Y}{\cong} Y$$

$$Y \xrightarrow{g} Y' \longmapsto G(Y) \xrightarrow{F^{-1}_{GY, GY'}(\varepsilon_{Y'}^{-1} g \varepsilon_Y)} G(Y') \quad \text{in } \mathcal{D}$$

By the fully faithfulness of F, the function $F^{-1}_{GY, GY'} : Mor_{\mathcal{C}}(GY, GY') \to Mor_{\mathcal{D}}(FGY, FGY')$ is a bijection, so it has an inverse which we used to define $G(g)$. Now we need to show G is a functor. Given morphisms $g: Y \to Y'$ and $g': Y' \to Y''$, we have

$$F(G(Y)) \xrightarrow{G(g)} F(G(Y')) \quad \text{in } \mathcal{C}$$

$$\varepsilon_Y \downarrow \qquad \circlearrowright \qquad \downarrow \varepsilon_{Y'}$$

$$Y \xrightarrow{g} Y'$$

$$G(g') \circ G(g) = F^{-1}_{GY', GY''}(\varepsilon_{Y''}^{-1} g' \varepsilon_{Y'}) \circ F^{-1}_{GY, GY'}(\varepsilon_{Y'}^{-1} g \varepsilon_Y)$$

$$= F^{-1}_{G\mathcal{Y}, G\mathcal{Y}''}\left(\varepsilon^{-1}_{\mathcal{Y}''} g' \varepsilon_{\mathcal{Y}'} \circ \varepsilon^{-1}_{\mathcal{Y}'} g \varepsilon_{\mathcal{Y}}\right)$$

$$= F^{-1}_{G\mathcal{Y}, G\mathcal{Y}''}\left(\varepsilon^{-1}_{\mathcal{Y}''} g'g \varepsilon_{\mathcal{Y}}\right)$$

$$= G(g'g)$$ (The second equality follows from the preservation of composition by F).

Likewise

$$G(1_{\mathcal{Y}}) = F^{-1}_{G\mathcal{Y}, G\mathcal{Y}}\left(\varepsilon^{-1}_{\mathcal{Y}} 1_{\mathcal{Y}} \varepsilon_{\mathcal{Y}}\right)$$

$$= F^{-1}_{G\mathcal{Y}, G\mathcal{Y}}\left(1_{F(G(\mathcal{Y}))}\right)$$

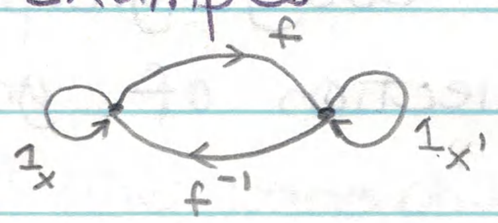
$$= 1_{G(\mathcal{Y})}.$$ Thus G is a functor.

Finally the functions $\pi_0 F : \pi_0(\mathcal{C}) \longrightarrow \pi_0(\mathcal{D})$ and $\pi_0 G : \pi_0(\mathcal{D}) \longrightarrow \pi_0(\mathcal{C})$ are mutually inverse to each other once we show that $\pi_0 F$ is injective, which by definition of G implies that $\pi_0 G$ is its inverse. Thus we have to show that for any objects $X, X' \in \mathcal{C}$ such that $F(X) \cong F(X')$ in $\mathcal{D}$, X and X' are themselves isomorphic in $\mathcal{C}$. Indeed if $g : F(X) \to F(X')$ is an isomorphism, then so is $F^{-1}_{X,X'}(g) : X \to X'$.

note

Equivalent categories are identical except that they might have different number of isomorphic copies of the same objects. One way of making this claim precise is as follows:
A category $\mathcal{C}$ is called "skeletal" if isomorphic objects of $\mathcal{C}$ are equal. Suppose $\mathcal{C}$ is a category and $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ such that any isomorphism class in $\mathcal{C}$ has exactly one representative that belongs to $\mathcal{D}$. Then the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ is an equivalance of categories. Furthermore if $\mathcal{C}$ and $\mathcal{D}$ are both skeletal, and $F: \mathcal{C} \to \mathcal{D}$ is an equivalance, then F is actually an isomorphism.
(Hence two categories are equivalent iff they have isomorphic skeleta)

Examples

① 

and 

are equivalent.

In general, given a groupoid G, we can take the full subcategory of G formed by some fixed isomorphism class of objects. The collection of all such full subcategories contains all objects

of G, and there are no morphisms between objects that belong to different full subcategories. Thus G splits as a disjoint union of its connected components. So it is suffices to classify connected groupoids inorder to classify groupoids. For any $* \in G$, the inclusion of the full subcategory on $*$ into G is an equivalance. Thus any connected groupoid is equivalent to a groupoid with one object, which is completely determined by the group $Aut_G(*)$ (whose elements form morphisms $* \longrightarrow *$ and the operation determine by composition of morphisms)

Vice versa, for any group G one can construct a groupoid BG with one object which has $Aut_{BG}(*) \cong G$. Furthermore functors $BG \to BH$ can be identify with group homomorphisms $G \to H$ Thus up to an equivalance, every groupoid G can be thought as a collection of groups indexed by $\pi_0(G)$.

(2) Hahn - Banach theorem

$$\mathrm{Ban}_1^{OP} \underset{\text{DUB}}{\overset{\Theta}{\rightleftarrows}} \mathrm{CompBall}$$

is an equivalance of categories.

$$\begin{array}{ccc} \mathbb{DUB} & & \mathcal{O} \\ X \longmapsto (X^*, \underbrace{X^*_{\leq 1}}) & & (V,B) \longmapsto V^* \\ \text{compact, } T_2 \text{ space} & & = \{V \xrightarrow{f} \mathbb{C} : \} \\ & & \|f\| = \sup |f(x)| \end{array}$$

① $X \longrightarrow \mathcal{O}(\mathbb{DUB}(X))$

$\leqslant X^{**} = \{X^* \xrightarrow{\varphi} \mathbb{C} : \sup_{\|f\| \leq 1} |\varphi(f)|\}^{x \in B}$

$x \longmapsto X^* \longrightarrow \mathbb{C}$

$f \longmapsto f(x)$

Given $X \in \mathrm{Ban}_1$, the isomorphism $ev: X \to \mathcal{O}(\mathbb{DUB}(X))$ sends an element $x \in X$ to the linear functional $ev_x : X^* \longrightarrow \mathbb{C}$ that sends an element $f \in X^*$ to $f(x) \in \mathbb{C}$. i.e. $ev_x(f) = f(x)$. we have

$$\begin{aligned} \|ev(x)\| &= \sup_{f \in X^*_{\leq 1}} |ev(x)(f)| \\ &= \sup_{f \in X^*_{\leq 1}} |f(x)| \\ &= \|x\|, \text{ so } ev \text{ is a contraction.} \end{aligned}$$

② the isomorphism $ev: Y \longrightarrow \mathbb{DUB}(\mathcal{O}(Y))$ of given $Y = (V,B) \in \mathrm{CompBall}$ is a morphism of CompBall $(V,B) \longmapsto (\mathcal{O}(V,B)^*, \mathcal{O}(V,B)^*_{\leq 1}) = (V^{**}, V^{**}_{\leq 1})$ that sends $v \in V$ to the continuous linear map

$\mathcal{O}(V, B) \longrightarrow \mathbb{C}$ that sends $f \in \mathcal{O}(V, B)$ to $f(V)$
i.e. $ev(v)(f) = f(v)$.
Since $\|ev(v)\| = \sup_{f \in B} |f(v)| \overset{?}{=} \|f\| \|v\|$, $ev(v)$
is indeed a bdd morphism of balls.

Remark
The traditional formulation of Hahn-Banach theorem states that any linear functional $B \to \mathbb{C}$ in $B^*_{\leq 1}$ on a closed Banach subspace $B \subseteq A$ can be extended to a functional $A \to \mathbb{C}$ with the same norm. We can deduce it from this stronger version as follows:
The inclusion $B \hookrightarrow A$ is a monomorphism in Ban_1, equivalently an epimorphism in Ban_1^{op}. The above equivalance send it to an epimorphism in CompBall. The forgetful functor CompBall $\longrightarrow$ CompHaus preserve epimorphisms. The resulting epimorphism in CompHaus is a map that sends a contractive functional $A \to \mathbb{C}$ to its restriction $B \to \mathbb{C}$. Epimorphisms in CompHaus are precisely surjective maps, which in our case means that any contractive map $B \to \mathbb{C}$ extends to a contractive functional $A \to \mathbb{C}$.

(3) Pontryagin Duality

LocCompHausAbTopGroup = LCHA $\rightsquigarrow$ Haar measure

Ob: locally compact Hausdorff abelian topological groups

Mor: continuous group homomorphisms.

Circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ over multiplication (closed topological subgroup of $\mathbb{C}^\times$).

$(\mathbb{R}/2\pi\mathbb{Z}, +)$

The dual group

If $G \in$ LACH, a character of G is a continuous group homomorphism $G \longrightarrow \mathbb{T}$. The set of all characters on G, $\mathrm{Hom}(G, \mathbb{T}) = \hat{G}$ can be made in to a group in LCHA called the dual group of G (this is analogous to the dual space in linear algebra). The group operation is given by pointwise multiplication of characters, the inverse of a character is its complex conjugation and the topology is given by uniform convergence on compact sets (Compact-Open topology, viewing $\hat{G}$ as a subset of the space of all continuous function from G to $\mathbb{T}$).

For $f, g \in \hat{G}$ $\quad \underbrace{f \cdot g}_{\in \hat{G}}(a) = f(a)\, g(a) \in \mathbb{T}$ $\forall a \in G$

Compact - Open topology

$X, Y \in \mathrm{Top}$

$C(X,Y) = \{ X \xrightarrow{f} Y : f \text{ is continuous} \}$

$U \subseteq C(X,Y)$ is open if

$U = \bigcup \left(\bigcap_{\text{Finite}} F_{K,V} \right)$, where

$K \subseteq X$ compact, $V \subseteq Y$ open

$F_{K,V} = \{ X \xrightarrow{f} Y : f(K) \subseteq V \}$

i.e. $\{ F_{K,V} \}$ form a subbasis for

Compact - open topology

$f_i \xrightarrow{\text{pointwise}} f$ in $C(X,Y)$ $\forall K \subseteq X$ compact

$f_i|_K \longrightarrow f|_K$ uniformly.

Examples of dual groups

(i) $\hat{\mathbb{T}} = \mathrm{Hom}(\mathbb{T}, \mathbb{T})$

$= (\mathbb{Z}, +)$ $\quad \forall n \in \mathbb{Z} : z \longmapsto z^n$

(ii) $\hat{\mathbb{R}} = \mathrm{Hom}(\mathbb{R}, \mathbb{T})$

$= (\mathbb{R}, +)$ $\quad x \longmapsto e^{2\pi a x}$ for some $a \in \mathbb{R}$

(iii) $\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}, \mathbb{T})$

$= (\mathbb{T}, \times)$ $\quad 1 \longmapsto z_0$ for some $z_0 \in \mathbb{T}$

$n \longmapsto z_0^n$

(iv) $\hat{\mathbb{Z}}_n = \mathrm{Hom}(\mathbb{Z}_n, \mathbb{T})$ $\quad 1 \longmapsto e^{2\pi i/n}$

$= (\mathbb{Z}_n, +)$

The functor $PD: LCHA^{op} \longrightarrow LCHA$

$$G \longmapsto \hat{G} = Hom(G, \mathbb{T})$$

Pontryagin dual of G

$$G \xrightarrow{f^{op}} G' \longmapsto \hat{G} \longrightarrow \hat{G}'$$

$$G' \xrightarrow{f} G \qquad \text{continuous group homo}^{m}$$

$$G \xrightarrow{\varphi} \mathbb{T} \longmapsto \hat{G}' \xrightarrow{\varphi f} \mathbb{T}$$

is an equivalance of categories.

Theorem (Pontryagin duality)

double dual $G \to \hat{\hat{G}}$ is covariant and naturally isomorphic to 1_{LCHA}

$LCHA^{op}$ $\underset{Hom(-,\mathbb{T})}{\overset{Hom(-,\mathbb{T})}{\rightleftarrows}}$ $LCHA$ is a equivalance of categories.

$\forall A \in LCHA$

$$A \xrightarrow{\cong} Hom(Hom(A, \mathbb{T}), \mathbb{T})$$

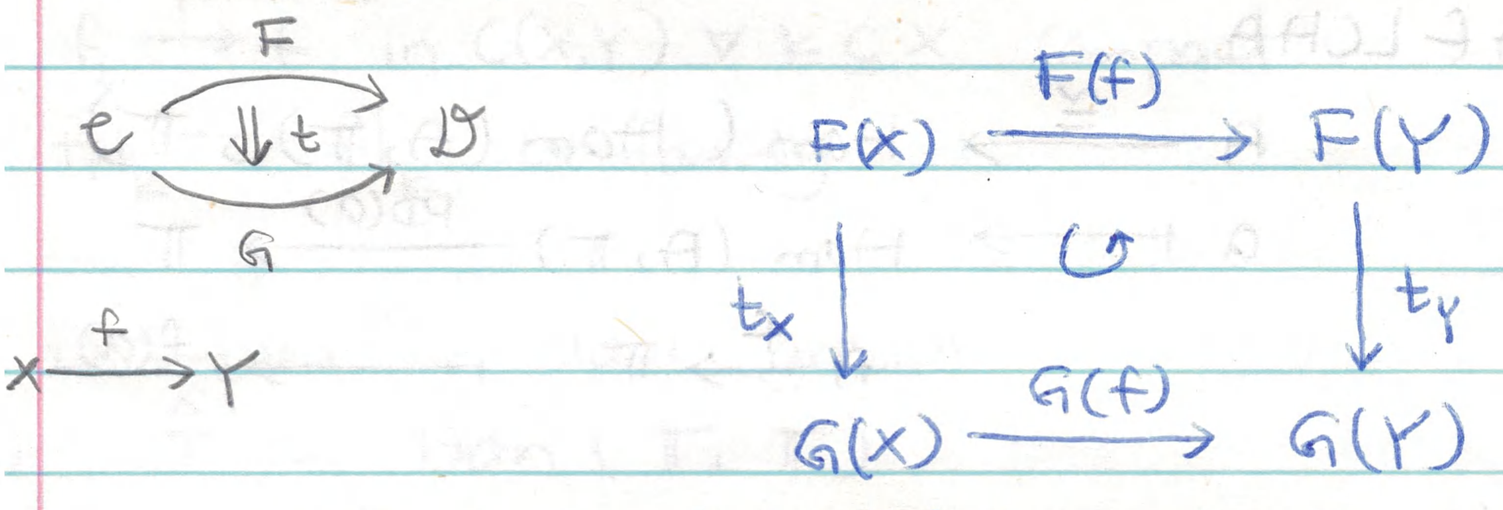
$$a \longmapsto Hom(A, \mathbb{T}) \xrightarrow{PD(a)} \mathbb{T}$$

$$A \xrightarrow{f} \mathbb{T} \longmapsto f(a)$$

oratural Transformations.

Def²

Let $\mathcal{C}, \mathcal{D}$ be categories and $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{C} \to \mathcal{D}$ be two functors between them. A natural transformation (morphism of functors) from F to G denoted by $t: F \Rightarrow G$ is given by a collection of morphisms $t_x : F(X) \longrightarrow G(X)$ for any $x \in \mathcal{C}$ such that the following diagram commutes for any morphism $f: X \to Y$ in $\mathcal{C}$:



i.e. $G(f) \circ t_x = t_y \circ F(f)$ for any $x \xrightarrow{f} y \in \mathcal{C}$

Def³ $\mathcal{C}^{\mathcal{D}}$

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. The category of functors $F(\mathcal{C}, \mathcal{D})$ has functors $F: \mathcal{C} \longrightarrow \mathcal{D}$ as objects and natural transformation $t: F \Rightarrow G$ as morphisms. The identity natural transformation $1_F : F \Rightarrow F$ (which assigns to every object $X \in \mathcal{C}$ the identity morphism

$\mathrm{Fun}(\mathcal{C},\mathcal{D})$

ob: $\mathcal{C} \xrightarrow{F} \mathcal{D}$ functors

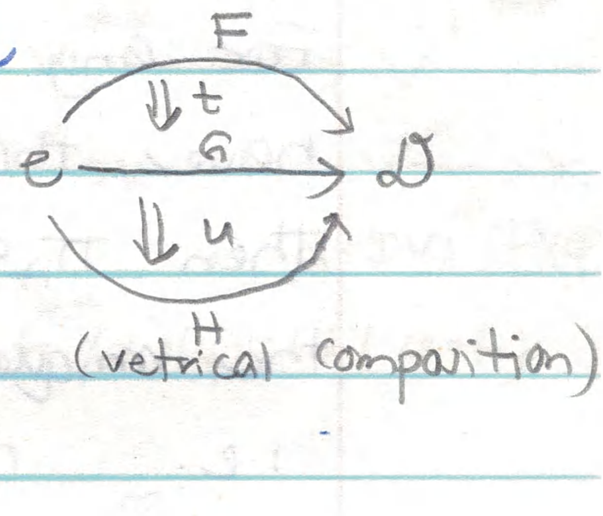
mor: $\mathcal{C} \overset{F}{\underset{G}{\Downarrow \alpha}} \mathcal{D}$ natural transformations

$F(X) \xrightarrow{\alpha_X} G(X) \xrightarrow{\beta_X} H(X)$

$(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$

$F(X) \xrightarrow{1_{F(X)}} F(X)$ $\quad F(X) \xrightarrow{G\alpha_X} G(X)$ $\quad \forall X \in \mathcal{C}$.

on $F(X)$) and the composition of two natural transformations $u: G \Rightarrow H$ and $t: F \Rightarrow G$ has $(u \circ t)_X = u_X \circ t_X$ for all $X \in \mathcal{C}$



(vetrical composition)

naturality of composition $u \circ t$ follows from naturality of u and t. Also associativity and unitality follow from those of $\mathcal{D}$.

Observation

Isomorphism in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ are precisely those natural transformations $t: F \Rightarrow G$ for which t_X is an isomorphism for all $X \in \mathcal{C}$. These are known as natural isomorphisms.

i.e. a natural transformation $t: F \Rightarrow G$ is a natural isomorphism iff there exists a natural transformation $u: G \Rightarrow F$ such that $ut = 1_F$ and $tu = 1_G$

Example

① Abelianization

$$\begin{array}{ccc} \mathrm{Group} & \xrightarrow{ab} & \mathrm{Ab} \\ G & \longmapsto & G^{ab} = G/[G,G] \\ G \xrightarrow{\varphi} H & \longmapsto & G^{ab} \xrightarrow{\varphi^{ab}} H^{ab} \\ g \longmapsto h & & gG' \longmapsto hH' \end{array}$$

is a functor. Let $\pi_G: G \to G^{ab}$ be the projection map onto the coset of $[G,G]$. For any group homomorphism $f: G \to H$ we have that $[G,G] \leq \ker(\pi_H \circ f)$ as H^{ab} is abelian. Then $\pi_H \circ f$ factor through G^{ab} as $f^{ab} \circ \pi_G$ for the unique homomorphism $f^{ab}: G^{ab} \to H^{ab}$

i.e.

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \pi_G}\downarrow & \circlearrowright & \downarrow{\scriptstyle \pi_H} \\ G^{ab} & \xrightarrow{f^{ab}} & H^{ab} \end{array}$$

universal property of quotient groups

This makes π a natural transformation (but not a natural isomorphism) from the identity functor on Group to $ab: \text{Group} \to \text{Group}$.

(2) Double dual of a vector space.

Recall that the functor $DV: \text{Vect}_k^{op} \to \text{Vect}_k$ sends a vector space V to $V^* = \text{Hom}(V, k)$ and a linear map $V \to V'$ to the induced map $\text{Hom}(V', k) \to \text{Hom}(V, k)$

$$\text{Vect}_k \xrightarrow{*^{op}} \text{Vect}_k^{op} \xrightarrow{*} \text{Vect}_k$$

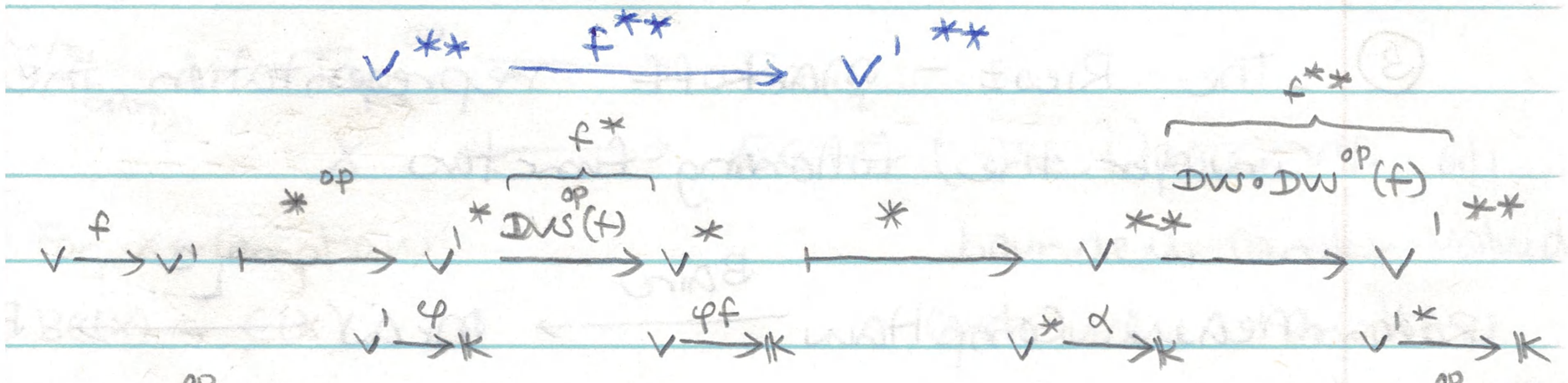
$$V \longmapsto V^* = \text{Hom}(V,k) \longmapsto V^{**} = \text{Hom}(\text{Hom}(V,k), k)$$

$$V \xrightarrow{f} V' \longmapsto V \xrightarrow{f} V' \longmapsto \text{Hom}(\text{Hom}(V,k), k)$$

$$f^* = \varphi f$$

$$V \xrightarrow{f} V' \xrightarrow{\varphi} k$$

Finite Calculus is a natural transformation
— wiki



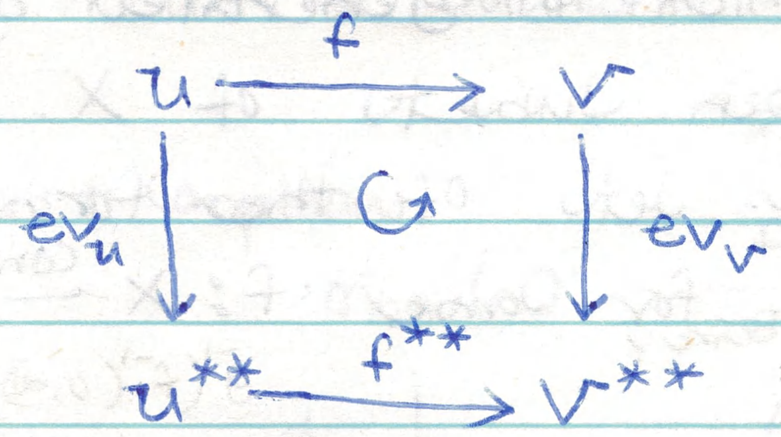
$D\mathbb{V}^{op}(f)(\varphi) = \varphi \circ f$

$\alpha \circ D\mathbb{V}^{op}(f)(\varphi)$

$f^{**}(\alpha)(\varphi) = \alpha(\varphi \circ f)$

We have a natural transformation

$ev: 1_{Vect_{\mathbb{K}}} \Rightarrow D\mathbb{V} \circ D\mathbb{V}^{op} = **^{op}$ whose value on a vector space V is the linear map $V \to V^{**}$ that sends $v \in V$ to the linear map $ev_v: V^* \to \mathbb{K}$ with $ev_v(\varphi) = \varphi(v) \quad \forall \varphi \in V^*$



$F = 1_{Vect_{\mathbb{K}}}$

$\Downarrow ev$

$G = **$

$f^{**}(ev_U(u))(g) = ev_U(u)(g \circ f) \quad g \in V^*$

$= g \circ f(u)$

$ev_V(f(u))(g) = g(f(u))$

$= g \circ f(u)$

(3) The Riesz - Markoff representation theorem

Consider the following functors

$$\text{Radon mea} : \text{CompHaus} \xrightarrow{\text{Baire}} \text{mea} \xrightarrow{L^1} \text{Ban}_1$$

and

$$\text{DualCont} : \text{CompHaus} \xrightarrow{\mathcal{O}_{\text{Ban}}} \text{Ban}_1^{op} \xrightarrow{\text{DV}} \text{Ban}_1$$

functionals

$$\text{Top} \xrightarrow{\text{Baire}} \text{mea}$$

$X \longmapsto (X, \text{Baire}_X, \{\phi\})$, where Baire_X is the σ-algebra of Baire sets of X, which is generated by functionally open subsets of X.

i.e. sets of the form $f^{-1}((0,\infty))$ for some $f : X \xrightarrow{\text{continuous}} \mathbb{R}$

$$X \xrightarrow{g} Y \longmapsto \text{Baire}_X \to \text{Baire}_Y$$

$$g^{-1}(f^{-1}(0,\infty)) = (f \circ g)^{-1}((0,\infty))$$

$$\text{mea} \xrightarrow{L^1} \text{Ban}_1$$

$(X, m, \sigma) \longmapsto$ (Complex) Banach space of finite measures on (X, m, σ).

$\{m \xrightarrow{\mu} \mathbb{C} :$ Countably additive, $\mu(\phi) = 0$, and $\mu|_N = 0\}$

$\|\mu\| = |\mu|(X) = \sup \int f \, d\mu\}$

$$(X, m, \sigma) \xrightarrow{f} (X', m', \sigma') \longmapsto L^1(X) \xrightarrow{f_*} L^1(X')$$

$|f| \leq 1$

twist forward

$$\mu \longmapsto f_* \mu$$

$$f_* \mu(m') = \mu(f^{-1}(m'))$$

$C_{Ban} : Top^{op} \longrightarrow Ban_1$

$X \longmapsto BC(X)$ (the space of all bounded, complex valued continuous functions on X)

For X ∈ Compltaw

$BC(X) = C(X)$

pointwise operations

$\|f\|_u = \sup_{x \in X} |f(x)|$

$X \xrightarrow{f} Y \longmapsto BC(Y) \xrightarrow{x \in X} BC(X)$

$x \xrightarrow{f} Y$

φf : $BC(Y) \xrightarrow{\varphi} \mathbb{C}$

$\|\varphi f\|_u = \sup_{x \in X} |\varphi(f(x))|$

we have a natural transformation α: Mean ⇒ DualCont

α_X : $Mean(X) \longrightarrow C(X)^*$

$\mu \longmapsto (g \longmapsto \int g \, d\mu)$

$\forall X \xrightarrow{f} Y$ ∈ Compltaw

$$\begin{array}{ccc} Mean(X) & \xrightarrow{f_*} & Mean(Y) \\ \downarrow \alpha_X & \circlearrowright & \downarrow \alpha_Y \\ C(X)^* & \xrightarrow[C(f)^*]{} & C(Y)^* \end{array}$$

naturality

$C(f)^* \alpha_X(\mu)(g) = \alpha_X(\mu)(g)(f)$

$= \alpha_X(\mu)(g \circ f)$

$= \int (g \circ f) \, d\mu$

$C(f)^* : C(Y) \longrightarrow C(X)$

$g \longmapsto g \circ f$

$$\alpha_Y(f_*(\mu))(g) = \int g \, d(f_*\mu)$$

$$= \int (g \circ f) \, d\mu$$

$f_*\mu(A) = \mu(f^{-1}(A))$

$\int \chi_A \, d(f_*\mu)$

$= \int \chi_{f^{-1}(A)} \, d\mu$

$= \int (\chi_A \circ f) \, d\mu$

Theorem (Riesz, Markov, Kakutani)

α is a natural isomorphism

i.e. $\forall X \in \mathrm{meas}(X)$, α_X is an isomorphism.

proof sketch:

easy - α_X is a norm preserving injection

difficult - α_X is surjective

surjectivity:

(i) when Y is extremely disconnected (closure of any open set is open) this is surjective

(ii) $\forall X \in \mathrm{CompHaus}$ $\exists Y \in \mathrm{CompHaus}$ extremely disconnected

i.e. $Y \xrightarrow{f} X$ Stone-Čech compactification of X with the discrete topology on X.

(iii) Use naturality square for f to show that

α_Y surjective $\Rightarrow$ α_X surjective

$Y \twoheadrightarrow X$

$\Rightarrow C(X) \hookrightarrow C(Y)$ (Hahn-Banach)

$\Rightarrow C(Y)^* \twoheadrightarrow C(X)^*$

$$\begin{array}{ccc} \mathrm{meas}(Y) & \xrightarrow{f_*} & \mathrm{meas}(X) \\ \downarrow \alpha_Y & \circlearrowright & \downarrow \alpha_X \\ C(Y)^* & \xrightarrow{C(f)^*} & C(X)^* \end{array}$$

⑦ Sheaves of Sets on Topological spaces.

$X \in$ Good Top "Good": locally path connected and semi locally simply connected.

Def$^{\underline{n}}$

path connected: $\pi_0(X) = \{*\}$

locally path connected: $\exists$ a base $\mathcal{B}$: $\forall U \in \mathcal{B}$ U is path connected.

simply connected: $\forall S^1 \xrightarrow{f} X \;\; \exists D^2 \xrightarrow{g} X : g|_{S^1} = f$

locally simply connected: $\exists$ a base $\mathcal{B}$: $\forall U \in \mathcal{B}$ U is simply connected.

semi locally simply connected: $\exists$ a base $\mathcal{B}$: $\forall U \in \mathcal{B}$ $\forall S^1 \xrightarrow{f} U \;\; \exists D^2 \xrightarrow{g} X : g_{S^1} = f$

(the contraction does not required to be in U)

Recall:

$\pi_{\leq 1}(X)$ objects: points in X

$\mathrm{Mor}(x, y)$: homotopy equivalance classes of paths $x \to y$.

Def$^{\underline{n}}$

Given $X \in$ Top, the category $\mathrm{Open}(X)$ is the category associated to the poset of Open subsets of X.

$\mathrm{Open}(X)$ Objects : open subsets of X

$$\mathrm{mor}(U,V) = \begin{cases} \{*\}, & U \subseteq V \\ \phi, & \text{otherwise} \end{cases}$$

Def^{n} (Presheaves)

the category $\mathrm{Preh}(X)$ on a $X \in \mathrm{Top}$ is defined as $\mathrm{Fun}(\mathrm{Open}(X)^{op}, \mathrm{Set})$

$F: \mathrm{Open}(X)^{op} \longrightarrow \mathrm{Set}$ (contravariant functor)

$\underbrace{U}_{\text{open}} \longmapsto F(U)$: sections of F over U $\in$ Set

$U \subseteq V \longmapsto F(V) \xrightarrow{\text{function}} F(U)$: restriction

$s \longmapsto s|_U$

Example

(i) $F(U) = C(U, \mathbb{R})$ (continuous real valued function on U)

$U \hookrightarrow V \longmapsto C(V,\mathbb{R}) \longrightarrow C(U,\mathbb{R})$

$f: V \to \mathbb{R} \longmapsto f|_U : U \to \mathbb{R}$

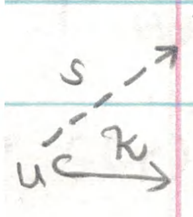
(restricted continuous function)

(ii) Bounded continuous maps

(iii) Borel measurable maps.

(iv) $\mathrm{SecPre}(P)$ (presheave of continuous section of p)

Given a continuous map $P: T \to X$ of topological spaces,



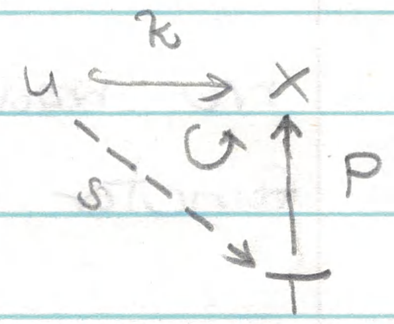
$\mathrm{SecPre}(P)(U) = \{ U \xrightarrow{s} T : Ps = t \}$ for any

↑ a right inverse of P on U

$Top_* = (Top^{op}/\{*\})^{op}$ $\quad X \xrightarrow{f} Y \xrightarrow{g} X$ $\quad gf = 1_X$

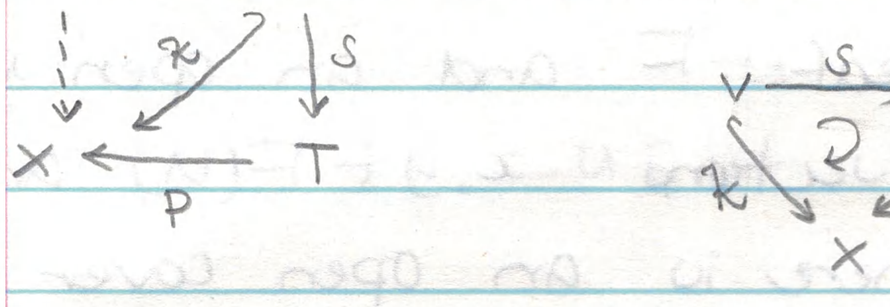
retraction, section

open set $k : U \to X$



$U \xrightarrow{c} V \longmapsto SecPre(P)(V) \longrightarrow SecPre(P)(U)$

$\longmapsto s \circ u$



Def² (slice category)

Given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, the slice category $\mathcal{C}/X$ has

Objects: $Y \xrightarrow{f} X$ (morphisms of $\mathcal{C}$ with codomain X

$\underset{\mathcal{C}/X}{Mor}(f,g)$: $Y \xrightarrow{h} Z$, $f = gh$, X

$h \in Mor_{\mathcal{C}}(Y,Z)$ such that $f = gh$

Note that $Open(X)$ is a full subcategory of Top/X. Infact we have a functor

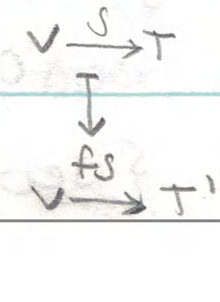
$SecPre : Top/X \longrightarrow Presh(X)$

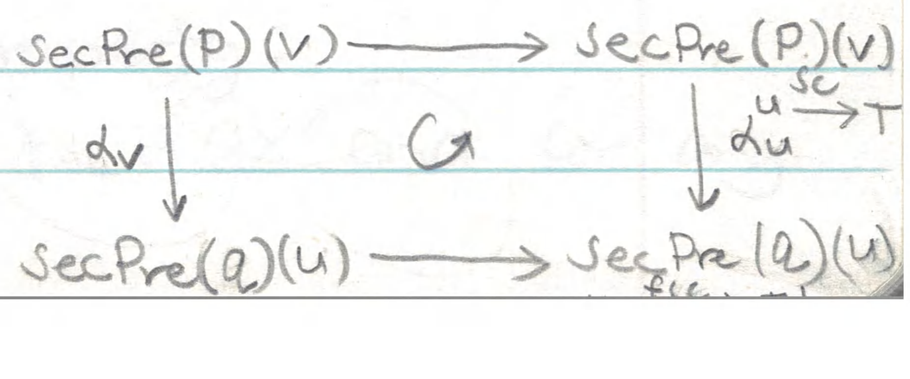
$T_1 \xrightarrow{P} X \longmapsto SecPre(P)$

$T \xrightarrow{f} T'$, p, q, X $\longmapsto SecPre(P) \Rightarrow SecPre(Q)$

$U, V \in Open(X)$

$U \xrightarrow{c} V$





we would like to characterize the image of functor SecPre in PreSh(X) for any $X \in$ Top.

Def^n

(i) Given a presheaf F and an open set $U \subseteq X$, we say that some sections $x, y \in F(U)$ are locally equal if there is an open cover $\{V_i\}$ of U such that $x|_{V_i} = y|_{V_i}$ $\forall i$. In this case we write $x \sim y$.

(ii) Given a presheaf on X, an open set $U \subseteq X$, and an open cover $\{V_i\}$ of U, a compatible family is a collection $\{t_i\}$ of sections $t_i \in F(V_i)$ such that $t_i|_{V_i \cap V_j} \sim t_j|_{V_i \cap V_j}$ $\forall i, j$. The set of all compatible families of sections of F over an open cover V is denoted by ComFam(F, V)

The gluing property

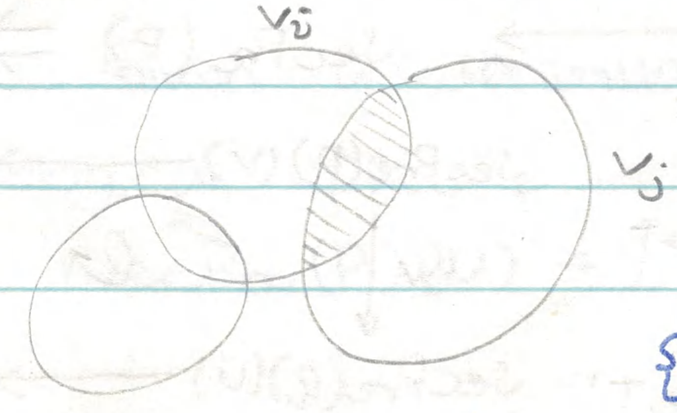
U is open in $X \in$ Top

$U = \bigcup_i V_i$ (open cover)

$V_i \hookrightarrow U$ $\forall i$

$F(U) \longrightarrow F(V_i)$

(sections of F over U) $\ni s \xmapsto{\text{restriction}} s|_{V_i} = t_i$



a compatible family

$\{$section t_i over V_i : $t_i|_{V_i \cap V_j} = t_j|_{V_i \cap V_j}$?

$$s|_{V_i \cap V_j} = t_i|_{V_j} = t_j|_{V_i} \qquad \forall i,j$$

Example

$$F: \mathrm{Open}(X)^{op} \longrightarrow \mathrm{Set}$$

$$U \longmapsto C(U, \mathbb{R})$$

$$U \subseteq V \longmapsto C(V, \mathbb{R}) \longrightarrow C(U, \mathbb{R})$$
$$f \longmapsto f|_U$$

Let $U = \bigcup_i V_i$, then

$$F(U) = \{f : U \xrightarrow{ctn} \mathbb{R}\} \xrightarrow{r} \left\{ f_i : V_i \longrightarrow \mathbb{R} \;\middle|\; f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j} \right\}$$
$$= \mathrm{ComFam}(F, V)$$

we have a canonical map $F(U) \hookrightarrow \mathrm{ComFam}(F,V)$, that sends a section $s \in F(U)$ to the compatible family $\{s|_{V_i}\}$. Indeed the compatibility condition satisfied because $s|_{V_i \cap V_j} = s|_{V_j \cap V_i}$.

Def^n

Given $X \in \mathrm{Top}$, the category $\mathrm{sh}(X)$ of sheaves of sets on X is defined as the full subcategory of $\mathrm{Presh}(X) = \mathrm{Fun}(\mathrm{open}(X)^{op}, \mathrm{Set})$ consisting of those functors $F: \mathrm{open}(X)^{op} \longrightarrow \mathrm{Set}$ that satisfy the gluing property:

For any open set $U \subseteq X$ and for any

Open cover $\{V_i\}$ of U, the canonical map $F(U) \longrightarrow \mathrm{ComFam}(F, V)$ is an isomorphism.

i.e. $Sh(X)$ = full subcategory of $Presh(X)$ such that for any compatible family $\exists!$ $f \in F(\bigcup_i V_i)$

$$f|_{U_i} = f_i$$

Example

① $\{f : U \xrightarrow{cts} \mathbb{R}\} \xrightarrow{\cong} \left\{ \begin{array}{l} f_i : V_i \longrightarrow \mathbb{R} \\ f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j} \end{array} \right\}$

injectivity : $r(f) = r(g)$

$\Rightarrow$ f and g are coincide on any point of U as $U = \bigcup_j V_j$ and $f|_{V_j} = g|_{V_j}$ $\forall j$

$\Rightarrow$ $f = g$

Surjectivity $\Rightarrow$ $\forall \{f_i\}$ compatible family $\exists!$ f such that $f|_{V_j} = f_i$

$f : U \to \mathbb{R}$ is well defined as a function since we know its value at any point of U without any ambiguity.

$W \subseteq U$ is open $\Leftrightarrow$ $W \cap V_i$ open $\forall i$ implies that f is continuous.

② non example

$$F(U) = \{ U \xrightarrow{\text{bounded}} \mathbb{R} \}$$

Recall

A prewheaf over X, i.e. a functor

$F: \text{Open}(X)^{op} \longrightarrow \text{Set}$ is a sheaf if

$\forall U \in \text{open}(X) \quad \forall \{V_i\}$

$V_i \subseteq U \ \forall i$ and $\bigcup_i V_i = U$

$F(U) \cong \{$ Compatible families, $t_i \in F(V_i) \}$

i.e. $\forall i, j$

$$V_i \cap V_j \xhookrightarrow{\iota_i} V_i$$

$\iota_j \downarrow$

V_j

$F(\iota_i)(t_i) \sim F(\iota_j)(t_j)$

$\in F(V_i \cap V_j)$

* Prewheaf
* gluability
* uniqueness

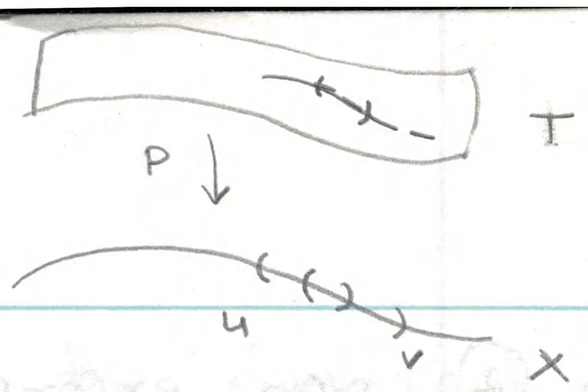
"$\sim$"

$W \subseteq X$

$x, y \in F(W)$

$x \sim y$ if $\exists \{Q_i\}$ such that $W = \bigcup_i Q_i$

$\forall i \quad x|_{Q_i} = y|_{Q_i}$.



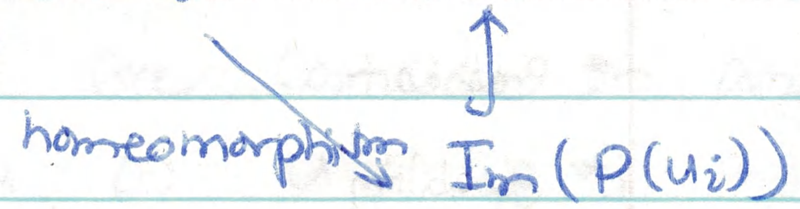
note

The functor $SecPre: Top/X \longrightarrow Presh(X)$ lands on $Sh(X)$. we denote the corestricted functor as $Sec: Top/X \longrightarrow Sh(X)$.

Def^{n}

A continuous map $p: Y \longrightarrow X$ is etale (also local homeomorphism) if it is an open map and a homeomorphism locally on Y.

i.e. $\exists$ an open covering $\{U_i\}$ of Y such that the corestriction $p|_{U_i} : U_i \longrightarrow X$ $\forall\ i$.



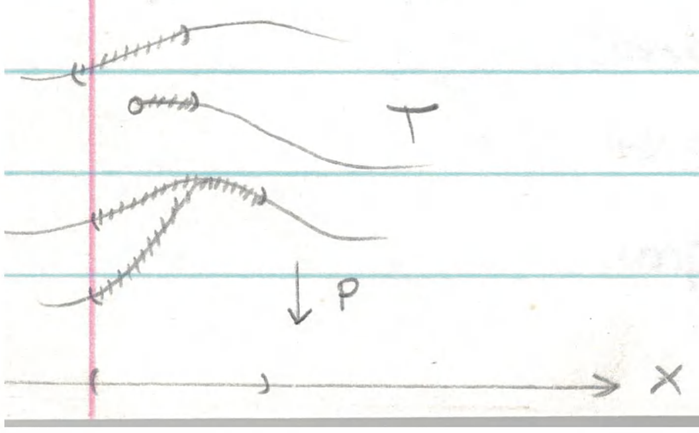
The category Et/X is the full subcategory of Top/X consisting of etale maps of X.

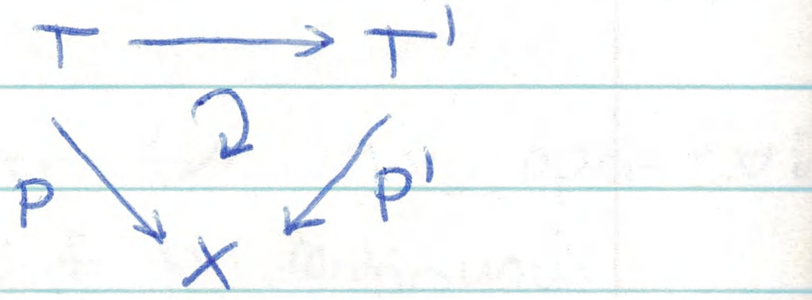
i.e

$X \in Top$ $\quad$ Et/X (full subcategory of Top/X)

Ob: $T \xrightarrow{p} X$ (etale)

$Mor\left(\begin{smallmatrix} T \\ \downarrow p \\ X \end{smallmatrix}, \begin{smallmatrix} T' \\ \downarrow p' \\ X \end{smallmatrix}\right) : T \longrightarrow T'$ such that

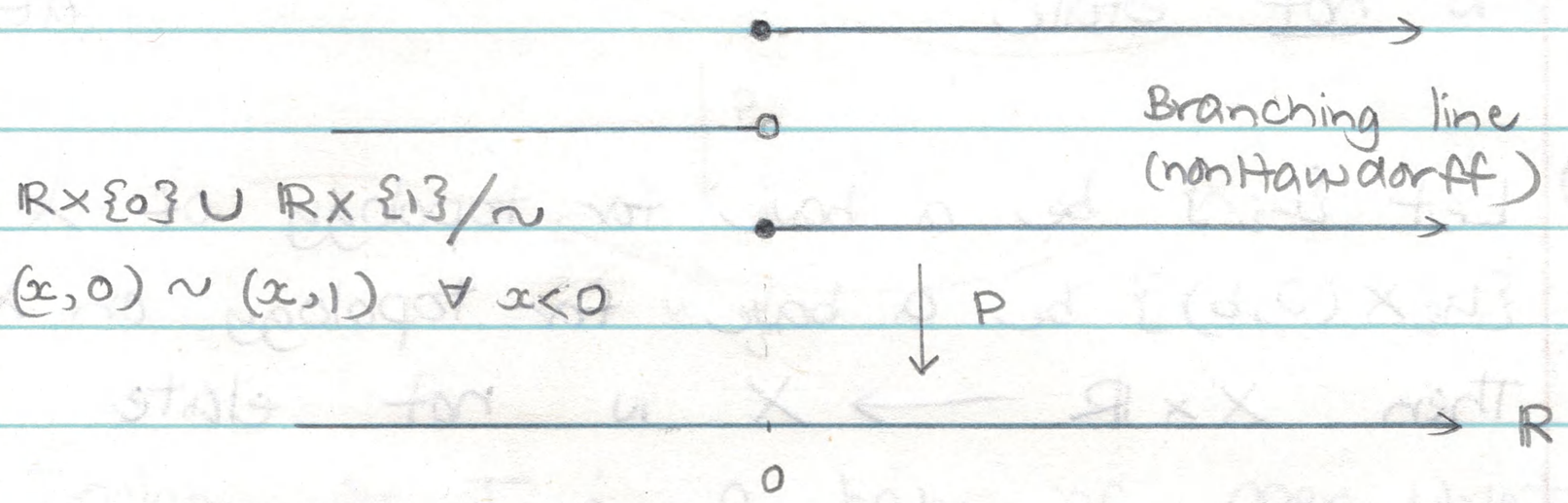




(continuous maps that preserve fibers)

Examples of etale maps

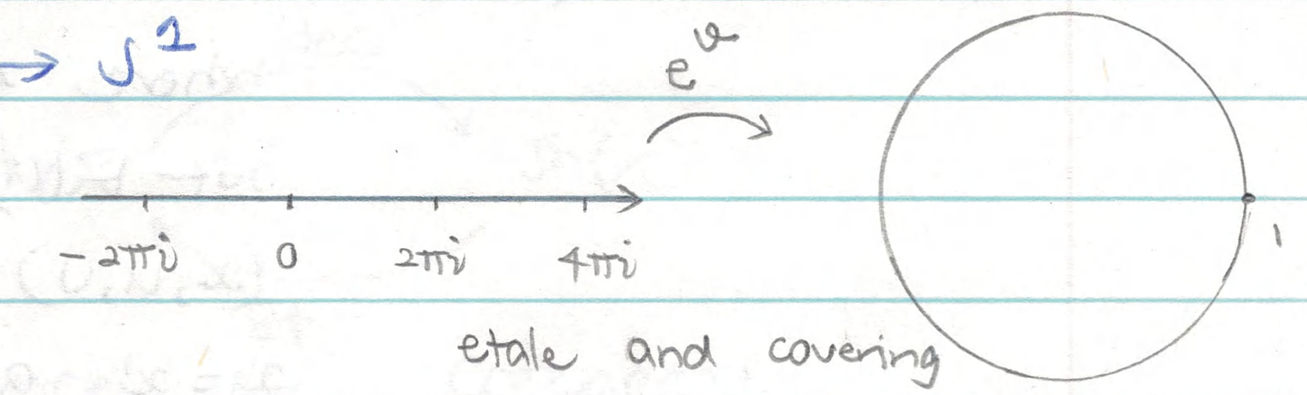
(1)



(2) Any covering map is etale

Let $B \in Top$. $P: E \to B$ is a covering map if every $b_0 \in B$ has an open neighborhood U such that $P^{-1}(U)$ consists of a disjoint union of some open sets (called slices) and P restricted to any of these is a homeomorphism on to the image (U is said evenly covered by P). E: a covering space of B.

(3) $\exp: i\mathbb{R} \longrightarrow S^1$



etale and covering

exp: Riemann surface of logarithm $\longrightarrow$ Punched plane $\mathbb{C}^\times$

④ $(\mathbb{R}, \text{standard topology}) \longrightarrow (\mathbb{R}, \{(a,\infty), \phi, \mathbb{R}\})$ $a \in \mathbb{R}$
is not etale

⑤ Let $\{U_i\}$ be a base for topology of X and $\{U_i \times (a,b)\}$ be a base for topology on $X \times \mathbb{R}$. Then $X \times \mathbb{R} \longrightarrow X$ is not etale.

Def^n

the etale space functor $Et : Sh(X) \longrightarrow Et/X$ sends a sheaf $F : \text{open}(X)^{op} \longrightarrow \text{Set}$ to the etale map $Et(F) : T \longrightarrow X$, where T is the space of F constructed as follows:

$F \in Sh(X)$

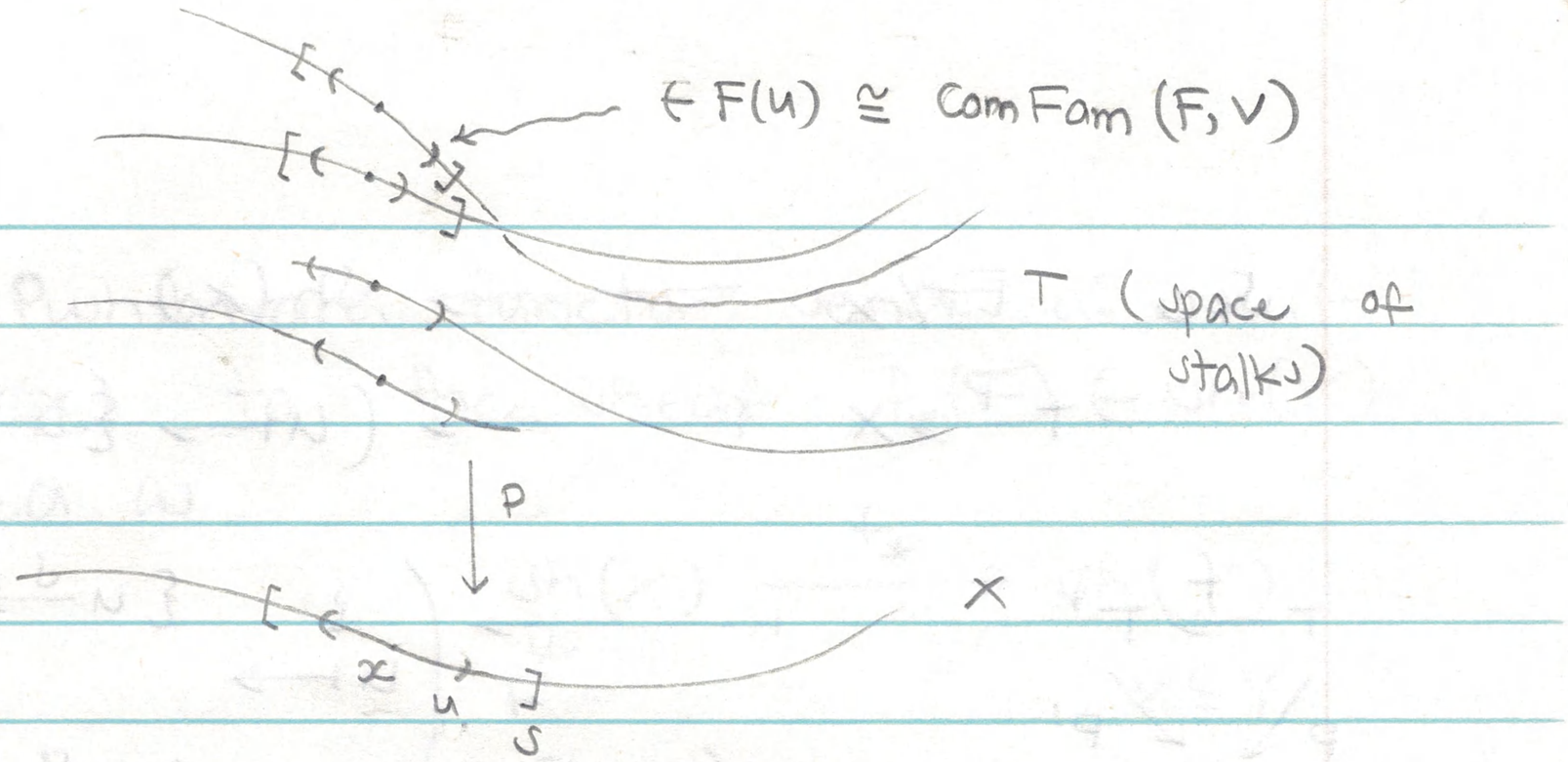
$x \in U \subseteq X$ open

$s \in F(U) \cong \text{ComFam}(F, V)$

$Et(F) \in Et/X$

space of equivalance classes of triples (x, U, s) where $x \in U$, $U \in \text{Open}(X)$ and $s \in F(U)$ wrt the relation $(x, U, s) \sim (x', U', s')$ if $x = x'$ and $\exists\ V \subseteq U \cap U'$ st $x \in V$ and $s|_V = s'|_V$.

These equivalance classes are known as stalks or germs.



Topology on T : a base of open subsets of T

Take any $U \in Open(X)$ and $s \in F(U)$

the set of equivalance classes

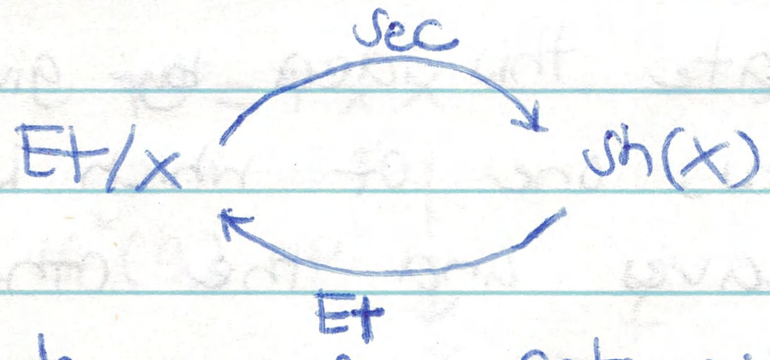
$\{[(x, U, s)] : x \in U\}$

Etale map : $T \xrightarrow{p} X$ given by $p([x, U, s]) = x$

is a continuous map.

Main Theorem

$X \in Top$. The functors



is an equivalance of categories.

$Sec : Et/X \longrightarrow Sh(X)$

$T \xrightarrow{P} X \longmapsto (U \mapsto \{U \xrightarrow{s} T : Ps = 1_U\})$

$T \xrightarrow{f} T'$, $P \searrow X \swarrow P'$ $\longmapsto$ ($U \mapsto$ $\{U \xrightarrow{s} T : Ps = 1_U\}$ $\downarrow$ $\overset{s}{\downarrow}_{fs}$ $\{U \xrightarrow{s'} T' : P's' = 1_U\}$)

$Sec Sh(P) \Rightarrow Sec Sh(P')$

naturality

$U \subseteq V$ $\quad Sec Sh(P)(V) \xrightarrow[\text{to } U]{\text{restriction}} Sec Sh(P)(U)$

$U \overset{c}{\subseteq} V$ $\quad \{V \xrightarrow{s} T : Ps = 1_V\} \quad \{U \xrightarrow{s} T : Ps = 1_U\}$

$\overset{s}{\downarrow}_{fs}$ $\quad \downarrow$ $\quad \circlearrowright$ $\quad \downarrow_{sc}$

$Sec Sh(P')(V) \xrightarrow[\text{to } U]{\text{restriction}} Sec Sh(P')(U)$

Above theorem provides us with a powerful dictionary that relates etale maps and sheaves. We illustrate this idea by giving two important constructions, one of which is easier to state using sheaves and the other using etale maps.

Pullback and Pushforward of sheaves.

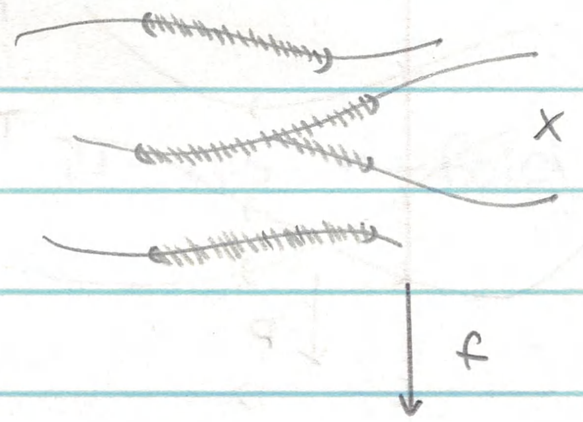
$X, Y \in Top$, $\quad f : X \xrightarrow{cts} Y$

Pushforward functor $f_* : Sh(X) \longrightarrow Sh(Y)$

The Pushforward functor sends a sheaf $F \in Sh(X)$ to the sheaf $f_*(F) \in Sh(Y)$ defined as

$$\begin{array}{ccc} Sh(X) & \xrightarrow{f_*} & Sh(Y) \\ \downarrow F & & V \subseteq Y \end{array}$$

$$f_*(F)(V) = F(f^{-1}(V))$$

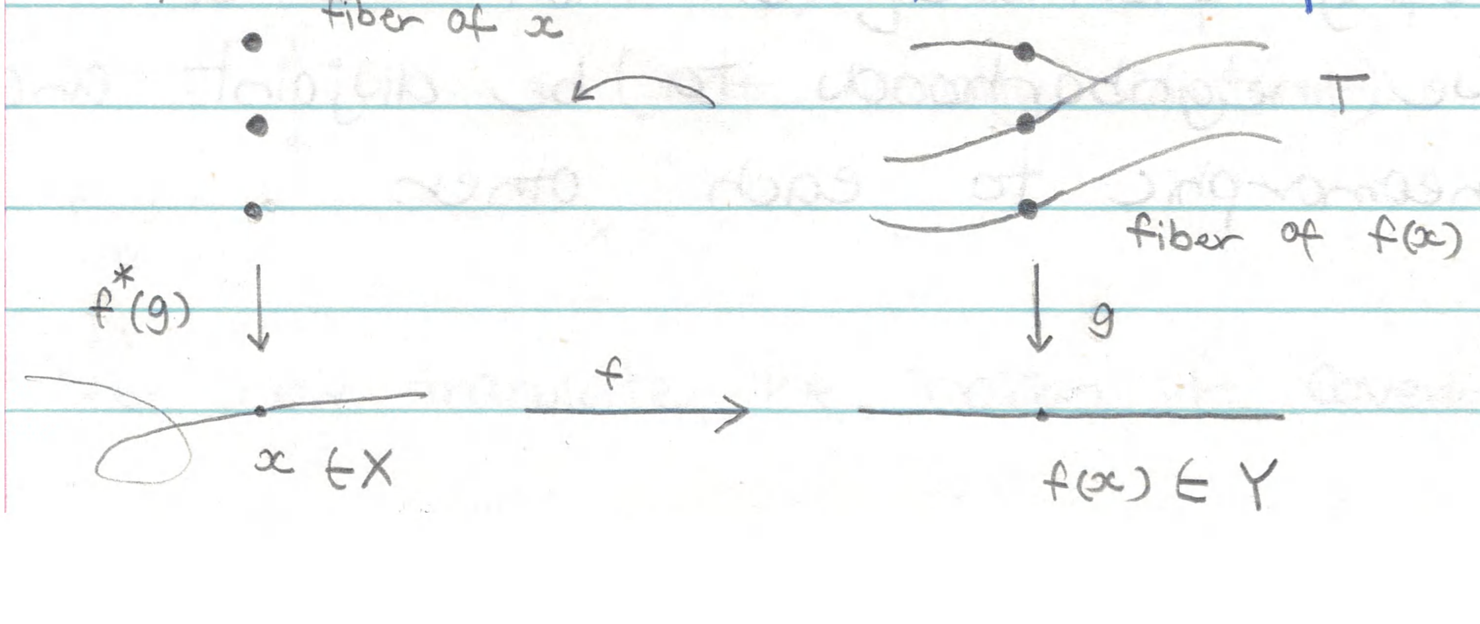


The Pushforward functor $Et/X \longrightarrow Et/Y$ is defined using the above equivalance.

The pullback functor $f^*: Et/Y \longrightarrow Et/X$ send an etale map $g: T \to Y$ to the etale map $f^*(g): T \underset{Y}{\times} X \longrightarrow X$, where

$T \underset{Y}{\times} X = \{(t, x) : g(t) = f(x)\}$ fiber product

$$\begin{array}{ccc} T' = T \underset{Y}{\times} X & \longrightarrow & T \\ \downarrow f^*(g) = p & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

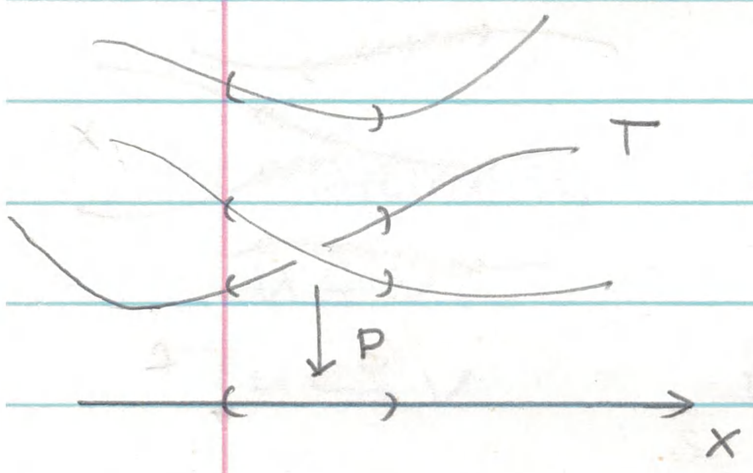


Galois theory of Coverings

Given $X \in Top$, the category Cov/X of covering maps (covering projections) of X is defined as

Objects: $T \xrightarrow{P} X$ continuous and there is a base B of X such that $\forall U \in B$ $P^{-1}(U) \cong U \times F$ (disjoint union of copies of U)

F - discrete set (need not be countable)



$Mor(P, P')$: continuous $T \xrightarrow{f} T'$ such that $P'f = P$

Recall

Cov/X is a full subcategory of Et/X. The definition an etale map $P: T \longrightarrow X$ requires that any point in $P^{-1}(x)$ for some $x \in X$ has an open neighborhood that maps homeomorphically to X. The definition of a covering then require that we can choose these neighborhoods to be disjoint and homeomorphic to each other.

Example

① $\forall X \in \mathrm{Top}$ $\forall$ open covering $\{U_i\}$ of X

$\coprod_i U_i \xrightarrow{\text{Projection}} X$

$U_i \xhookrightarrow{\text{inclusion}} X$

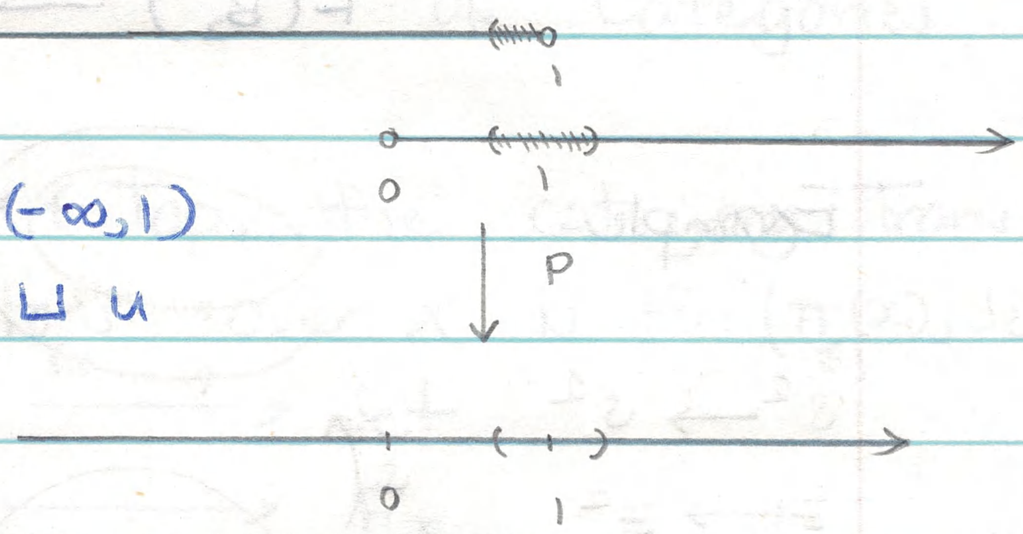
If every U_i is a union of connected components of X, then P is a covering map.

② $S^1 \longrightarrow S^1$ given by $x \longmapsto x^n$ is an n-fold covering map.

③ $(-\infty, 1) \coprod (0, \infty) \xrightarrow{\text{Projection}} (-\infty, \infty)$ is not a covering map

$\forall$ nbhd U $\quad P^{-1}(U) = U \cap (-\infty, 1) \coprod U$

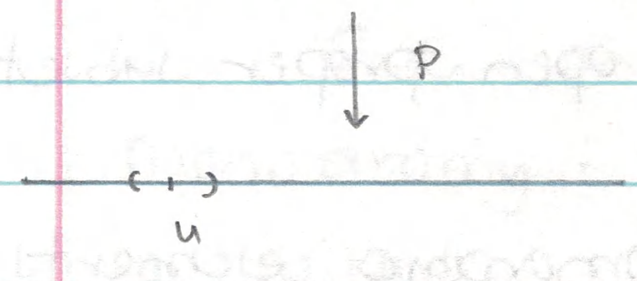
$\not\cong U$



④ $\neq$ (different U for different points in the fiber)

$\in \mathrm{Et}/X$ but $\notin \mathrm{Cov}/X$

(not locally constant)



We now translate the notion of covering spaces

into the language of sheaves.

Def ≥

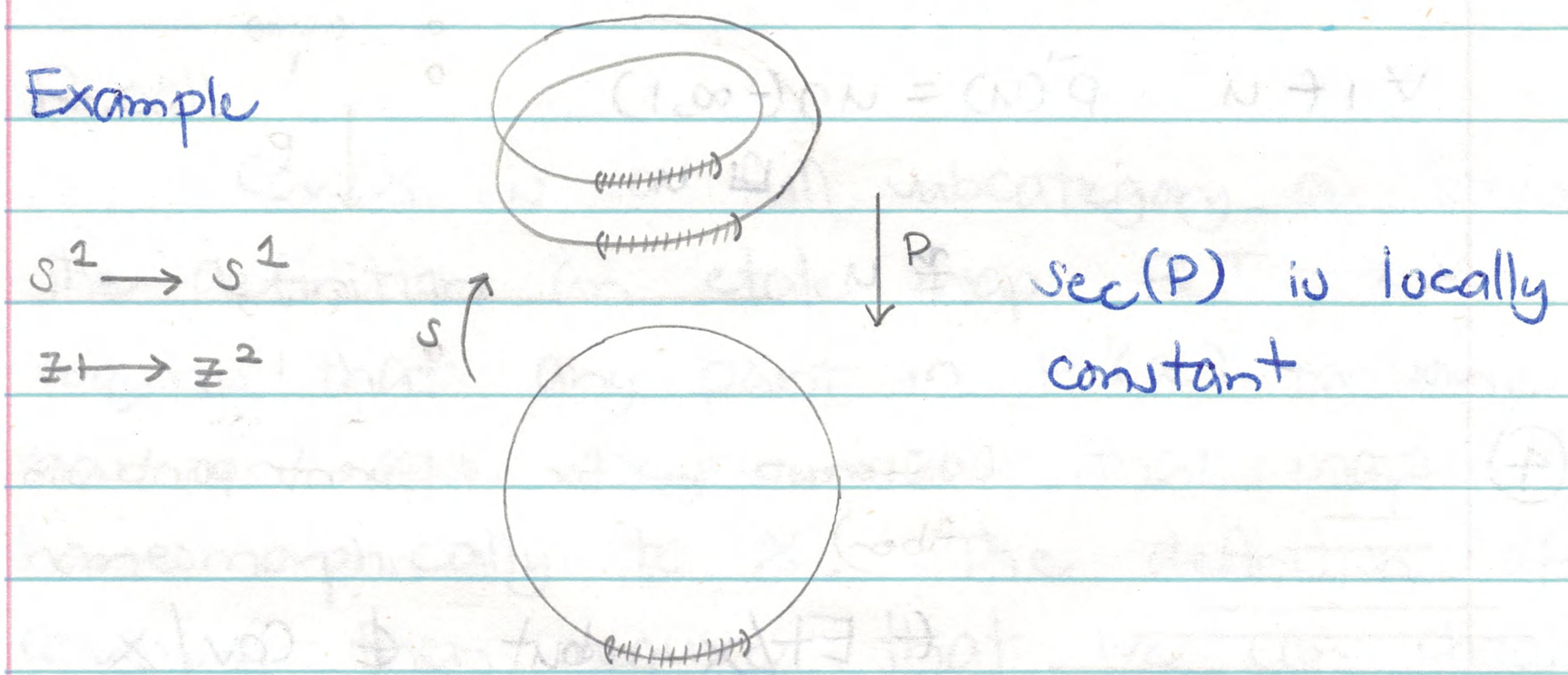
LCSh(X) of locally constant sheaves over X is the full subcategory of Sh(X) consisting of locally constant sheaves over X.

$F: Open(X)^{op} \longrightarrow Set$ $F \in Sh(X)$ is locally constant if $\exists$ a base B of X such that $F|_B$ (a full subcategory of $Open(X)$) is a constant functor.

i.e. $\forall B_1, B_2 \in B$ such that $B_1 \subseteq B_2$

$$F(B_2) \xrightarrow{\cong} F(B_1)$$

Example



Sec(P) is locally constant

$B = (a,b) \subseteq S^1$, connected open proper subsets of S^1

$\forall B \in B$: $F(B)$ has 2 isomorphic elements.

$\therefore F \notin LCSh(S^1)$

Proposition

The functors Sec and Et restricted to the full subcategories Cov/X of Et/X and $LCsh(X)$ of $sh(X)$ are form an equivalance of these full subcategories.

$$\begin{array}{ccc} Cov/X & \hookrightarrow & Et/X \\ {\scriptstyle Sec}\downarrow & & \uparrow{\scriptstyle Et} \\ LCsh(X) & \hookrightarrow & sh(X) \end{array}$$

is an equivalance of categories.

Def^n

Given an $X \in Top$, the category $Trans(X)$ of transport functors over X is $Fun(\pi_{\leq 1}(X), Set)$.

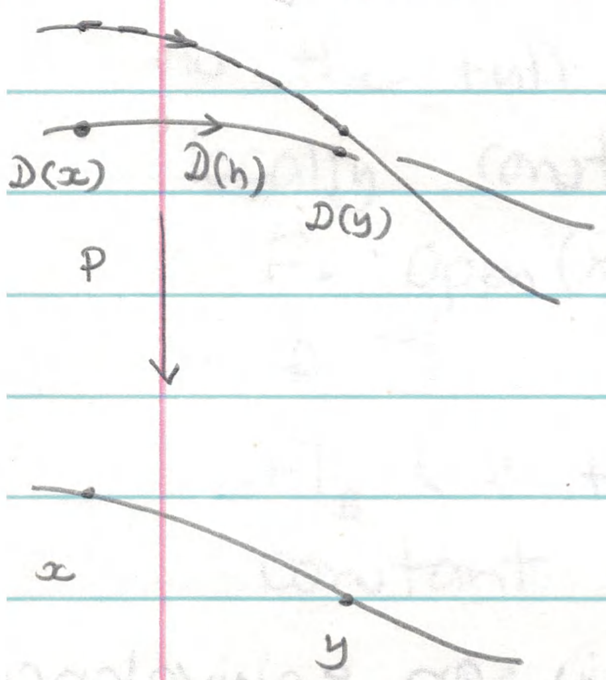
$$F: \pi_{\leq 1}(X) \longrightarrow Set$$

$$x \longmapsto F_x$$

$$x \xrightarrow{\text{path}} y \longmapsto F_x \xrightarrow{\text{function}} F_y$$

Def^n

Give an object $X \in Top$, the functor monodromy: $Cov/X \longrightarrow Trans(X)$ is defined as follows



$$T \xrightarrow{p} X \qquad \text{Monodromy } (p)$$

$$D : \pi_1(X) \longrightarrow \text{Set}$$

$$x \in X \longmapsto D_p(x) = p^{-1}(\{x\}) \quad \text{fiber of } x$$

$$x \longrightarrow y \longmapsto D(h) : D(x) \longrightarrow D(y)$$

$$h : [0,1] \xrightarrow{\text{ctu}} X \qquad p^{-1}(\{x\}) \longrightarrow p^{-1}(\{y\})$$

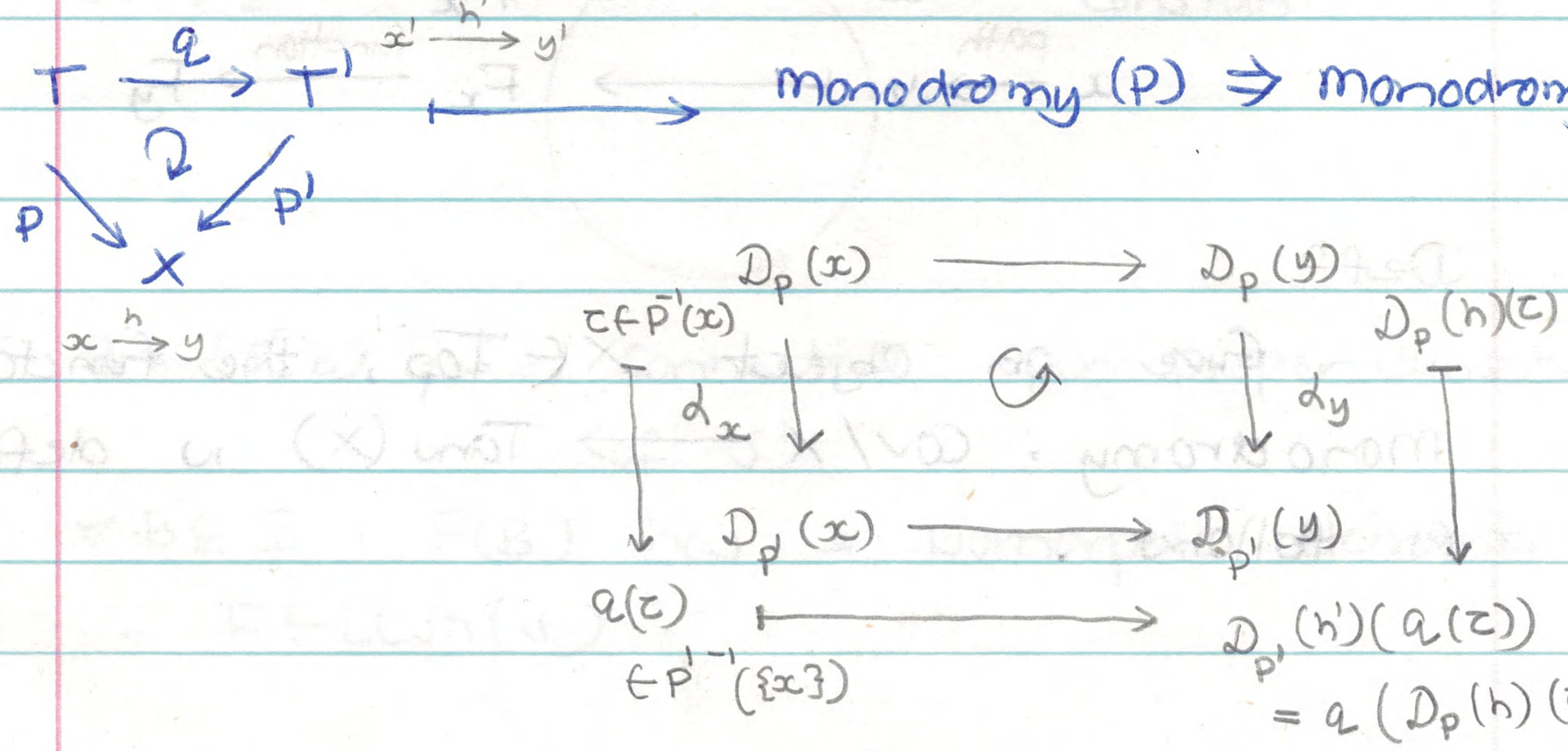
$$h(0) = x \qquad \tau \longmapsto D(h)(\tau)$$

$$h(1) = y \qquad \exists!\ h' : [0,1] \longrightarrow T$$

(homotopy equivalance class of paths) such that $p \circ h' = h$ and $h'(0) = \tau$

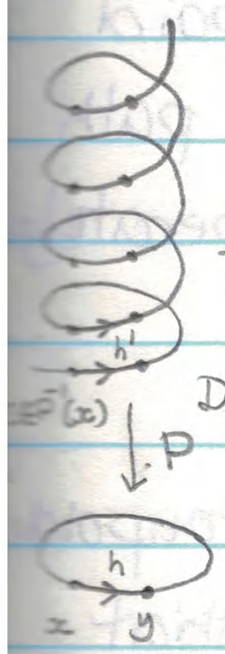
$$h'(1) = D(h)(\tau)$$

the definition of a covering space is designed in such a way as to guarantee that h' is unique and the map $p^{-1}(\{x\}) \longrightarrow p^{-1}(\{y\})$ does not depend on the choice of h in the equivalance class.



Def ≈

Given $X \in Top$, the reconstruction functor $Recon : Trans(X) \longrightarrow Cov/X$ sends a transport functor $F: \pi_{\leq 1}(X) \longrightarrow Set$ to the covering space $P: T \longrightarrow X$ defined as follows:



The underlying set of T is the disjoint union of sets $F(x)$ $\forall x \in X$ which is equipped with a canonical map to X.

A base of open subsets of T is constructed by taking any $x \in X$, $w \in F(x)$ and $U \in Open(X)$ such that $x \in U$ the set $\{F(h)(w) \mid h: [0,1] \longrightarrow U\}$

A natural transformation $F \Rightarrow G$ induces a map $T_F \longrightarrow T_G$, which turns out to be continuous.

To exclude pathologies we work in the category GoodTop of locally path connected and semi locally simply connected topological spaces.

Theorem

For $X \in GoodTop$, Cov/X $\underset{Recon}{\overset{Monodromy}{\rightleftarrows}}$ $Trans(X)$ form an equivalance of categories.

In particular both categories are equivalent to $LCSh(X)$.

Recall that any groupoid is equivalent to a disjoint union of groupoids of the form BG for some groups G. In the case of groupoid $\pi_{\leq 1}(X)$ the disjoint union is indexed by path components of X, which we can study seperately

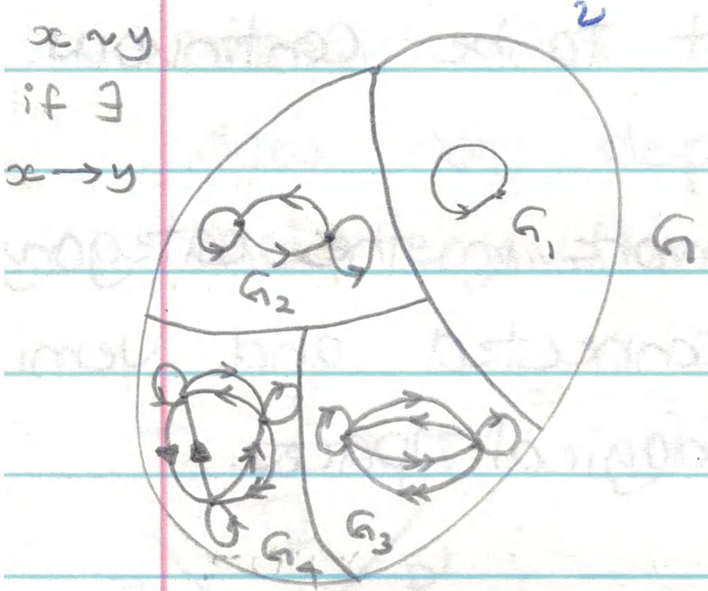
Proposition

For any groupoid G $\exists!$ collection of groupoids $\{G_i\}$: full subcategories of G such that

(i) $ob(G_i) \cap ob(G_j) = \phi$ for $i \neq j$

(ii) G_i is connected i.e $\pi_0(G_i) = \{*\}$
(there is only one isomorphism class)

(iii) $\bigcup_i ob(G_i) = G$



If $G = \pi_{\leq 1}(X)$ for some $X \in Top$, then $G_i = \pi_{\leq 1}(X_i)$, where $X = \bigsqcup_i X_i$ are path components of X.

Thus for the sake of brevity we consider only connected groupoids.

i.e. BG for some group G (which is $\pi_1(X, *)$ in the case of a path connected space X and some base point $* \in X$).

$G \curvearrowright X$ $\qquad$ $\varphi: G \times X \longrightarrow X$

$\rho: G \longrightarrow \Sigma_X$ $\qquad$ $(g, x) \longmapsto gx$

$g \longmapsto \rho_g$

$x \longmapsto gx$

permutational representation

such that $ex = x \quad \forall x \in X$

$g_1(g_2 x) = (g_1 g_2)x \quad \forall g_1, g_2 \in G$

In this case the category $\mathrm{Fun}(BG, \mathrm{Set})$ is equivalent to the category $G\mathrm{Set}$.

Def^n

Given $G \in \mathrm{Group}$, we have the category $G\mathrm{Set}$ of sets with a G-action.

Objects : (S, ρ), $S \in \mathrm{Set}$ (ρ: action of G on S)

$G \xrightarrow{\rho} \Sigma_S$ group homomorphism

i.e. every element of G acts on S via a permutation on S.

morphism : $(S, \rho) \xrightarrow{f} (S', \rho')$ are functions $S \xrightarrow{f} S'$ such that $f(\rho(g)(s)) = \rho'(g)(f(s))$

i.e. f commutes with the action of G such that $g f(s) = f(gs)$.

The equivalance is implemented by the functor

$\mathrm{Fun}(BG, \mathrm{Set}) \longrightarrow G\mathrm{Set}$

$F \longmapsto (F(*), \rho)$, where $* \in BG$

and

$\rho: \mathrm{Mor}_{BG}(*, *) \longrightarrow \mathrm{Mor}_{\mathrm{Set}}(F*, F*)$

$(G) \qquad\qquad (\Sigma_{F*})$

$f \longmapsto F(f)$

Proposition

$\forall G$ - groupoid $\exists X \in Top$: $G \cong \pi_{\leq 1}(X)$

Proposition

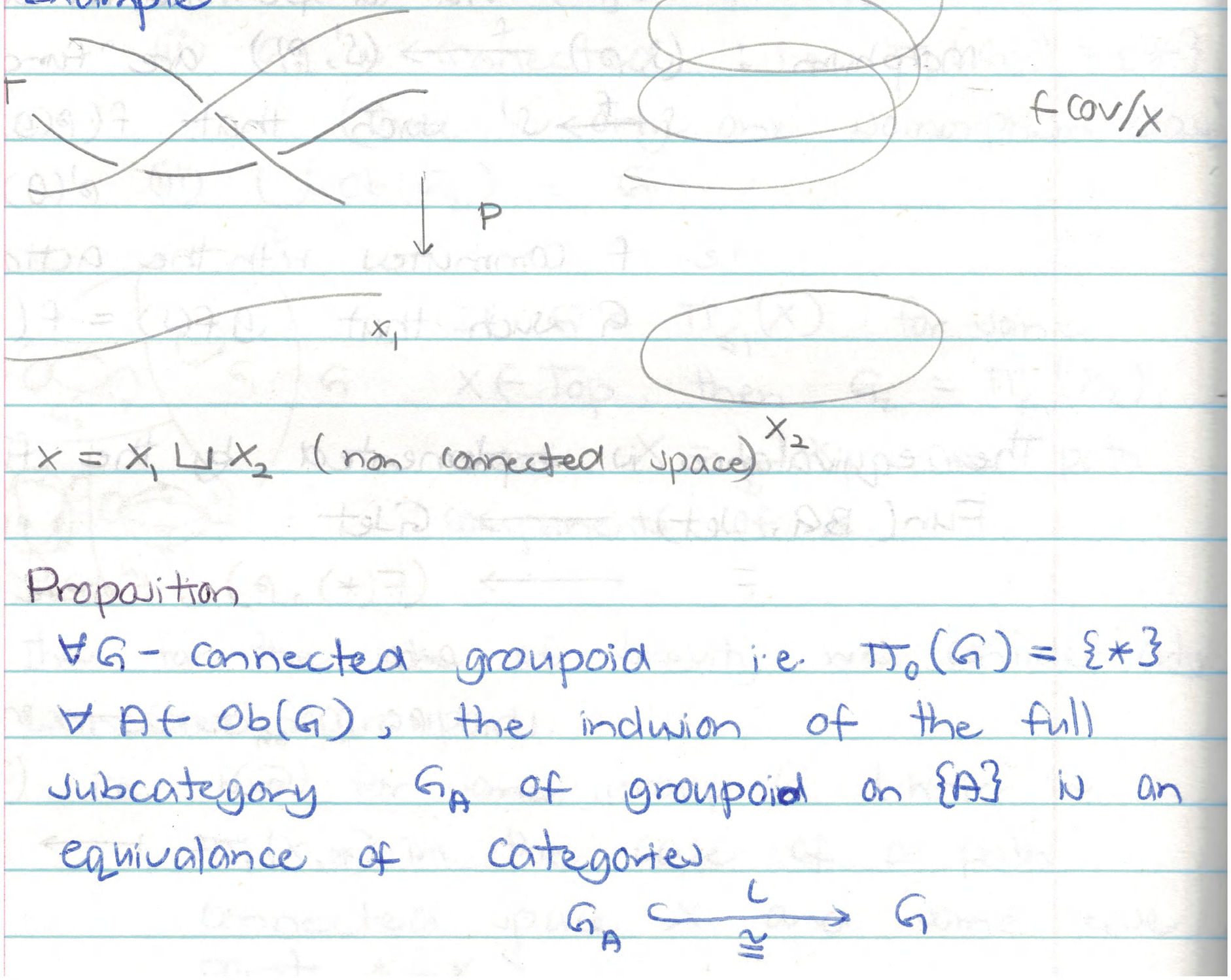
$Fun(\pi_{\leq 1}(X), Set) = \prod_i Fun(\pi_{\leq 1}(X_i), Set)$

$\forall\ x \in X \quad F(x) \in Set \qquad \exists!\ i : x \in X_i \subseteq X$

$\forall\ h : [0,1] \xrightarrow{\text{path}} X$

$F(h) : F(h(0)) \longrightarrow F(h(1)) \qquad \exists!\ i : [0,1] \xrightarrow{h} X_i$ (path component)

Example



$X = X_1 \sqcup X_2$ (non connected space)

Proposition

$\forall G$ - connected groupoid i.e. $\pi_0(G) = \{*\}$

$\forall A \in Ob(G)$, the inclusion of the full subcategory G_A of groupoid on $\{A\}$ is an equivalance of categories

$$G_A \xrightarrow[\cong]{L} G$$

proof :

c is fully faithful by construction
(any full subcategory has this property).
Also c is essentially surjective as

$$\begin{array}{ccc} \pi_0(G_A) & \longrightarrow & \pi_0(G) \\ \| & & \| \\ \{*\} & & \{*\} \end{array}.$$

Hence any connected groupoid is (non canonically) equivalent to a groupoid with a single object.

Proposition

$\forall H$-group $\exists!$ groupoid BH with a single object $*$ such that $\mathrm{Mor}_{BH}(*,*) = H = \mathrm{Aut}_G(*)$

Composition $\circ$ = $\cdot$ (multiplication) in H

1_* = identity of H

Thus

B: Group $\longrightarrow$ Groupoid is a functor

$$H \longmapsto BH$$

$$H_1 \xrightarrow{\varphi} H_2 \longmapsto BH_1 \xrightarrow{B\varphi} BH_2$$

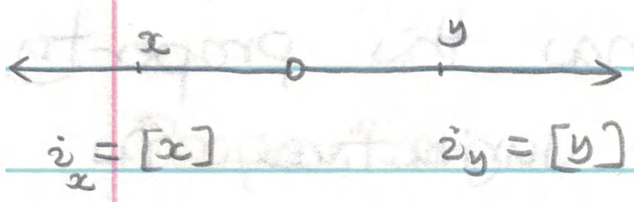
Corollary

$\forall$ groupoid G,

$$G \overset{\text{noncanonical}}{\cong} \coprod_{[x] \in \pi_0 G} B\mathrm{Aut}_G(x), \quad x \in \mathrm{Ob}(G)$$

Example : $X \in \mathrm{Top}$

$$\pi_{\leq 1}(X) \cong \coprod_{[x] \in \pi_0(X)} B\pi_1(X, x)$$



$\in \pi_0(X)$ Assume X is connected i.e. $\pi_0(X) = \{*\}$,

if we choose some base point $x \in X$,

then $\pi_{\leq 1}(X) \cong B\pi_1(X, x)$

↖ canonically.

Recall

$\forall G \in \mathrm{Group}$

$$\mathrm{Fun}(BG, \mathrm{Set}) \cong G\text{-Set}$$

ob: (X, ρ) ↖ Set ↖ action of G on X

$\rho : G \times X \longrightarrow X$

$(g, x) \longmapsto \rho(g, x) = gx$

mor: $(X, \rho) \longrightarrow (X', \rho')$

function $X \xrightarrow{f} X'$ such that

$f(g, x) = g f(x)$

proof

$F \longmapsto (F(*), \rho)$

$BG \to \mathrm{Set}$

$\mathrm{Mor}_{BG}(*, *) \xrightarrow{\rho} \mathrm{Mor}_{\mathrm{Set}}(F*, F*)$

$\mathrm{Aut}_G(*) = G \longrightarrow \Gamma_{F*}$

Other direction

$(X, \rho) \longmapsto F$

$BG \longrightarrow \mathrm{Set}$

$* \longmapsto F(*) = X$

$* \xrightarrow{g \in G} * \longmapsto F(h) : X \longrightarrow X$, $x \longmapsto hx$

Transitive action: $\forall x, y \in X \ \exists g \in G$ such that $y = gx$ (single orbit)

If $X \in \mathrm{GoodTop}_*$ with $\pi_0(X) = \{*\}$, $\mathrm{Trans}(X) = \mathrm{Fun}(G, \mathrm{Set})$ is canonically equivalent to $G\mathrm{Set}$, where $G = \pi_1(X, x)$ for some base point $x \in X$.

Proposition

$$\begin{array}{ccc} \mathrm{LCov}(X) & \underset{\text{Sec}}{\overset{\text{Et}}{\rightleftarrows}} & \mathrm{Cov}/X \\ \text{Monodromy} \downarrow & & \uparrow \text{Recon} \\ \mathrm{Trans}(X) = \mathrm{Fun}(\pi_{\leq 1}(X), \mathrm{Set}) & \simeq & \end{array}$$

$$\simeq \quad \pi_1(X, x) - \mathrm{Set}.$$

(the Galois theory of covering spaces)

Coverings of a nonconnected $X \in \mathrm{GoodTop}$ can be treated separately for each connected component of X, so we lose no generality by assuming X is connected. Furthermore, for any covering $p: T \to X$ we may decompose T into its connected components. Such a decomposition corresponds (in the category $G\mathrm{Set}$) to the decomposition of a G-set in to a disjoin union of transitive G-sets.

i.e. Any group action is decomposible into a disjoint union of transitive actions.

Again, we lose no generality if we assume T is connected.

thus, the category of connected coverings of a connected base X is equivalent to the category $G\text{Set}^{\text{Trans}}$ of transitive G-sets, where $G = \pi_1(X, x)$ once we choose a basepoint $x \in X$.

i.e. $\forall\ (X, \rho) \in G\text{-Set}$

$$(X, \rho) = \coprod_{i \in \pi_0 X} \underbrace{(X_i, \rho_i)}_{\text{transitive}}$$

$$\cong \coprod_{G_i \leq G} G/G_i \quad \text{(fixed on } G_i\text{)}$$

Propasition (Existance of universal covering space)

Any $X \in \text{GoodTop}$ admits a universal covering which is an object $P \in \text{Cov}/X$ such that $\pi_0(P): \pi_0(T) \to \pi_0(X)$ is an isomorphism and whose total space T has a property that any continuous map $S^1 \to T$ extends to a continuous map $D^2 \to T$

(i.e. if X is path connected, T is simply connected).

proof

It is suffices to treat the case $\pi_0(X) = \{*\}$ so $\pi_{\leq 1}(X)$ is noncanonically isomorphic to BG for some G which can be taken to be the fundamental group of X with respect to some

basepoint of X. In this case categories $\mathrm{Trav}(X)$ and $G\text{-}\mathbf{Set}$ are equivalent. The latter category contains the left regular action of G (on itself).

i.e. $G \curvearrowright G$

$$\begin{array}{ccc} G \times G & \xrightarrow{\text{gp action}} & G \\ (g, h) & \longmapsto & gh \ (\text{gp multiplication}) \end{array}$$

This G-set gives us an object in $\mathrm{Trav}(X)$, hence an object in $\mathbf{Cov}/X$, which is the universal cover of X.

Note

$X \in \mathbf{GoodTop}$

$$\begin{array}{ccc} \mathrm{Fun}(\pi_{\leq 1}(X), \mathbf{Set}) & \underset{\text{total space}}{\overset{\text{fibers}}{\rightleftarrows}} & \mathbf{CovGrpd}/\pi_{\leq 1}(X) \\ \text{stalks} \uparrow & & \pi_{\leq 1} \uparrow \quad \downarrow \text{topologize} \\ \mathrm{Fun}(\mathrm{Open}(X)^{op}, \mathbf{Set}) & \xleftarrow{\text{sections}} & \mathbf{Cov}/X \end{array}$$

Fork
Cone } diagram

Limits and Colimits

Initial object and Terminal object

(i) An object $I \in \mathcal{C}$ is initial if for every $c \in \mathcal{C}$ there is a unique morphism $I \to c$

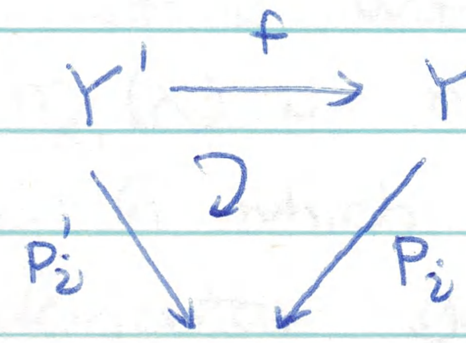
(ii) Dually, an object $T \in \mathcal{C}$ is terminal if for every $c \in \mathcal{C}$ there is a unique morphism $c \to T$

Def^n

Given a category $\mathcal{C}$ and a family $\{X_i\}_{i \in I}$ of objects in $\mathcal{C}$, the product of $\{X_i\}_{i \in I}$ (if it exists) is the following collection of data that satisfies the following properties:

- an object $Y \in \mathcal{C}$ (usually denoted by $\prod_{i \in I} X_i$)
- for each i, a projection morphism $P_i : Y \to X_i$

such that for any object $Y' \in \mathcal{C}$ and a family of morphism $\{P_i' : Y' \to X_i\}_{i \in I}$ there is a unique morphism $f : Y' \to Y$ satisfying



$P_i \circ f = P_i' \quad \forall i \in I$

: Universal property.

Example

$I = \{0, 1\}$ $\quad\quad$ $\mathcal{C} = Set$

Then $X_0, X_1 \in Set$ (objects)

$Y = X_0 \times X_1$ (binary or Cartesian product)

$= \{(x_0, x_1) : x_0 \in X_0,\ x_1 \in X_1\}$

set of all ordered pairs of elements of X_0 and X_1.

$$P_0((x_0, x_1)) = x_0$$

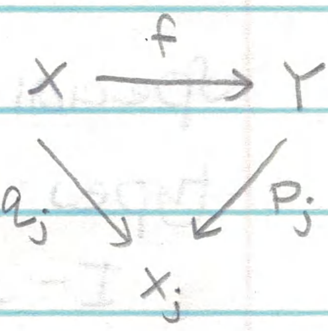
$$P_1((x_0, x_1)) = x_1$$

$\forall X \in Set \quad \forall\, q_0 : X \longrightarrow X_0$

$q_1 : X \longrightarrow X_1$

$\exists!\ f : X \longrightarrow Y = X_0 \times X_1$

$$\left.\begin{aligned} P_0 \circ f &= q_0 \\ P_1 \circ f &= q_1 \end{aligned}\right\} \Leftrightarrow \quad f(x) = (q_0(x), q_1(x)) \quad \forall\, x \in X$$



So, f exist and unique.

Recall

$(x_0, x_1) = \{\{\{x_0\}, \phi\}, \{x_1\}\} \in 2^{2^{X_0 \cup X_1}}$: Wiener

$= \{\{x_0\}, \{x_0, x_1\}\} \in 2^{X_0 \cup X_1}$: Kuratowski

Both of these definitions follow the following property of ordered pairs

$$(a, b) = (a', b') \Leftrightarrow a = a' \text{ and } b = b'.$$

This allows us to define

$X_0 \times X_1 = \{ z \in 2^{2^{X_0 \cup X_1}} : \exists\ x_0, x_1$ such that

$z = (x_0, x_1)$ and

$x_0 \in X_0,\ x_1 \in X_1 \}$

↑ by Kuratowski definition

Thus the existance of binary products of sets follows from Zermelo - Frankel axioms

(i) axiom of separation (comprehension)

$\exists \{x|P\} = \{a \in X : P(a)\}$

(ii) axiom of Power set

$\exists 2^X$

(iii) axiom of pairing

$\exists A \cup B$

(iv) axiom of extensionality

$\forall A \forall B (\forall X (X \in A \Leftrightarrow X \in B) \Rightarrow A = B)$

Special indexing classes I give rise to special types of products.

I- set (as opposed to a class) : small products

I- finite, infinite, countable, uncountable

: finite, infinite, countable, uncountable product

$I = \{0,1\}$: binary product

$I = \{*\}$: $X_* \xrightarrow{P_*} X_*$ is an isomorphism

$I = \phi$: terminal object

If the indexing class I is empty then the def^n of an I- indexed product says

"product of an empty family is an $Y \in \mathcal{C}$ such that for any $Y' \in \mathcal{C}$ $\exists! Y' \to Y$ "

Recall

The terminal object $1 \in \mathcal{C}$ is such that $\forall X \in \mathcal{C}$ $\exists!$ $f: X \to 1$.

Example

① $\mathcal{C} = \mathbf{Set}$ $\quad 1 = \{*\}$

Any singleton set is terminal.

② Top $\quad 1 = \{*\}$

Singleton space is terminal

③ Top/X ; $X \in$ Top (fixed)

ob: $Y \xrightarrow{f} X$

mor: $Y \xrightarrow{h} Z$, $f = g \circ h$, $g: Z \to X$

$X \xrightarrow{1_X} X$ is the terminal object

④ $\mathbf{Vect}_{\mathbb{K}}$ $\quad 1 = \{0\}$: Zero vector space

⑤ Cat $\quad 1 =$ $*$ ⟲ 1_* is terminal object

Proposition

Any pair of terminal objects are isomorphic up to a unique isomorphism.

i.e. $\mathcal{C}$: category $1, 1'$: terminal objects of $\mathcal{C}$

$\exists!$ $f: 1 \to 1'$

$g: 1' \to 1$ such that $gf = id_1$

$fg = id_{1'}$

proof

$gf: 1 \to 1$ is unique and $id_1: 1 \to 1$

$\therefore$ 1 and 1' are canonically (f and g are unique) isomorphic.

This is the reason why we talk about the terminal object. Although the terminal object is not unique, it is unique up to a unique isomorphism which is all what we really care about. For instance, in Group the terminal object is the trivial group with one element. Even though there are more than one such group (because there are many sets with one element) they are all uniquely isomorphic to each other.

There is no reason for terminal object to exist. For example Field has no terminal object.

Def^{n}

Let $\mathcal{C}$ be a category and $\{X_i\}_{i\in I}$ be an I-indexed family of objects in $\mathcal{C}$.

$\text{Cone}_{\{X_i\}_{i\in I}}$ object : (Y, P_i)

$Y \in Ob(\mathcal{C})$ and $P_i : Y \to X_i$

morphism : $(Y', P_i') \xrightarrow{f} (Y, P_i)$

$Y' \xrightarrow{f} Y$ such that

$P_i' = P_i f \quad \forall i \in I.$

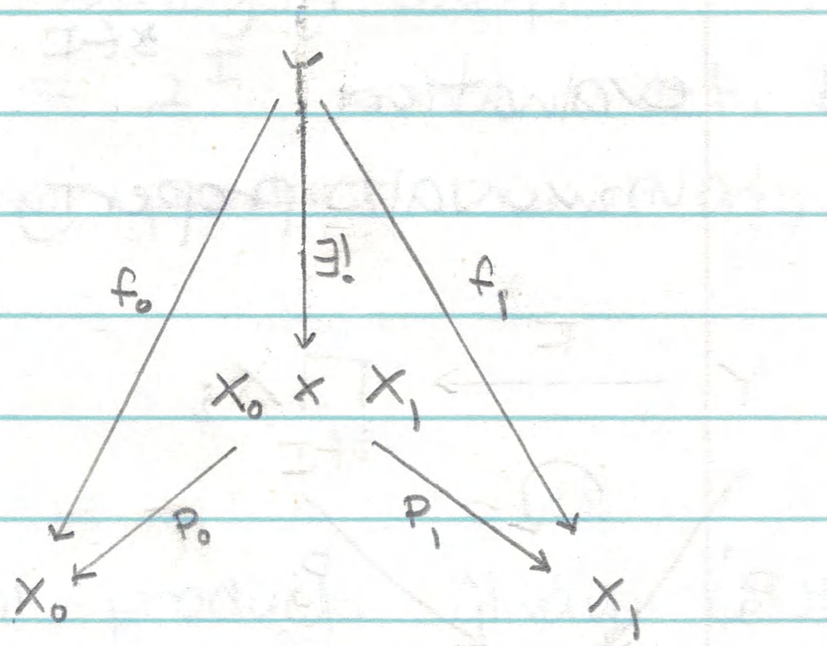
The product of $\{X_i\}_{i\in I}$, if it exists, is the terminal object in this category.

Corollary

Product of $\{X_i\}_{i\in I}$ is unique upto a unique isomorphism.

Example

① Set



(binary product)

The case of finite products of sets can be treated similarly to the case of binary products.

the product of $\{X_i\}_{0 \leq i \leq n-1}$ is denoted by $X_0 \times X_1 \times \dots \times X_{n-1}$ can be written as $\{(x_0, x_1, \dots, x_{n-1}) : x_i \in X_i\}$ and the projection map P_i sends $(x_0, x_1, \dots, x_{n-1})$ to x_i. Of particular interest is the case $n=0$: the product of the empty family of sets is a set containing a single element, namely the empty tuple which is the terminal object in the category of sets.

Infinite small products (not over a proper class). Given a family $\{X_i\}_{i \in I}$ of sets we get

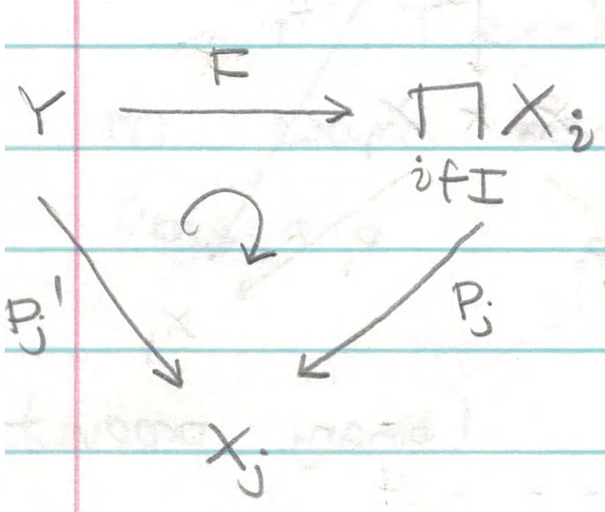
$$\prod_{i \in I} X_i = \left\{ f : I \longrightarrow \bigcup_{i \in I} X_i \;\middle|\; \forall i \in I : f(x_i) \in X_i \right\}$$

The projection map

$$P_j : \prod_{i \in I} X_i \longrightarrow X_j$$ are given by evaluation $f \longmapsto f(j)$

universal property:

Given a family of function $P_j' : Y \longrightarrow X_j$, the unique function $F : Y \longrightarrow \prod_{i \in I} X_i$ such that $P_j' = P_j F$ is given by $F(y)(j) = P_j'(y)$



(i) If $X_i \neq \phi \ \forall i \in I$, then $\prod_{i \in I} X_i \neq \phi$ if and only if the axiom of choice satisfied for the family $\{X_i\}_{i \in I}$

(ii) If $X_i = A \quad \forall i \in I$, then

$$\prod_{i \in I} X_i = \{f : I \longrightarrow A\} = A^I$$

In particular,

if $I = \{0,1\}$, $A^2 = A^{\{0,1\}} = A \times A$

if $A = \{0,1\}$ $\prod_{i \in I} 2 = 2^I = \mathcal{P}(I)$

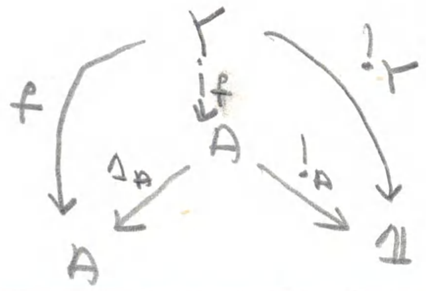
(iii) If $X_i = \phi$ for some $i \in I$, then $\prod_{i \in I} X_i = \phi$.

If the indexing family I is a proper class, the product $\prod_{i \in I} X_i$ of an I-indexed family $\{X_i\}_{i \in I}$ need not always exist.
For instance, $\prod_{i \in I} 2 = 2^I$ is not a set, but a proper class. So, this product exist only when I is a set.

(2) Groups
In this category small products always exists.
Observation

$$\{\mathbb{Z} \xrightarrow{f} G\} \cong G$$

$$f \longmapsto f(1)$$



$\Rightarrow A \cong A \times \mathbb{1}$

Thus $\mathbb{1}$ act like a unit

Consider a family $\{G_i\}_{i \in I}$ of groups with the underlying sets $\{U(G_i)\}_{i \in I}$, where U: Group $\longrightarrow$ Set is the forgetful functor. By the universal property of products, if the product $\prod_{i \in I} G_i$ exist, must satisfy $U(\prod_{i \in I} G_i) \cong \prod_{i \in I} U(G_i)$. Using the same trick with homomorphisms from $\mathbb{Z}$

$$\{\mathbb{Z} \longrightarrow \prod_{i \in I} G_i\} \cong \prod_{i \in I} \{\mathbb{Z} \longrightarrow G_i\},$$

we see that $U(P_j)$ must be the projection map $\prod_{i \in I} U(G_i) \longrightarrow U(G_j)$. The projection maps $\{P_i\}_{i \in I}$ are group homomorphisms and this forces the group operations on $\prod_{i \in I} U(G_i)$ to be defined indexwise.

$I = \{0, 1\}$

$$\{\mathbb{Z} \longrightarrow G_0 \times G_1\} \overset{\text{by universal property}}{\cong} \left\{ \begin{matrix} \mathbb{Z} \longrightarrow G_0 \\ \mathbb{Z} \longrightarrow G_1 \end{matrix} \right\}$$

$$\cong U(G_0) \times U(G_1)$$

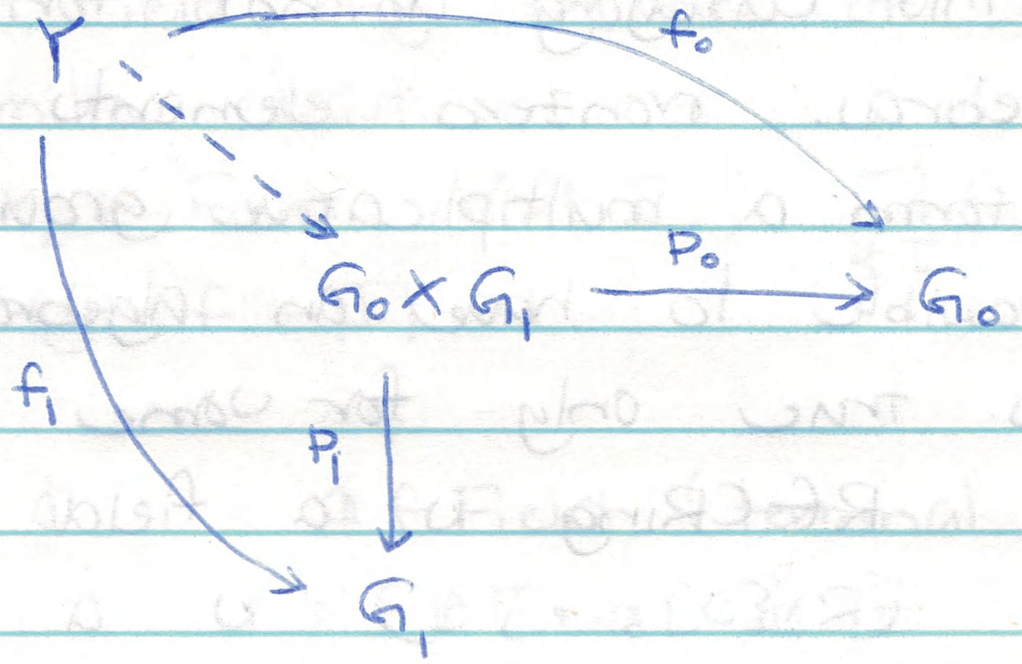
$$\begin{array}{ccc} G_0 \times G_1 & \xrightarrow{P_1} & G_1 \\ \downarrow{\scriptstyle P_0} & & \\ G_0 & & \end{array}$$

$$\begin{aligned} P_0((g,h) \cdot (g',h')) &= P_0((g,h)) P_0((g',h')) \\ &= gg' \\ &= P_0((gg', hh')) \end{aligned}$$

Same holds for P_1. Thus P_0 and P_1 are group homomorphisms.

In Ab, both products and coproducts are isomorphic, and called direct sum.

Universal property:



$$Y \longrightarrow G_0 \times G_1$$
$$y \longmapsto (f_0(y), f_1(y))$$

$yy' \longmapsto (f_0(yy'), f_1(yy'))$ this is a gp homo^{m}.

$= (f_0(y) f_0(y'), f_1(y) f_1(y'))$

$= (f_0(y), f_1(y))(f_0(y'), f_1(y'))$

Any pair of group homomorphisms from Y to G_0 and G_1 uniquely factor through $G_0 \times G_1$.

nothing in this example is specific to Groups. same argument works for monoids, rings, modules, vector spaces and other algebraic structures that are define using only algebraic identities which are known as varieties of algebras.

(3) Field (no free field generated by a set)

This category do not form a variety of algebras. nonzero elements of a field must form a multiplicative group. and it is not possible to have an algebraic identity that is true only for some elements.

$R \in CRing$ is a field if

$(R \setminus \{0\}, \cdot, 1)$ is a group

$$ch(R) = \begin{cases} p, & p \cdot 1 = 0 \text{ for smallest } p \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\eta} & R \\ 1 & \longmapsto & 1_R \end{array}$$ is a ring homo

$\eta(\mathbb{Z}) = \mathbb{Z}/\ker\eta$ is a subring of a field therefore an integral domain

thus $\ker\eta$ is a prime ideal in $\mathbb{Z}$.

$$\ker\eta = \begin{cases} (p) & \Rightarrow \mathbb{Q} \subseteq R \\ 0 & \Rightarrow \mathbb{Z}/p\mathbb{Z} \subseteq R \end{cases}$$

If $f: R \to R'$ is a maphism in Field, then $R \cong f(R) \subseteq R'$ implies $ch(R) = ch(R')$

$ch(\mathbb{Z}/p\mathbb{Z}) = p$, $ch(\mathbb{Q}) = 0$, $ch(\mathbb{C}) = 0$.



is a category with no products

Propaition

In Field only product of one element familiw exist.

i.e. $X_i \in$ Field $\forall i \in I$, $\prod_{i \in I} X_i$ exist if and only if $|I| = 1$.

proof:

(i) $I = \phi$ $\quad \mathbb{1} \in$ Field (terminal Object)

$\forall F \in$ Field $\exists!$ $F \longrightarrow \mathbb{1}$

choose F such that $ch(F) \neq ch(\mathbb{1})$

for a contradiction.

(ii) $|I| \geq 2$

a) If $ch(X_j) \neq ch(X_k)$ for some j, k

$\{Y \longrightarrow \prod_{i \in I} X_i\} \cong \{Y \longrightarrow X_i\}_{i \in I}$

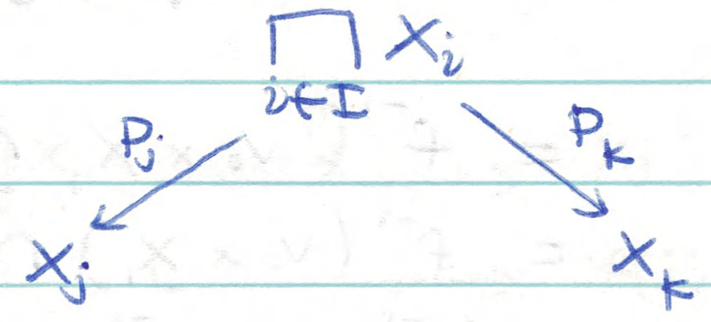
by universal property.

when $Y = \prod_{i \in I} X_i$, 1_Y (identity on Y)

is in LHS, but RHS is empty.

b) Let $ch(X_j) = ch(X_k)$ for all j, k

Assuming $\prod_{i \in I} X_i$ exist, we have following Projection maps



for some $j \neq k$.

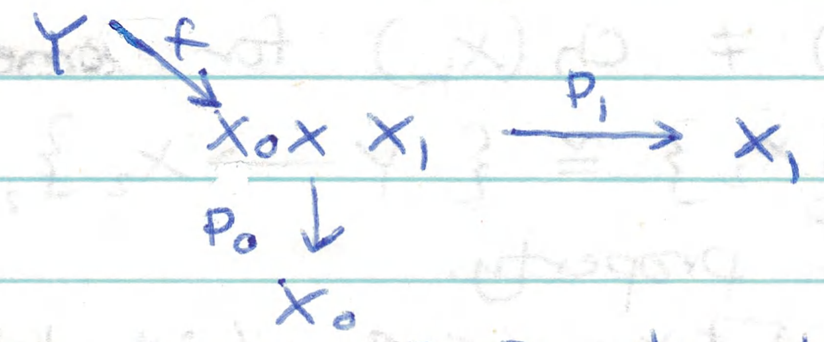
In Field any morphism is a monomorphism (injective) and since

$$U\left(\prod_{i\in I} X_i\right) \cong \prod_{i\in I} U(X_i) \longrightarrow\!\!\!\!\!\rightarrow U(X_i)$$

morphism P_j and P_k are surjective. Hence they are isomorphism of fields.

(b) Top

$X_0, X_1 \in$ Top $\qquad U(X_0 \times X_1) = U(X_0) \times U(X_1)$



(set theoritic projections)

we need P_0, P_1 to be continuous. i.e. $P_0^{-1}(V_0) = V_0 \times$ must be open in $X_0 \times X_1$ $\forall V_0 \in$ Open(X_0). Since same must holds for P_1, $P_0^{-1}(V_0) \cap P_1^{-1}(V_1) = V_0 \times V_1$ should be open. Take $\{V_0 \times V_1\}$ as a basis for product topology on $X_0 \times X_1$.

Universal property : $y \longmapsto (f_0(y), f_1(y))$ is cts.

$\Leftrightarrow f^{-1}(V_0 \times V_1)$ is open in Y.

$= f^{-1}(V_0 \times X_1 \cap X_0 \times V_1)$

$= f^{-1}(V_0 \times X_1) \cap f^{-1}(X_0 \times V_1)$ $\qquad f_0^{-1}(X_0) = f_1^{-1}(V_1) = Y$.

$= (f_0^{-1}(V_0) \cap f_1^{-1}(X_1)) \cap (f_0^{-1}(X_0) \cap f_1^{-1}(V_1))$

$= f_0^{-1}(V_0) \cap f_1^{-1}(V_1)$ is open in Y.

④ $\mathbf{Ban}_1$

Banach spaces with contractive maps. i.e. $\|f\| \leq 1$

Let $\{X_i\}_{i \in I}$, $X_i \in \mathbf{Ban}_1$

Recall

$$\{\mathbb{C} \longrightarrow X\} \xrightarrow{\cong} X_{\leq 1} = \{x \in X : \|x\| \leq 1\}$$
$$f \longmapsto f(1)$$

Therefore if $\prod_{i \in I} X_i$ exist, its unit ball must be isomorphic to product of unit balls. i.e.

$$\left(\prod_{i \in I} X_i\right)_{\leq 1} \cong \mathbf{Ban}_1\left(\mathbb{C}, \prod_{i \in I} X_i\right) \cong \prod_{i \in I} \mathbf{Ban}_1(\mathbb{C}, X_i)$$

(unit ball in product)

$$\cong \prod_{i \in I} (X_i)_{\leq 1}$$

(product of unit balls)

Reconstructing a Banach space from its unit ball, we get that $\prod_{i \in I} X_i$ is a Banach space whose elements are families $\{x_i \in X_i\}_{i \in I}$ such that $i \longmapsto \|x_i\|$ is (uniformely) bounded. And algebraic operations are defined indexwise and the norm of $f \in \prod_{i \in I} X_i$ is defined as

$\|f\| = \sup_{i \in I} \|f(i)\| < \infty$ which is forced upon us by the fact that projection maps $\{P_i\}_{i \in I}$ are contractive maps.

$$\prod_{i \in I} X_i \xrightarrow{P_j} X_j$$
$$f \longmapsto f(j)$$

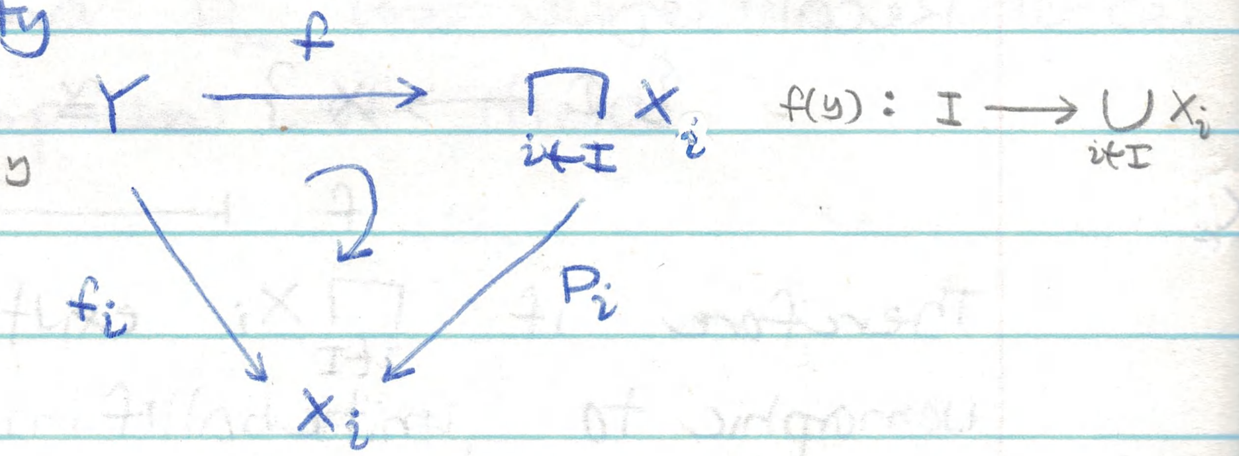
$\|f\| \geq \|f(j)\|$ as P_j is contractive $\forall j$.

Def$^{\underline{n}}$ $\|f\| = \sup_{\|x\|\leq 1} \|f(x)\|$ is better than $\sup_{\|x\|=1} \|f(x)\|$ as it counts the zero vector space $\{0\}$ as a Banach space.

For example

$$X_i = \mathbb{C} \text{ f } \mathrm{Ban}_1 \text{ , then } \prod_{i\in I} X_i = \ell^\infty(I)$$

Universal property



$\{Y \xrightarrow{f_i} X_i\}_{i\in I}$ in Ban_1 $\exists!$ $Y \xrightarrow{f} \prod_{i\in I} X_i$ such that $P_i f = f_i$ $\forall i \in I$

Here $f(y) = (i \longmapsto f_i(y))$

Clearly f is linear and contractive

$$\|f\| = \sup_{\|y\|\leq 1} \|f(y)\| = \sup_{y\in Y, \|y\|\leq 1} \sup_{i\in I} \|f_i(y)\| \leq 1$$

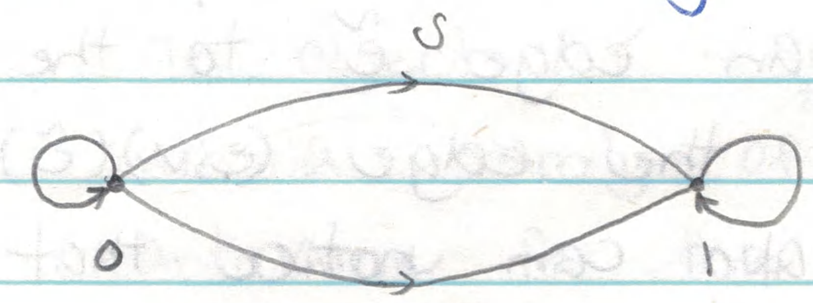
Ban (Banach spaces with bounded linear maps) does have finite products, but does not have infinite products.

(5) Graph

Recall that the category Graph of (directed) graphs has graphs (i.e. pairs of functions $s,t : E \to V$) as objects and homomorphisms of graphs (i.e. functions $v : V \to V'$ and $e : E \to E'$ such that $vs = s'e$ and $vt = t'e$) as morphisms.

$$\begin{array}{ccc} E & \xrightarrow{s,t} & V \\ {\scriptstyle e}\downarrow & \circlearrowright & \downarrow{\scriptstyle v} \\ E' & \xrightarrow{s',t'} & V' \end{array}$$

Consider the following category $\mathcal{C}$

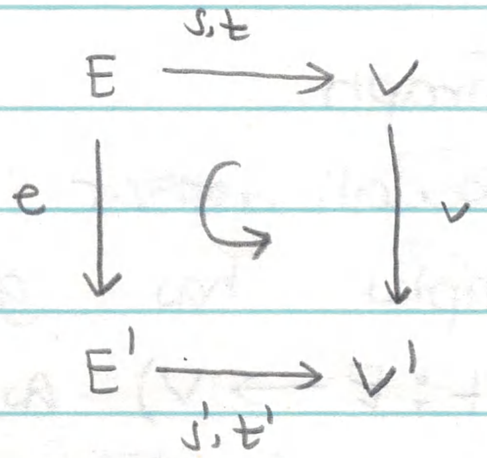


and lets look at Presheaves over $\mathcal{C}$ i.e $Fun(\mathcal{C}^{op}, Set)$. Suppose $F \in Fun(\mathcal{C}^{op}, Set)$ Then $F(0) = V$ and $F(1) = E$ for some arbitrary sets V and E. Also $F(S)$, $F(T)$ are two functions from E to V say s, t respectively. This object $F(\mathcal{C})$ is called a graph (specifically directed multigraph with loops). The arrows in the category of graphs are just natural transformations between functors. In this case that means if we have two graphs $(E \xrightarrow{s,t} V)$ and $(E' \xrightarrow{s',t'} V')$ a graph hom^{m} is a pair of functions $e : E \to E'$ and $v : V \to V'$

Such that

$vs = s'e$

$vt = t'e$



Adjacency matrix of tensor product of G and H

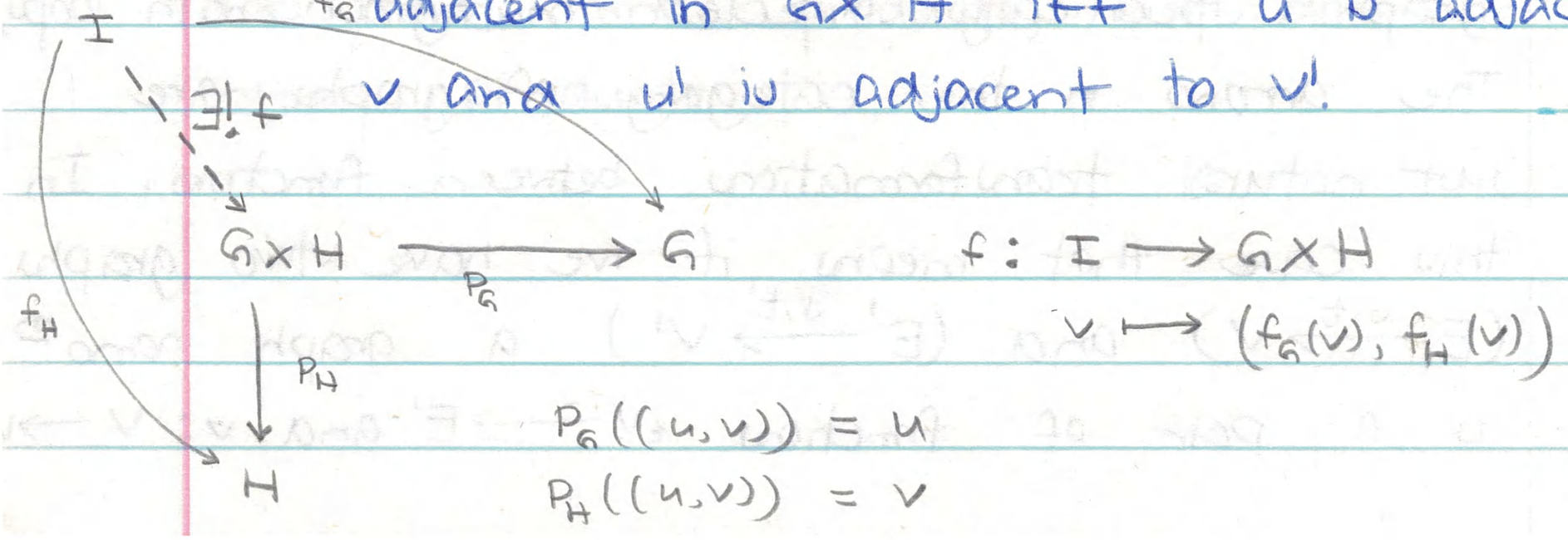
the adjacency matrix of GxH

with a bit of thought we can see that all this says that s is a fnction that take an edge and gives its source vertex and similarly t gives the target vertex. Through a graph homomorphism (e,v) we must map the source and target of an edge $\tilde{e}$ to the source and target of the edge $(e,v)(\tilde{e}) = e(\tilde{e})$.

we also can notice that any pair of sets and pair of functions between them is a graph.

The product of graphs G and H is given by their tensor product which is defined by

- $V(G \times H) = V(G) \times V(H)$
- distinct vertices (u,u') and (v,v') are adjacent in $G \times H$ iff u is adjacent to v and u' is adjacent to v'.



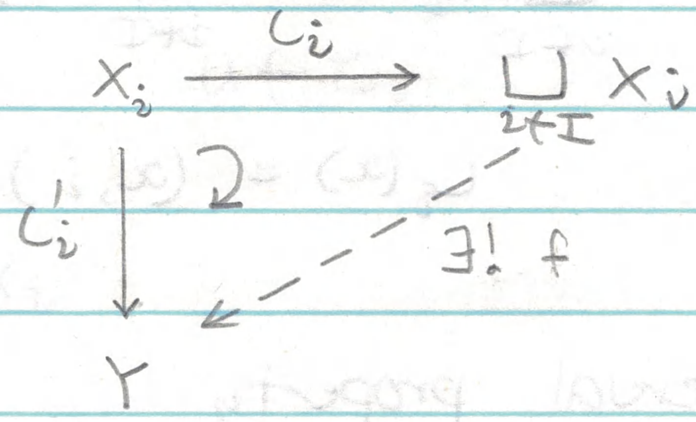
Coproducts

Coproducts are defined as products in the opposite category.

Defⁿ

Given a category $\mathcal{C}$ and a family $\{X_i\}_{i\in I}$ of objects in $\mathcal{C}$, their coproduct $\coprod_{i\in I} X_i$ is the product of $\{X_i\}_{i\in I}$ in $\mathcal{C}^{op}$.

i.e. an object $Y = \coprod_{i\in I} X_i$ in $\mathcal{C}$ together with a family of injection morphisms (antiname for the projection morphisms) $c_i : X_i \longrightarrow Y$ such that (universal property) for any $Y' \in \mathcal{C}$ with morphisms $c_i' : X_i \to Y'$ for all i there is a unique morphism $f : Y \to Y'$ such that $f c_i = c_i'$ $\forall\, i \in I$.



Remark

Coproducts are unique up to a unique isomorphism.

Remark

The initial object in $\mathcal{C}$ is the coproduct of empty family

i.e. an object $0 \in \mathcal{C}$: $\forall X \in \mathcal{C}\ \exists!\ 0 \longrightarrow X$

$0 = \coprod_{\phi}$

Examples

① In Set empty set is initial

② Top : empty space is initial

③ Set_* : Any singleton set is both initial and terminal

④ The integers $\mathbb{Z}$ define an initial object in Ring, but not in Rng in which the zero ring is initial.

⑤ $Vect_k$: $\{0\}$ the zero vector space $\forall V\ \exists!\ 0 \to V$

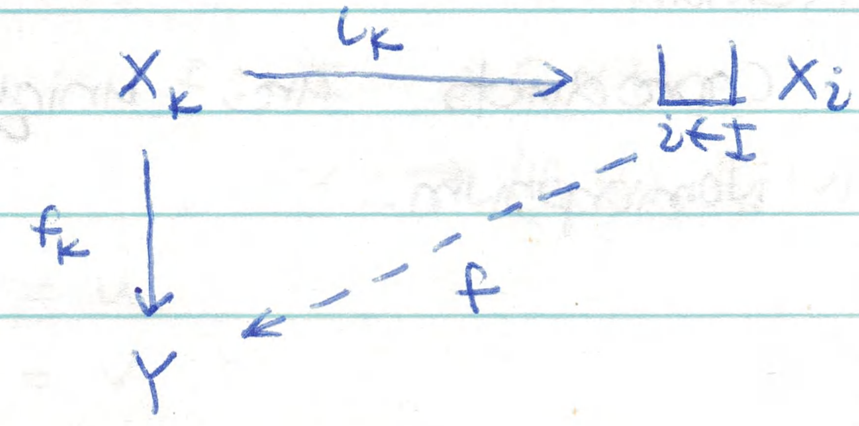
Examples of Coproducts

① Set

Then $\coprod_{i \in I} X_i = \bigcup_{i \in I} X_i \times \{i\}$ (disjoint union)

and $\iota_i(x) = (x, i)$ for any $i \in I$

Universal property



$$X \cong \{\{*\} \xrightarrow{\text{function}} X\}$$

$$2^X \cong \{X \xrightarrow{\text{function}} \{0,1\}\}$$

$$X^2 \cong \{\{*,\bullet\} \xrightarrow{\text{function}} X\}$$

$f((x,k)) := f_k(x) \quad \forall x \in X_k, \; k \in I$

satisfies $f \circ c_k = f_k \quad \forall k \in I.$

(2) Top

$$U\left(\coprod_{i\in I} X_i\right) = \coprod_{i\in I} U(X_i)$$

U: Top $\longrightarrow$ set (underlying set functor)

The topology on $\coprod_{i\in I} U(X_i)$ can be recover by taking

Y to be the Sierpiński space

$$S = \{0,1\}$$

$$\tau = \{\phi, \{0\}, \{0,1\}\}$$

there is a bijection between open sets of X and continuous functions from X to S

$$C(X,S) \qquad X \xrightarrow{f} S$$

$U \mapsto 0$, $X \setminus U \mapsto 1$, $f^{-1}(\{0\})$

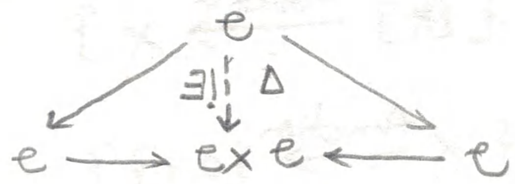
$\text{Open}(X) \qquad U \in \tau_X$

$$\begin{array}{ccc} X_k & \xrightarrow{c_k} & \coprod_{i\in I} X_i \\ & \searrow^{f_k} & \downarrow \exists! f \text{ continuous} \\ & & S \end{array}$$

This tells that

$$\left\{\coprod_{i\in I} X_i \xrightarrow{c_i} S\right\} \cong \left\{X_i \xrightarrow{c_i} S\right\}_{i\in I}$$

In Cat



called the diagonal map

$$C\left(\coprod_{i\in I} X_i, U\right) \cong \prod_{i\in I} C(X_i, U)$$

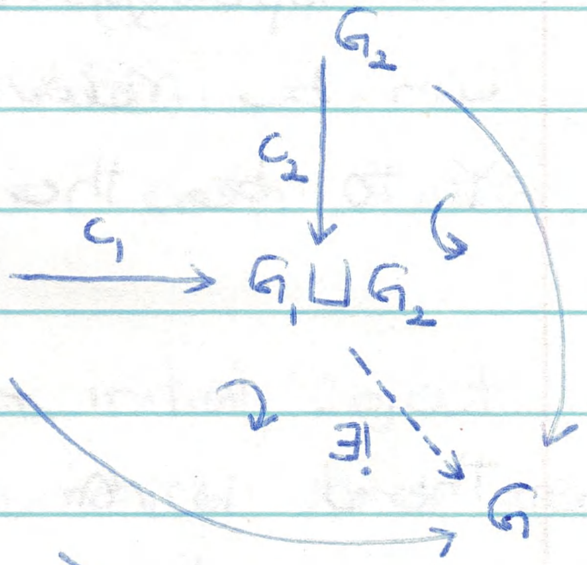
$$f \longmapsto \{f \circ c_i\}_{i\in I}$$

Therefore open subsets of $\coprod_{i\in I} U(X_i)$ are precisely unions of $c_i(U)$, where U is open in X_i.

(3) Group

$\langle x \mid R \rangle$ - group representation $G_1 \xrightarrow{c_1} G_1 \sqcup G_2$

(x: generators, R: relation)



Example

$\langle x, y, z \mid x^2 = zy^3,\ xz^3 = y,\ z^5 = 1 \rangle$

$$\langle x, R \rangle \xrightarrow{f} G \cong \langle f_x, f_y, f_z \mid f_x^2 = f_z f_y^3,\ f_x f_z^3 = f_y,\ f_z^5 = 1 \rangle$$

Any group has a representation

$G = \langle G \mid \underbrace{g_1 g_2}_{\text{formal product}} = \underbrace{g_1 \cdot g_2}_{\text{product in } G} \rangle$

Free product of groups

$G_1 * G_2 = G_1 \sqcup G_2$

$= \langle X_1 \sqcup X_2 \mid R_1, R_2 \rangle$

(Coproduct of G_1 and G_2)

If $G_1, G_2 \in Ab$, then

$$G_1 \amalg G_2 = (G_1 * G_2)^{ab}$$
$$= G_1 \oplus G_2$$

Ex: $\mathbb{Z} = \langle x \mid \phi \rangle$

$\mathbb{Z} * \mathbb{Z} = \langle x, y \mid \phi \rangle$ (free product of 2 generators)

$\coprod_{i \in I} \mathbb{Z}_i$ is known as free group on I.

- Elements are finite words (whose letters are generators) and their formal inverses.
- Identify all identity elements
- composition given by concatenation (several letters next to each other in the same group can collapse to their product) modulo the relation $x.x^{-1} = 1$.

Injective maps:

$$\langle X_1, R_1 \rangle \xrightarrow{\iota_1} \langle X_1 \amalg X_2 \mid R_1, R_2 \rangle$$
$$X_1 \longrightarrow X_1 \amalg X_2$$

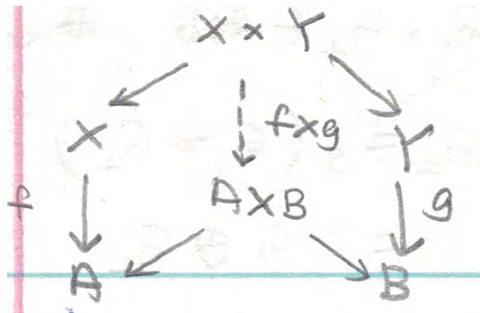
Universal property:

$$\langle X_1 \amalg X_2 \mid R_1, R_2 \rangle \xrightarrow{f} G$$

$\cong$ $\{ X_1 \amalg X_2 \xrightarrow{f} U(G)$ a function of sets satisfies R_1 and $R_2 \}$

$\cong$ $\left\{ \begin{array}{l} f_1 : X_1 \longrightarrow U(G) \} \\ f_1 \text{ satisfies } R_1 \end{array} \right\} \times \left\{ \begin{array}{l} f_2 : X_2 \longrightarrow U(G) \\ f_2 \text{ satisfies } R_2 \end{array} \right\}$

$\cong$ $\{ \langle X_1 \mid R_1 \rangle \xrightarrow{f_1} G \} \times \{ \langle X_2 \mid R_2 \rangle \xrightarrow{f_2} G \rangle$



$(f \times g)(x,y) = (f(x), g(y))$
product of morphisms

The construction with commutative rings

i.e. $\langle X|R \rangle \in CRing$

Example: $\mathbb{Z}[x,y,z] / (\underbrace{x^3+2yz-7=0}_{r_1}, \underbrace{z^5-xy+3z=0}_{r_2}, \underbrace{x+y=0}_{r_3})$

Elements of this ring are equivalance classes of polynomials with integer coefficients: $P_1, P_2 \in \mathbb{Z}[x,y,z]/(r_1,r_2,r_3)$

$P_1 \sim P_2$ if

$P_1 - P_2 = q_1 r_1 + q_2 r_2 + q_3 r_3$,

where q_1, q_2, q_3 are arbitrary polynomials in $\mathbb{Z}[x,y,z]$

Let $R_1, R_2 \in CRing$

$R_1 \sqcup R_2 = R_1 \otimes R_2$ - tensor product

$F_{\mathbb{Z}}(R_1 \times R_2) = \{ \sum_{\text{finite}} (r_1, r_2) : r_1 \in R_1, r_2 \in R_2 \}$

$H = \langle (r_1' + r_1'', r_2) - (r_1', r_2) - (r_1'', r_2),$
$(r_1, r_2' + r_2'') - (r_1, r_2') - (r_1, r_2''),$
$(r_1, n r_2) - (n r_1, r_2),\ (r_1, r_2)(r_1', r_2') - (r_1 r_1', r_2 r_2') \rangle$
$, n \in \mathbb{Z}$

Then

$R_1 \underset{\mathbb{Z}}{\otimes} R_2 = F_{\mathbb{Z}}(R_1 \times R_2) / H$

$\langle X_1 | I_1 \rangle \otimes \langle X_2 | I_2 \rangle = \langle X_1 \sqcup X_2 \mid I_1, I_2 \rangle$

- We can repeate the same construction for other algebraic structures such as monoids, noncommutative rings, Banach spaces... but not for fields

④ Vect_K (generally Mod_R or Ab)

Here $\bigsqcup_{i \in I} X_i$ can be computed as the vector space of finitely supported function $I \longrightarrow \bigsqcup_{i \in I} U(X_i)$ with indexwise operations.

Let $X_i \in \mathrm{Vect}_K$ $\quad i \in I$ $\quad U : \mathrm{Vect}_K \xrightarrow[\text{functor}]{\text{forgetful}} \mathrm{Set}$

$$\bigsqcup_{i \in I} X_i = \{ h : I \longrightarrow \bigsqcup_{i \in I} U(X_i) \mid h(i) \in U(X_i) \text{ and } \mathrm{supp}(h) = \{ i \in I \mid h(i) \neq 0 \} < \infty \}$$

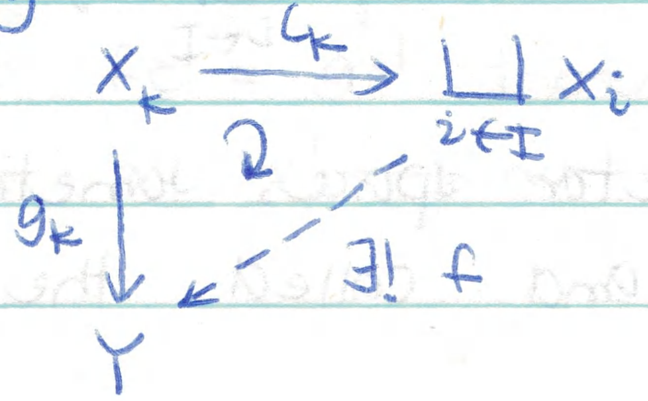
Injective maphisms:

$$X_k \xrightarrow{\iota_k} \bigsqcup_{i \in I} X_i$$

$$\iota_k(x) = h_{x,k}$$

$$h_{x,k}(i) = \begin{cases} x, & \text{if } i = k \\ 0, & \text{otherwise} \end{cases}$$

Universal property



In Top_*

$X \vee Y = X \sqcup Y / x_0 \sim y_0$ (wedge sum) is the coproduct

$$g_k(x) := f(h_{x,k}(k)) \quad \text{extends linearly}$$
$$= f(c_k(x)) \quad \forall k.$$

$$\forall h \in \bigsqcup_{i \in I} X_i \qquad h(k) = \sum_{i \in I} \alpha_i h_{x,i}(k) \quad , \alpha_i \in k$$

$$I \longmapsto \bigsqcup_{i \in I} U(X_i)$$

For example:

Let $I = \{1,2,3\}$ and $X_i = \mathbb{R}$

$$\bigsqcup_{i \in I} U(X_i) = \bigcup_{i \in I} U(X_i) \times \{i\}$$

$$= \bigcup_{i \in I} \mathbb{R} \times \{i\}$$

$h: I \longrightarrow \bigsqcup_{i \in I} U(X_i)$

$h(i) = (a,b,c)$

$= a(1,0,0) + b(0,1,0) + c(0,0,1)$

Here $\bigsqcup_{i \in I} X_i = \{(x_i)_{i \in I} : \#\{i \in I : x_i \neq 0\} < \infty\}$

If I is finite

$$\bigsqcup_{i \in I} X_i = \prod_{i \in I} X_i$$

or otherwise

$$\bigsqcup_{i \in I} X_i \subsetneq \prod_{i \in I} X_i$$

this vector space, sometimes denoted by $\bigoplus_{i \in I} X_i$ and called the direct sum.

(5) Field

no nonsingleton coproducts exist, essentially for the same reason as products. If we restrict to the full subcategory of some fixed characteristic p, it does have an initial object

$$0 = \begin{cases} \mathbb{Q}, & \text{if } P=0 \\ \mathbb{Z}/P\mathbb{Z}, & \text{if } P>0 . \end{cases}$$

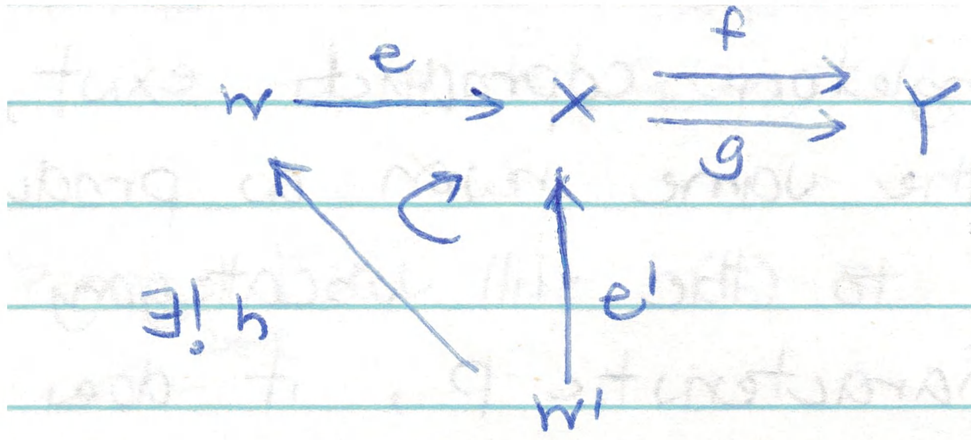
Equalizers

Equalizers generalize notion of kernels of group homomorphisms, and fixed points of maps.

Def^{n}

Suppose $\mathcal{C}$ is a category and $f,g: X \to Y$ is a pair of parallel morphisms in $\mathcal{C}$. The equalizer of f and g is an object $W \in \mathcal{C}$ together a morphism $e: W \longrightarrow X$ such that $fe = ge$ satisfying the universal property:

For any other morphism $e': W' \longrightarrow X$ such that $fe' = ge'$ there is a unique morphism $h: W' \longrightarrow W$ such that $eh = e'$



Just like the case of products, we can define equalizer of f and g as a terminal object of certain category

Def^{n}

Fork$_{f,g}$

Objects: $W \xrightarrow{e} X$ such that

$fe = ge$

i.e. Fork $W \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$

$\mathrm{mor}(W \xrightarrow{e} X, W' \xrightarrow{e'} X)$

: $W \xrightarrow{h} W'$ such that

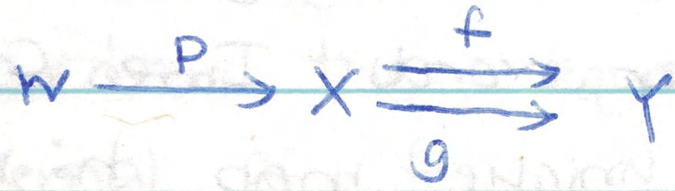
$e'h = h$.

Corollary

Equalizers are unique up to a unique isomorphism.

Example

① Set



$$f(P(w)) = g(P(w)) \quad \forall w \in W$$

$$f|_{ImP} = g|_{ImP}$$

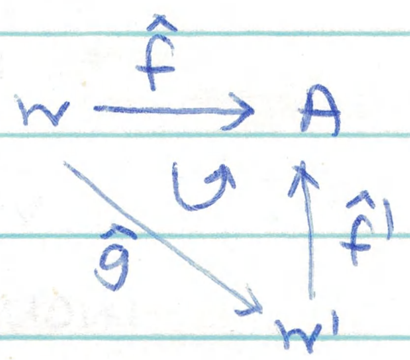
$$ImP \subseteq \{x \in X : f(x) = g(x)\} = A$$

Let's look at the $\hookrightarrow X$

slice category Set/A

Objects : $(W, \hat{f})$ $\quad W \xrightarrow{\hat{f}} A$

mor : $(W, \hat{f}) \longrightarrow (W', \hat{f}')$



this category is equal (equivalent) to $Fork_{f,g}$ and has the terminal object $A \xrightarrow{1_A} A$

Conclusion : The equalizer of f and g is $A = \{x : f(x) = g(x)\}$ and a $P : A \longrightarrow X$ is the canonical inclusion.

Indeed, if $e' : w' \to X$ such that $fe' = ge'$, then f and g are equal in the image of e', or in other words $\mathrm{Im} e' \subseteq A$. Therefore there is a unique map $h : w' \to A$ such that $ph = e'$, namely the corestriction of e' to A.

Special cases : (I) $X \overset{f}{\underset{0}{\rightrightarrows}} X$

$$A = \{x \in X : f(x) = 0\}$$
$$= \text{Zero set of } f \text{ in } X$$
$$= eq(f, 0)$$

(II) $X \overset{f}{\underset{\Delta_X}{\rightrightarrows}} X$

$$A = \{x \in X : f(x) = x\}$$
$$= \text{fixed points of } f \text{ in } X$$
$$= eq(f, id)$$

(2) $\mathbf{Ab}$, $\mathbf{Mod}_R$, $\mathbf{Vect}_K$, $\mathbf{Group}$, $\mathbf{Monoid}$, $\mathbf{Ring}$, $\mathbf{CRing}$

$w = \{x \in X : f(x) = g(x)\}$ with its induced algebraic operations and canonical inclusion map to X.

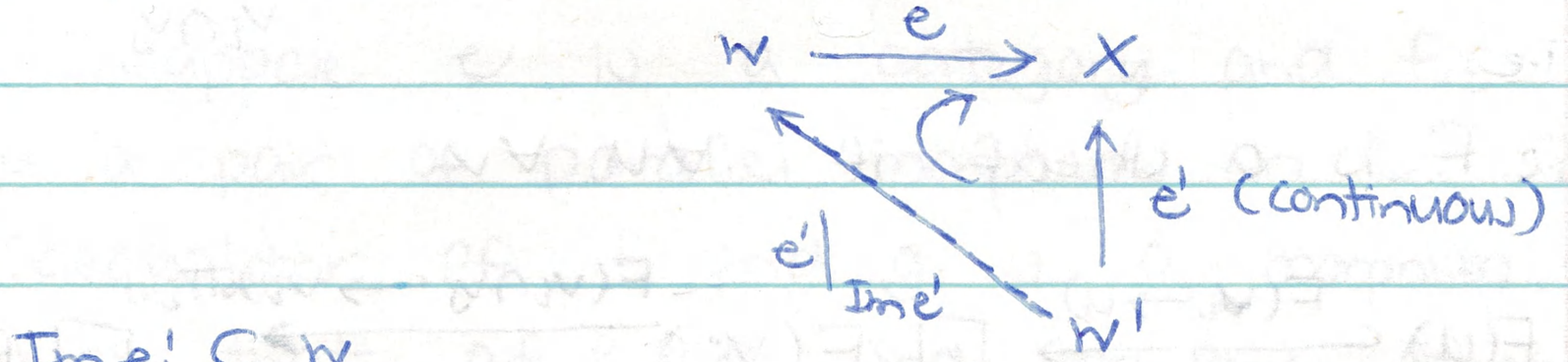
Indeed $w \in \mathcal{E}$ because w is closed under all

operations like addition, scalar multiplication because f and g are homomorphisms.

moreover the equalizer of $X \overset{f}{\underset{0}{\rightrightarrows}} Y$ is the kernal of f.

(3) Top

$W = \{x \in X : f(x) = g(x)\}$ with the induced (subspace) topology and the canonical inclusion map to X.



$Im\, e' \subseteq W$

e' is continuous $\Rightarrow W' \xrightarrow{e'|_{Im\, e'}} W$ is continuous

(4) meas : (X, m_X, n_X)

m_X: σ-algebra of measurable sets

n_X: σ-ideal of negligible sets

$W = \{x \in X : f(x) = g(x)\} \in m_X$?

This is true for localizable measurable spaces as m_X / n_X is complete.

⑤ HoTop

Here equalizers do not exist

$$W \longrightarrow S^1 \underset{n}{\overset{m}{\rightrightarrows}} S^1$$

⑥ Presh(X)

Let F be a presheaf on $X \in Top$

i.e. $F: Open(X)^{op} \longrightarrow Set$. Then F is a sheaf

if $\forall U \in Open(X)$ $\forall \{V_i\}_I$ such that $U = \bigcup_{i \in I} V_i$

$$F(U) \xrightarrow{\nu} ComFam(F, V)$$

$$= \{ x_i \in F(V_i) : x_i|_{V_i \cap V_j} = x_j|_{V_i \cap V_j} \}$$

i.e. F is a sheaf if $\forall U$ $\forall V$

$$F(U) \xrightarrow{F(V_i \to U)} \prod_{i \in I} F(V_i) \underset{g = F(V_i \cap V_j \hookrightarrow V_j) \circ \pi_j}{\overset{f = F(V_i \cap V_j \hookrightarrow V_i) \circ \pi_i}{\rightrightarrows}} \prod_{i,j \in I} F(V_i \cap V_j)$$

is an equalizer.

$F(V_i \hookrightarrow U)$ is the map induced by $F(U) \to F(V_i)$ restriction maps and $F(V_i \cap V_j \hookrightarrow V_i) \circ \pi_i$ is induced by the morphism

$$\prod_{i \in I} F(V_i) \xrightarrow{f} F(V_i \cap V_j)$$

$$\pi_i \searrow \qquad \nearrow F \circ \iota$$

$$F(V_i)$$

(universal property of products)

$\iota : V_i \cap V_j \longrightarrow V_i$

$$f(x) = g(x)$$

$$f(x)_{j,k} = g(x)_{j,k} \qquad \forall\, j,k$$

$$x_j|_{V_j \cap V_k} = x_k|_{V_j \cap V_k} \qquad \forall\, j,k$$

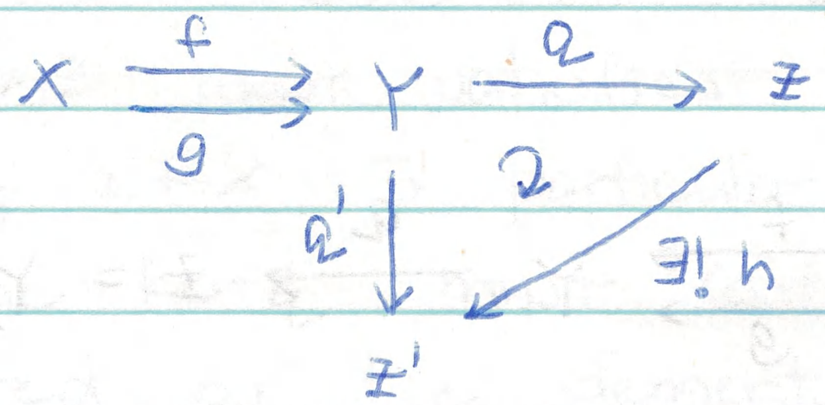
(by the def^n of f and g).

Coequalizers

Coequalizers in $\mathcal{C}$ are precisely equalizers in $\mathcal{C}^{op}$.

Def^n

Suppose $\mathcal{C}$ is a category and $f, g : X \to Y$ is a pair of parallel morphisms in $\mathcal{C}$. The coequalizer of f and g is a morphism $q : Y \to Z$ such that $qf = qg$ and for any other morphism $q' : Y \to Z'$ such that $q'f = q'g$ there is a unique morphism $h : Z \to Z'$ such that $hq = q'$



Coequalizers are precisely initial objects in the category $\mathrm{Cofork}_{f,g}$ defined entirely analogously to $\mathrm{Fork}_{f,g}$.

Example

① Equivalance relation

Any equivalance relation R on a set X gives rise to a coequalizer diagram

$$R \underset{p_1}{\overset{p_0}{\rightrightarrows}} X \xrightarrow{\pi} X/R \qquad R \subseteq X \times X$$

The universal property of coequalizers in this case boils down to precisely the universal property of X/R.

i.e. Function $X/R \to Z$ can be identified with function $f: X \to Z$ such that $f(x) = f(y)$ whenever $x \sim y$

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle \pi}\downarrow & \nearrow_{\exists! F} & \\ X/R & & \end{array}$$

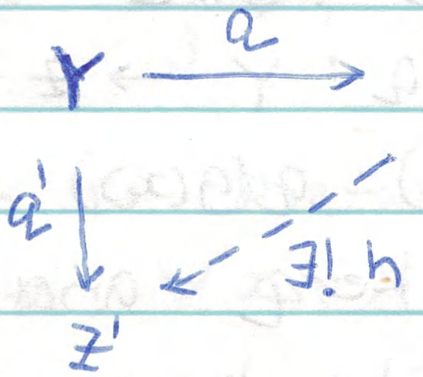
② Set

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Z = Y/\sim$$

$\sim$: Equivalance relation on Y generated by pairs $f(x) \sim g(x)$ for all $x \in X$.

$q(f(x)) = q(g(x))$ because $f(x)$ and $g(x)$ belong to the same equivalance class.

The universal property:



maps $h: Z \to Z'$ are in a bijection with maps $q': Y \to Z'$

universal property of quotients

$\Rightarrow$ Uniqueness and existance of maps that respects the equivalance relation

$q' = hq$

$q'(f(x)) = h(q(f(x)))$
$= h(q(g(x)))$
$= q'(g(x))$

Again the universal property of Coequalizers coincides with that of quotient maps.

(3) Group

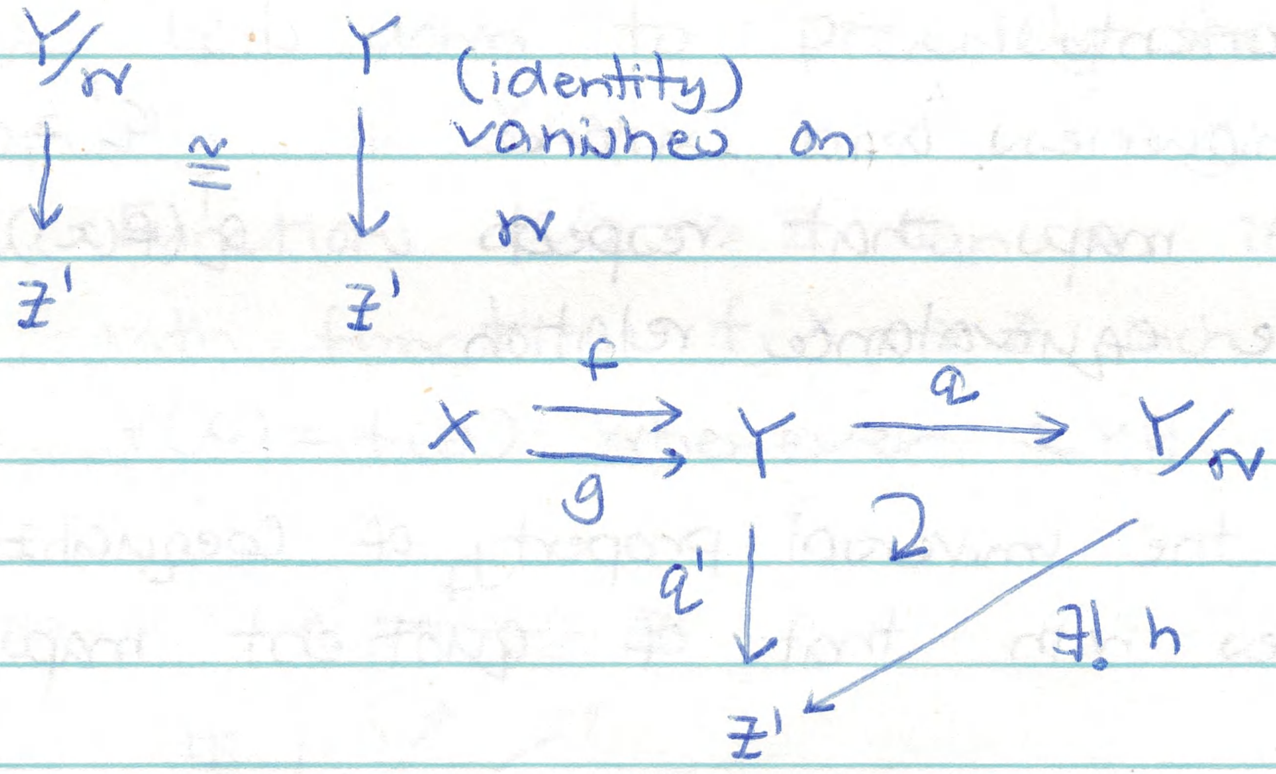
First observe that the homomorphism $q: Y \to Z$ must satisfies $q(f(x)) = q(g(x))$ for any $x \in X$. In particular $f(x)g(x)^{-1} \in \ker q$ Thus the $\ker q$ must contain the subgroup generated by the elements $f(x)g(x)^{-1}$, $x \in X$.

Let $N = \langle f(x)g(x)^{-1} : x \in X \rangle \trianglelefteq Y$

i.e. the closure of $\{f(x)g(x)^{-1} : x \in X\}$ under multiplication, inversion and conjugation by arbitrary elements of Y.

we claim that the quotient homomorphism $Y \to Y/N$ is the desire coequalizer.

If $q: Y \to Z$ such that $q(f(x)) = q(g(x)) \quad \forall x \in X$, then $N \leq \ker q$ and when the equality happens, the universal property satisfied.



$q' \circ f = q' \circ g$

$\Rightarrow q'|_N = 1$

$\Rightarrow \exists!\ h: Y/N \to Z'$ such that $h \circ q = q'$.

④ Ban and Ban_1

Using an entirely similar arguement, the coequalizer of f and g can be computed by the quotien map $q: Y \to Y/N$, where $N = \langle f(x) - g(x) : x \in X \rangle$ is the norm closure (closure under norm topology) of linear span.

(5) Top

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Y/R$$

where R is the subspace of Y generated by $f(x) \sim g(x) \quad \forall x \in X$.

$$\mathcal{H} : Y \longrightarrow \mathcal{H}(Y/\sim)$$

$\mathcal{H}$: free Hausdorffization functor

(left adjoint to the forgetful functor) $\sim$: equivalance relation generated by $x \sim y$ iff x, y can not seperated by disjoint open sets.

Pullback (fibered product)

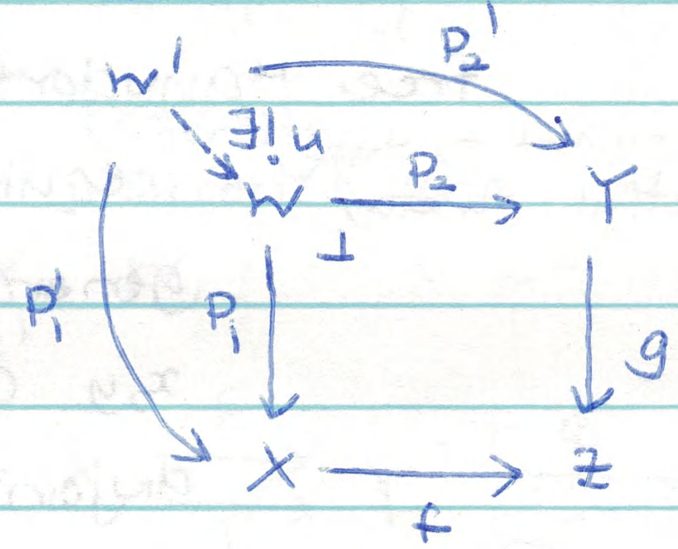
Given a category $\mathcal{C}$ and two morphisms $f: X \to Z$, $g: Y \to Z$ in it, the pullback of f and g is an object W (usually denoted by $X \underset{Z}{\times} Y$) equipped with two morphisms $P_1: W \to X$, $P_2: W \to Y$ such that $fP_1 = gP_2$

i.e

$$\begin{array}{ccc} W & \xrightarrow{P_2} & Y \\ {\scriptstyle P_1}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

moreover the pullback (W, P_1, P_2) must satisfy the following universal property:

For any other such triple (W', P_1', P_2') satisfying $fP_1' = gP_2'$ there is a unique morphism (called mediating morphism) $u: W' \to W$ such that $P_1 \circ u = P_1'$ and $P_2 \circ u = P_2'$



Example

(1) In any category with terminal object T, the pullback $X \underset{T}{\times} Y$ is just the product $X \sqcap Y$ (binary product)

(2) Set

Here pullback of f and g are given by $X \underset{Z}{\times} Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$ together with the restrictions of projection maps $X \times Y \xrightarrow{\pi_1} X$, $X \times Y \xrightarrow{\pi_2} Y$ to $X \underset{Z}{\times} Y$. Further if $Z = \{*\}$, then f and g are unique ($\{*\}$ is the terminal object of Set)

and $X \underset{\{*\}}{\times} Y = X \times Y$ (cartesian product)

If instead $Y = \{*\}$ is a singleton set and Z is any set, then $* \xrightarrow{g} Z$ pick some element $z \in Z$, then the pullback is the subset of X that maps to z under f

i.e. $f^{-1}(\{z\}) \subseteq X$ ← fiber of z in X

Suppose X and Y are subsets of Z and f, g are canonical inclusions, then

$$X \underset{Z}{\times} Y = \{(x,y) \in X \times Y : x = y\}$$
$$\cong X \cap Y \subseteq Z$$

suppose $g: Y \hookrightarrow Z$ is the inclusion $Y \subseteq Z$

$$X \underset{Z}{\times} Y = \{(x,y) : f(x) = y\} = f^{-1}(x) \subseteq X$$

③ Top

(Forgetfull functor Top ⟶ Set has both left and right adjoints, hence preserve both limits and colimits)

$X \underset{Z}{\times} Y = \{(x,y) \in X \times Y : f(x) = g(y)\}$ equipped with the subspace topology induced by the product topology on $X \times Y$.

④ Poset

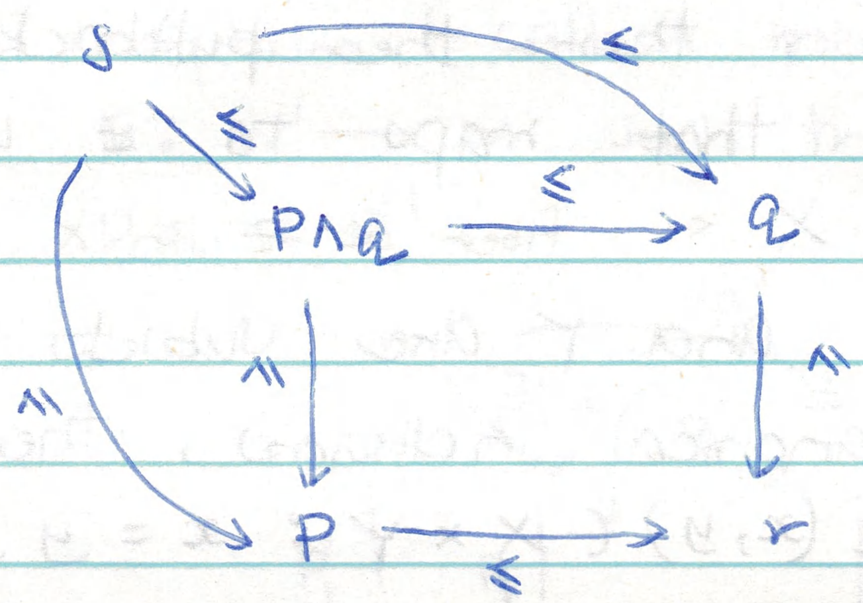
Let $(P, \leqslant)$ be a partially ordered set. Then we can define a category with

Obj: $x \in P$ elements of P

$$\text{Mor}(x,y) = \begin{cases} *, \text{ if } x \leqslant y \\ \phi, \text{ otherwise} \end{cases}$$

Then $(P, \leqslant)$ becomes a skeletal, thin category and monotone functions are precisely the functors between such poset categories.

In a poset P



Hence $P \underset{r}{\times} q = P \wedge q$ (meet of p and q)

$= glb\ (P, q)$

For example

(i) $(\mathbb{N}, \leqslant)$ where $m \leqslant n$ if $m \mid n$

i.e. $\exists\ r \in \mathbb{N}$ such that $n = rm$

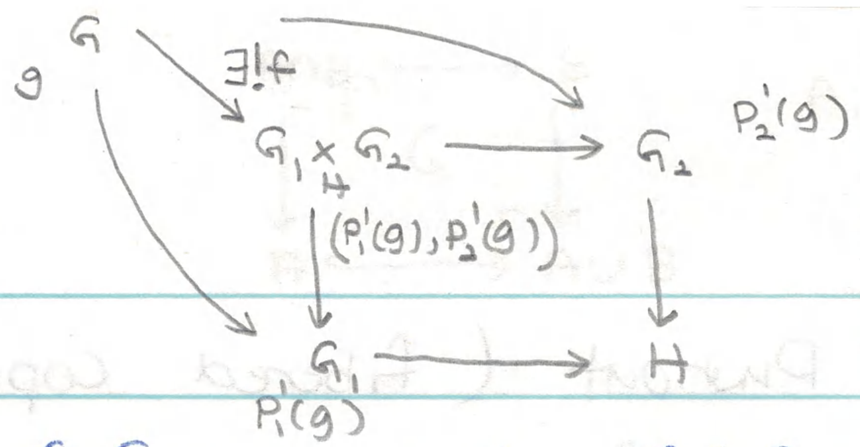
$P \wedge q = gcd\ (P, q)$

(ii) $(\mathbb{R}_0^+, \leqslant)$, $x \leqslant y$ if $x - y \in \mathbb{R}_0^+$

$P \wedge q = \min \{P, q\}$

universal property:

$f(g) = (P_1'(g), P_2'(g))$ is a gp homo^{m} s.t. $P_1 \circ f = P_1'$ and $P_2 \circ f = P_2'$



⑤ Group

Let G_1, G_2 ∈ Group and $\varphi_1 : G_1 \to H$, $\varphi_2 : G_2 \to H$ be group homomorphisms. Then

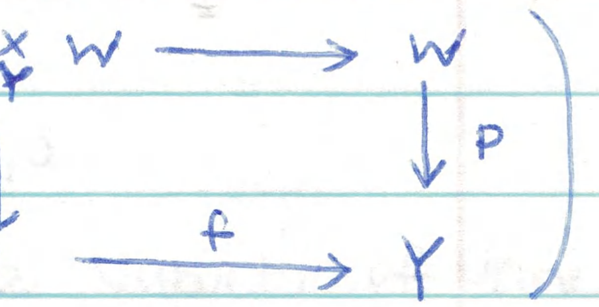
$$G_1 \times_H G_2 = \{(g_1, g_2) \in G_1 \times G_2 : \varphi_1(g_1) = \varphi_2(g_2)\}$$

we can generalize the same construction for CRing, mod_R and Vect_k easily.

note

For a morphism $f : X \to Y$ in category $\mathcal{C}$ with pullbacks, there is an induced functor $f^* : \mathcal{C}/Y \to \mathcal{C}/X$ between slice categories given by

$$(P : W \to Y) \longmapsto \left(f^*(P) = P^* : X \times_Y W \to X \right)$$



called the base change.

⑥ Fiber bundles

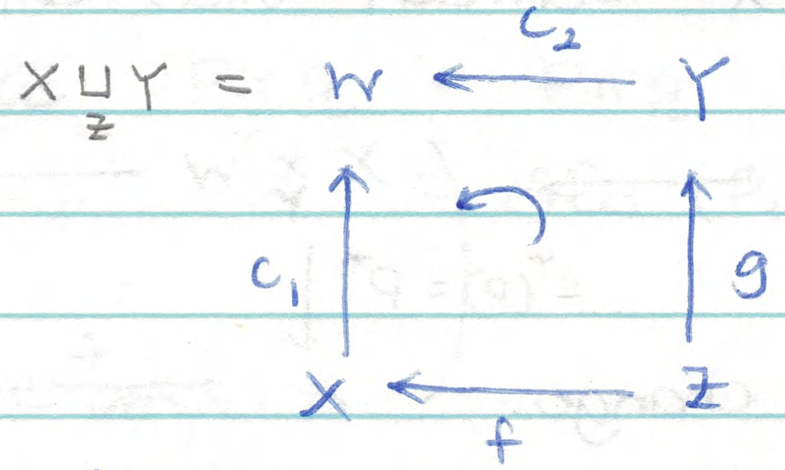
Given a bundle projection $\pi : E \to B$ and a continuous map $f : X \to B$, the pullback $X \times_B E$ is a fiber bundel over X called the pullback bundle. The associated commutative diagram is a morphism of fiber bundles.

Pushout (fibered coproduct or amalgamated sum)

A pushout in $\mathcal{C}$ is same as a pullback in $\mathcal{C}^{op}$.

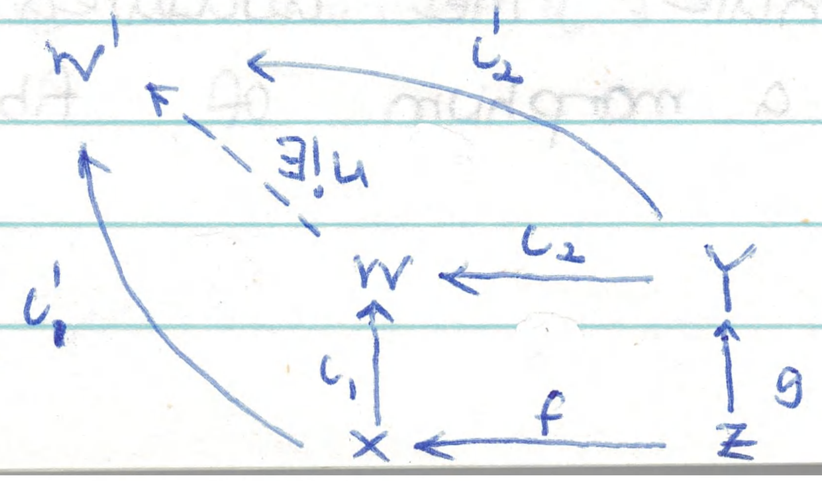
Def^n

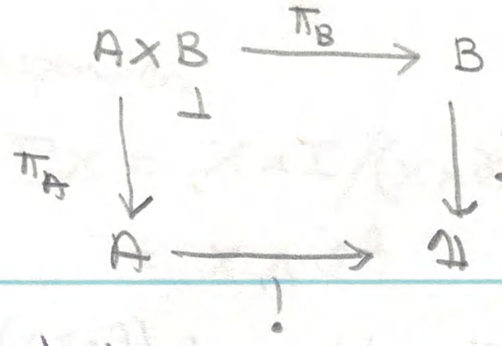
Let $\mathcal{C}$ be a category with two morphism $f: Z \to X$ and $g: Z \to Y$. The pushout of f and g consists of an object W and two morphism $c_1: X \to W$ and $c_2: Y \to W$ such that the diagram

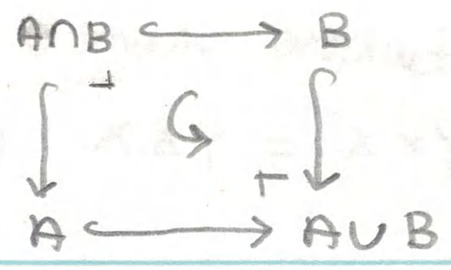


commute and such that (W, c_1, c_2) is universal with respect to this diagram.

i.e. For any other such set (W', c_1', c_2') there is a unique morphism $u: W \to W'$ such that $c_1' = u c_1$ and $c_2' = u c_2$.







$A, B \subseteq C$

Examples

① Set

The pushout of f and g in the disjoint union of X and Y, where elements sharing the same preimage in Z are identified, together with inclusion maps ι_1, ι_2 from X and Y.

i.e. $W = (X \sqcup Y)/\sim$, where $\sim$ is the smallest equivalance relation such that $f(z) \sim g(z)$ for all $z \in Z$.

$\iota_1(x) = [x]$ and $\iota_2(y) = [y]$ makes the pushout square commute and satisfy the universal property.

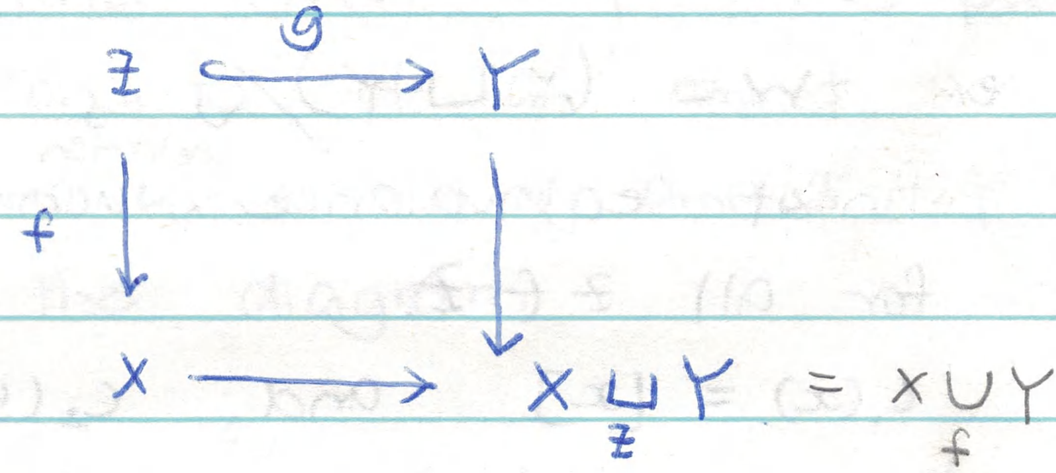
In particular if X and Y are subsets of some larger set and Z is their intersection with f and g are inclusion maps of Z into X and Y, then the pushout can be canonically identified with the union $X \cup Y$.

② Top

The pushout of $X \xleftarrow{f} Z \xrightarrow{g} Y$ exist and given by $(X \sqcup Y)/\sim$ equiped with quotient topology of topology on disjoint union

where $\sim$ is generated by $i_X(f(z)) \sim i_Y(g(z))$ for all $z \in Z$.

The construction of adjunction spaces is an common example of pushouts in Top. more precisely, if Z is a subspace of Y and g is the inclusion map



then the pushout is given by $(X \sqcup Y)/_{f(z) \sim z}$
Informally, we glue Y to X along Z using an attaching map $f: Z \to X$. The result is the adjunction space $X \cup_f Y$.
(All identification spaces may be regarded as pushouts in this way)

Special cases

(i) If Z is a one point set, the $X \vee Y$ adjunction is the wedge sum of X and Y

(ii) If X is a one point set, then the adjunction is the quotient Y/Z.

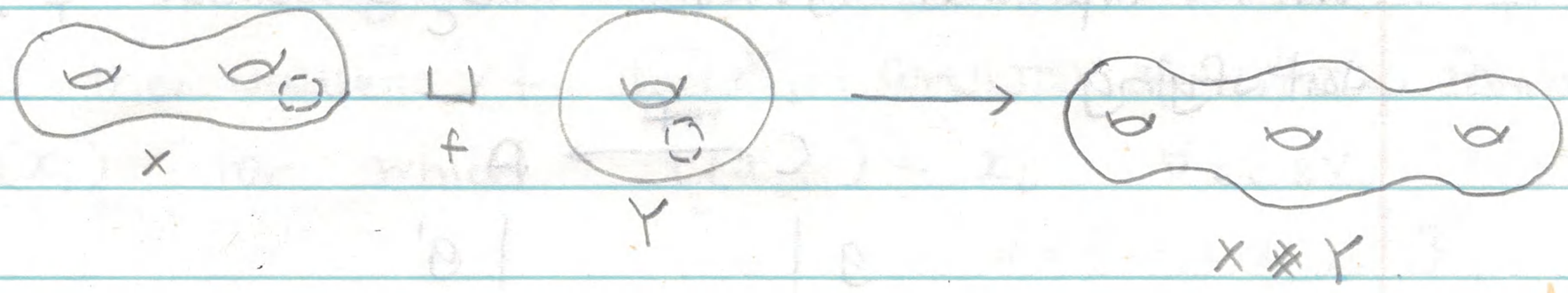
⊙ Suspension

$$\Sigma X = X \times I / (X \times \partial I \cup \{x_0\} \times I)$$
$$= X \wedge S^1$$

⊙ Smash product

$$X \wedge Y = X \times Y / X \vee Y$$

(iii) Adjunctions are also used to define connected sums of manifolds. Here, one first removes open balls from X and Y before attaching the boundaries of removed balls along an attaching map.



(3) Group

Here the pushouts are called free products with amalgamation or amalgamation product.

Let $\varphi_1 : H \to G_1$, $\varphi_2 : H \to G_2$ be two group homomorphisms

$$G_1 *_H G_2 = G_1 * G_2 / N,$$

N: smallest normal subgroup of $G_1 * G_2$ generated by elements of the form $\varphi_1(h)\,\varphi_2(h)^{-1}$ for all $h \in H$.

HNN extension

i.e. Amalgamation has forced an identification between $\varphi_1(H)$ in G_1 and $\varphi_2(H)$ in G_2, element by element.

This construction shows up in the Seifert - van Kampen theorem of homotopy theory.

④ CRing

The pushout is given by the tensor product of rings

$$A \otimes_C B = \left\{ \sum_{i \in I} (a_i, b_i) : \begin{matrix} a_i \in A \\ b_i \in B \end{matrix} \right\} / \langle (f(c)a, b) - (a, g(c)b) : \begin{matrix} a \in A \\ b \in B \\ c \in C \end{matrix} \rangle$$

with morphisms $g' : A \longrightarrow A \otimes_C B$ and $f' : B \longrightarrow A \otimes_C B$ satisfying

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ B & \xrightarrow[f']{} & A \otimes_C B \end{array}$$

Sequential limits (Inverse limits, Projective limits) (in general on directed sets)

Suppose $\mathcal{C}$ is a category, $\{X_i\}_{i \in \mathbb{N}}$ be an infinite family of objects in $\mathcal{C}$ with $t_i : X_{i+1} \longrightarrow X_i$ some morphism of $\mathcal{C}$ (called transition maps). The sequential limit of this system is an object $X \in \mathcal{C}$ (bad notation) together with morphism $P_i : X \longrightarrow X_i$ (called projections) satisfying $P_i = t_i \circ P_{i+1}$ $\forall i \in \mathbb{N}$. The pair (X, P_i) must be universal in the sense that, for any other such pair (Y, q_i) there exist a unique morphism $f : Y \longrightarrow X$ such that $q_i = P_i \circ f$ $\forall i$. The sequential limit is denoted by

p-adic numbers
cantor set
profinite completion

$X = \varprojlim_{i \in \mathbb{N}} X_i = \varprojlim X_i$ with the inverse system (X_i, t_i) being understood.

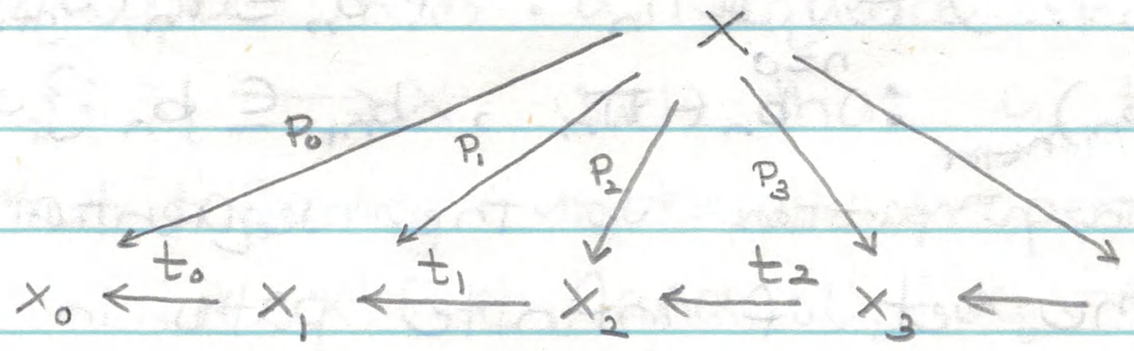
Diagram
$\mathcal{D}: \omega^{op} \to \mathcal{C}$

Examples

① Set

The sequential limit of (X,t) can be computed as the subset of $\prod_{i \in \mathbb{N}} X_i$ consisting of those families $(x_i)_{i \in \mathbb{N}}$ for which $t_i(x_{i+1}) = x_i \quad \forall i \in \mathbb{N}$.

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$



If all transition maps are inclusions, then the sequential limit can be computed as the intersection.

*

② Top

The sequential limit of (X,t) is given by the sequential limit of $(U(X), U(t))$ in Sets equipped with the topology generated by a subbase of sets given by the preimages of open sets of X_i under the projection maps p_i
(initial topology)

Top $\cdots \longrightarrow S^1 \xrightarrow{p} S^1 \xrightarrow{p} S^1$ (p^{th} power map) the covering map that wraps the domain circle uniformely p times around the codomain circle

the sequential limit is called p-adic solenoid

* Let U be a set and $(\Pi_n)_{n\in\mathbb{N}}$ be a sequence of partitions of U such that Π_{n+1} is finer than Π_n for all n. i.e. every cell in Π_n is a union of cells in Π_{n+1}. Thus there is a canonical transition $t_n : \Pi_{n+1} \longrightarrow \Pi_n$ that maps each cell in Π_{n+1} to the unique cell in Π_n that contains the former as a subset.

$$\Pi_0 \longleftarrow \Pi_1 \longleftarrow \Pi_2 \longleftarrow \Pi_3 \longleftarrow \cdots$$

The sequential limit of (Π, t) is

$$\{(b_0, b_1, b_2, \ldots) \in \prod_{n=0}^{\infty} \Pi_n : \quad b_n = t_n(b_{n+1}) \ \forall n\}$$

$$= \{(b_n)_{n\in\mathbb{N}} : b_n \in \Pi_n, \ b_{n+1} \subseteq b_n\}$$

(b_n: block in n^{th} partition) so the sequential limit is simply the set of infinite paths in the tree formed from this sequence of partitions. ⌋

③ Group

p-adic integers

The ring of p-adic integers is the sequential limit of the rings

$$\mathbb{Z}_p \xleftarrow{t_1} \mathbb{Z}_{p^2} \xleftarrow{t_2} \mathbb{Z}_{p^3} \longleftarrow \cdots$$

$$t_i(x) = x \pmod{p}$$

i.e.

$$\mathbb{Q}_p = \{(n_1, n_2, n_3, \ldots) : n_i \in \mathbb{Z}_{p^i}, \ n_{i+1} \equiv n_i \pmod{p}$$

$\mathbb{Q}_p$ has a solution for $x^2 = -1$.

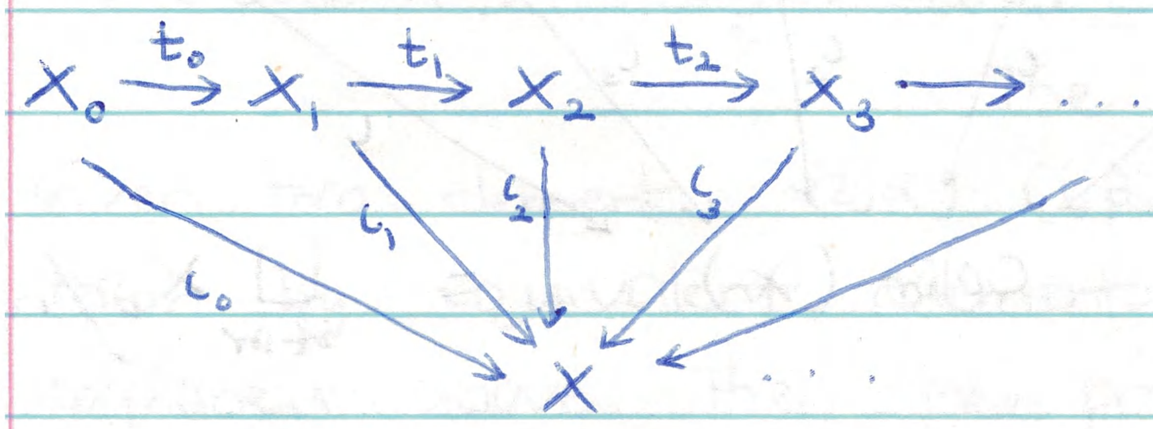
Sequential colimits (Direct, or inductive limit)

Sequantial colimits are sequential limits in the opposite category.

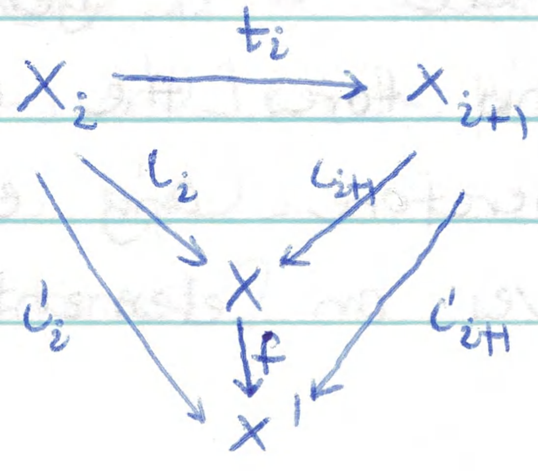
Def^n

Suppose $\mathcal{C}$ is a category and $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of objects in $\mathcal{C}$ with transition morphisms $t_i : X_i \longrightarrow X_{i+1}$ in $\mathcal{C}$. The sequential colimit of the collection (X,t) is an object X (bad notation) equipped with a sequence of injection morphisms $c_i : X_i \longrightarrow X$ such that $c_{i+1} t_i = c_i \ \forall i$ and the following universal property is satisfied:

If $X' \in \mathcal{C}$ and $c'_i : X_i \longrightarrow X'$ satisfy $c'_{i+1} t_i = c'_i \ \forall i$, then there is a unique morphism $f : X \longrightarrow X'$ such that $f c_i = c'_i$ for all i.



Universal property:



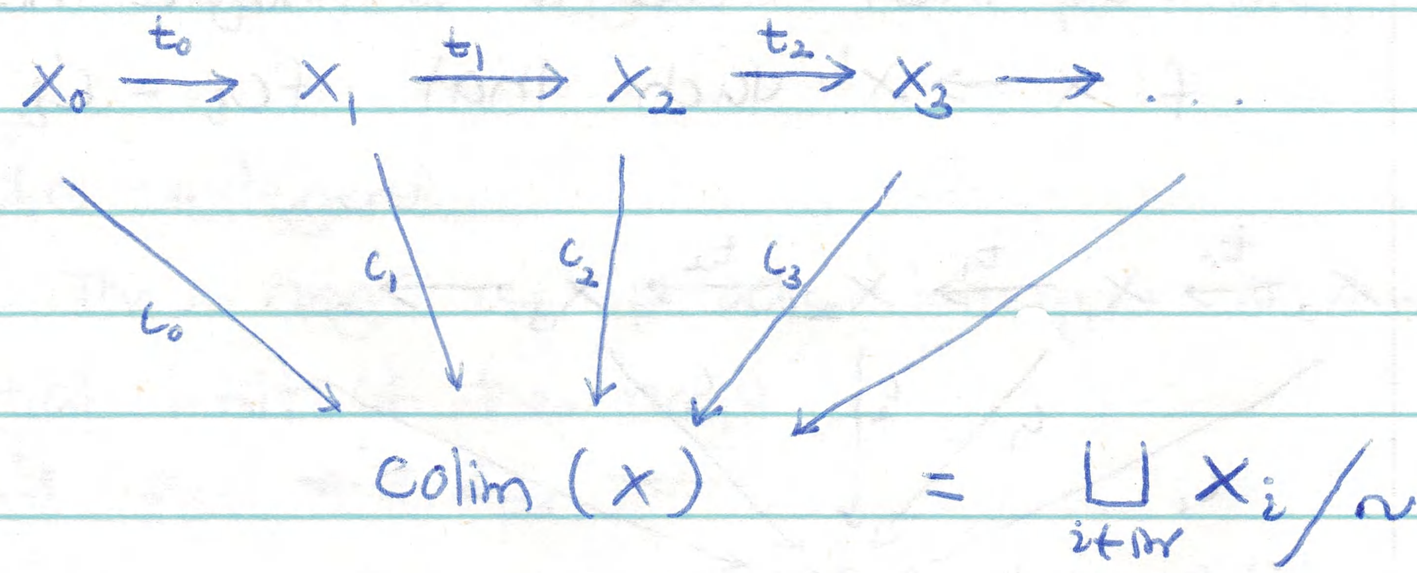
In complete analogy with coproducts and coequalizers, the sequential colimit of (X,t) can be discribed as the initial cocone under (X,t)

Def: (Cocone under (X,t))

A cocone under (X,t) is a pair (U, C), where $U \in \mathcal{C}$ and $c_i : X_i \to U$ such that $c_{i+1} t_i = c_i$ for all i. A maphism of cocones $(U, C) \to (U', C')$ is a maphism $f: U \to U'$ in $\mathcal{C}$ such that $f c_i = c'_i$ for all $i \in \mathbb{N}$.

Examples

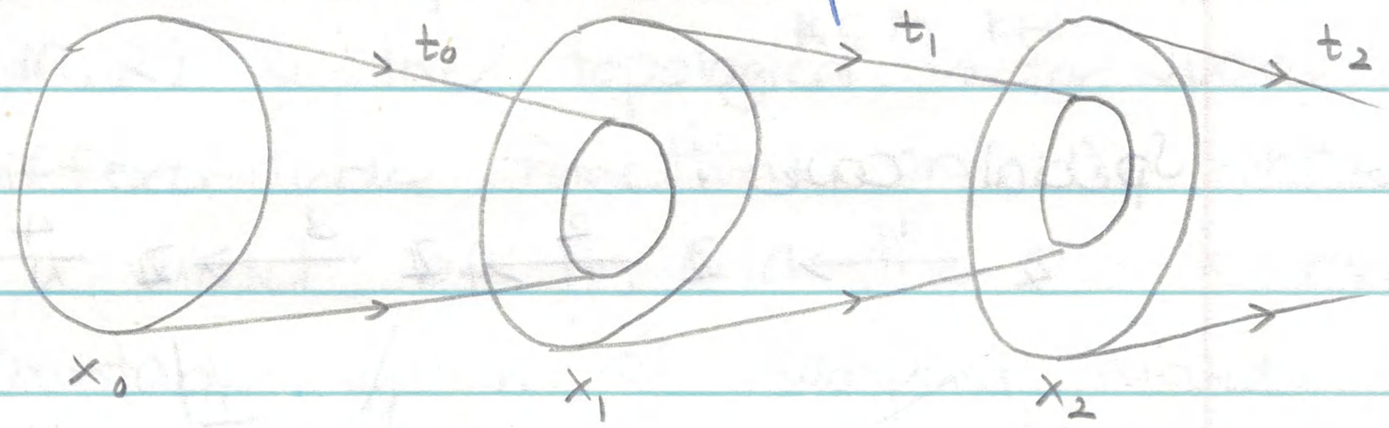
① Set



where $\sim$ is generated by $(i, \alpha) \sim (i+1, t(\alpha))$ (this forces the commutativity of triangles). Therefore any element of X_i for any $i \in \mathbb{N}$ gives an element in the sequantial colimit

(we say $x \in X_i$ lives at stage i), and two such elements (i,α) and (j,β) are equivalent if we can send both of them to a latter stage k such that $k \geq i$ and $k \geq j$ $(k, t(t(\dots(\alpha)))) = (k, t(t(\dots(\beta))))$

If all transition maps are inclusion of sets,



then $\coprod_{i \in \mathbb{N}} X_i / \sim \; = \bigcup_{i \in \mathbb{N}} X_i$

(2) Group (or any other variety of algebras)

$$X_0 \xrightarrow{t_0} X_1 \xrightarrow{t_1} X_2 \longrightarrow \dots$$

$\operatorname{colim} X \in$ Group

U: Group $\xrightarrow{\text{forgetful}}$ Set

$$U(\operatorname{colim} X) = \operatorname*{colim}_i U(X_i)$$

The group operation $= ?$

Given two elements (i,α), (j,β), first we replace them by equivalent elements (k,α'), (k,β') as explained above. Then the product of equivalance classes of (i,α) and (j,β) is defined as

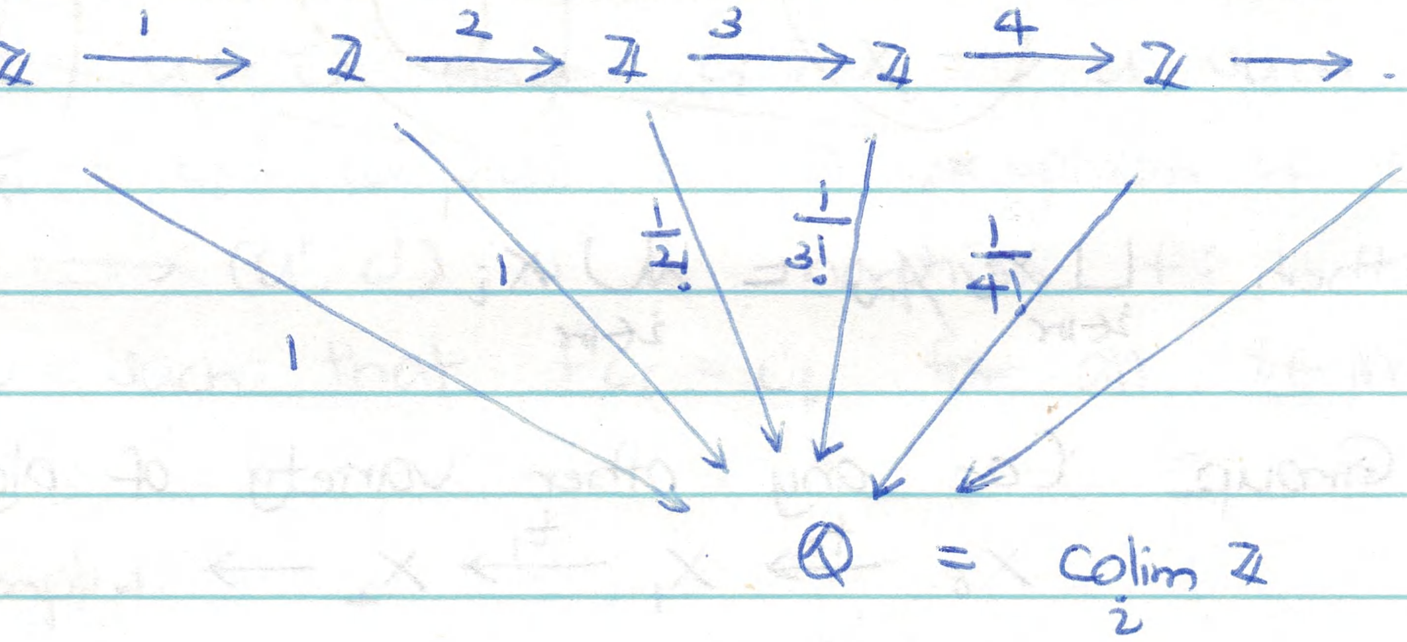
$$[(i,\alpha)] \cdot [(j,\beta)] = [(k,\alpha')] \cdot [(k,\beta')]$$
$$= [(k, \alpha'\beta')]$$

Is this product well defined?

Let $k' \geq i, j$, then $[(i,\alpha)] \cdot [(j,\beta)] = [(k',\alpha'')][(k',\beta'')]$
$= [(k', \alpha''\beta'')]$

Since $(k',\alpha'') \sim (k,\alpha')$ and $(k',\beta'') \sim (k,\beta')$
$(k,\alpha'\beta') \sim (k',\alpha''\beta'')$, and infact it equivalent to some $(\ell, t(t(t(t\cdots(\alpha'\beta'))))$ for any $\ell \geq k, k'$.

Special case:



③ Top

The sequential colimit of (X,t) can be computed by taking the sequential colimit of underlying sets in Set and equipping it with the topology in which a subset is open iff all of its preimages under injection maps are open.

④ Topvect

Here sequential limits and colimits are

computed like for vector spaces and topological spaces (taking closures of subspaces whenever quotients are needed). The topological vector space of compactly supported smooth functions on a smooth manifold M is defined as $\operatorname{Colim}_{K \subseteq M} \lim_{k \in \mathbb{N}} C_K^k(M, \mathbb{R})$. Here $C_K^k(M, \mathbb{R})$ is the topological vector space of k times differentiable functions on M whose support is a subset of K, and the set K runs over a countable system of compact subsets of M that covers M. The maps $t_k : C_K^{k+1}(M, \mathbb{R}) \to C_K^k(M, \mathbb{R})$ are inclusions. The limit $\lim_{k \in \mathbb{N}} C_K^k(M, \mathbb{R})$ has $C_K^\infty(M, \mathbb{R})$ as its underlying vector space. The topology is generated from a local subbase of open neighborhoods of zero given by

$$\{f \in C_K^\infty(M, \mathbb{R}) : \|f\|_k < \varepsilon \quad \forall k \in \mathbb{N} \ \forall \varepsilon > 0\}.$$

The topological vector space $C_K^\infty(M, \mathbb{R})$ is not a Banach space even though $C_K^k(M, \mathbb{R})$ is a Banach space for all k. The maps $C_K^\infty(M, \mathbb{R}) \to C_{K'}^\infty(M, \mathbb{R})$ for $K \subseteq K'$ are inclusions, then the colimit $\operatorname{Colim}_{K \subseteq M} C_K^\infty(M, \mathbb{R})$ has $C_{cs}^\infty(M, \mathbb{R})$ (the space of compactly supported smooth functions) as its underlying vector space. A subset of $C_{cs}^\infty(M, \mathbb{R})$ is open if its intersection with $C_K^\infty(M, \mathbb{R})$ are open for all compact $K \subseteq M$.

Limits and colimits

Def^n

A diagram of shape J in a category $\mathcal{C}$ is a functor $D: J \to \mathcal{C}$, where J is called the indexing category.

If J is small, finite and so on, we talk about small diagrams, finite diagrams, and so on.

Example

① Consider the discrete category on set I: $\mathrm{disc}(I)$ (its objects are objects of I and only morphisms are identity morphisms).
An I indexed diagram in $\mathcal{C}$ is precisely the data we need to define products and coproducts in $\mathcal{C}$. i.e. an I indexed family of objects in $\mathcal{C}$

② Consider the category $I = \{0 \rightrightarrows 1\}$. An I indexed diagram in $\mathcal{C}$ is exactly the data that we need to define equalizers and coequalizers.

i.e. Two objects $X, Y \in \mathcal{C}$ and two parallel morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

(3) Consider the poset $\mathbb{N} = \{0 < 1 < 2 < 3 < \dots\}$ as a category. An $\mathbb{N}$ indexed diagram in $\mathcal{C}$ is precisely the data that we used to define sequential colimits in $\mathcal{C}$.

i.e. $X_0 \xrightarrow{t_0} X_1 \xrightarrow{t_1} X_2 \xrightarrow{t_2} X_3 \longrightarrow \dots$

Likewise an $\mathbb{N}^{op}$ indexed diagram in $\mathcal{C}$ is the data used to define sequential limits in $\mathcal{C}$

i.e. $X_0 \xleftarrow{t_0} X_1 \xleftarrow{t_1} X_2 \xleftarrow{t_2} X_3 \longleftarrow \dots$

Defⁿ

For any object $c \in \mathcal{C}$ and any category J, the constant functor $C: J \longrightarrow \mathcal{C}$ sends

$$j \longmapsto c$$

$$j \to j' \longmapsto c \xrightarrow{1_c} c.$$

The constant functors define an embedding

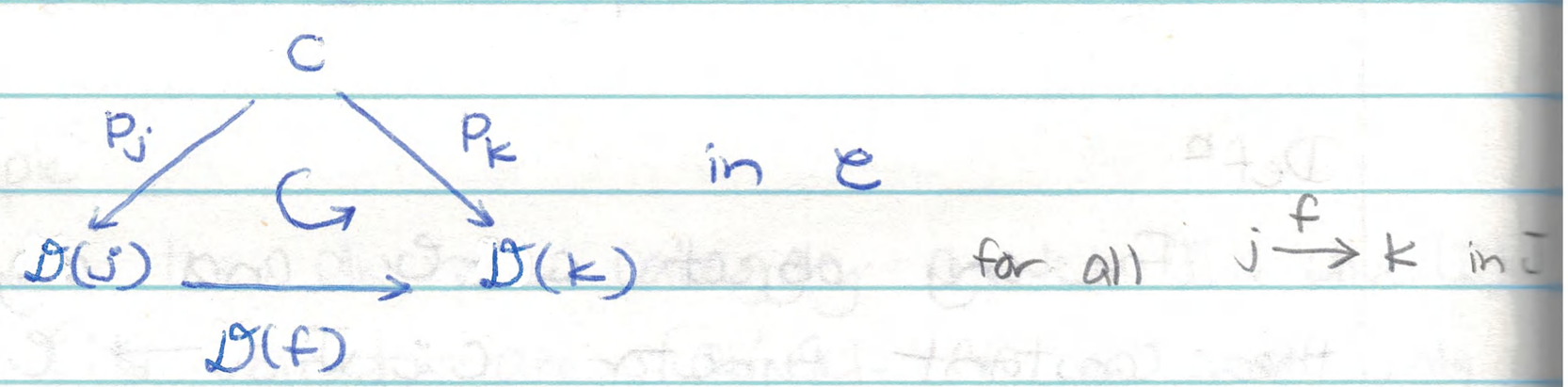
$$\Delta: \mathcal{C} \longrightarrow Fun(J, \mathcal{C}) = \mathcal{C}^J$$

$$c \longmapsto c$$

$$c \xrightarrow{f} c' \longmapsto c \overset{f}{\Rightarrow} c' \quad \text{(the constant natural transformation, in which each component is } f\text{)}$$

Def^n

Given a diagram $D: J \to \mathcal{C}$, a Cone over D with summit or apex $c \in \mathcal{C}$ is a natural transformation $P: c \Rightarrow D$ whose domain is the constant functor at c. The components $(P_j : c \to D(j))_{j \in J}$ of the natural transformation are called the legs of the cone.

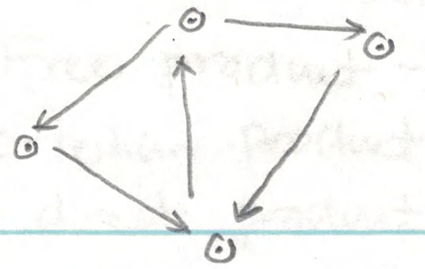


As a comma category, the category Cones of cones over D is defined as follows:

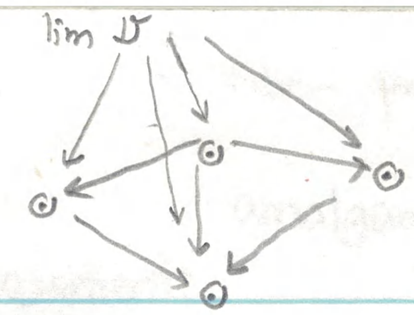
The objects of Cones are pairs $(U, \{P_j\}_{j \in J})$, where $U \in \mathcal{C}$ and for any object $j \in J$, the projection morphism $P_j : U \to D(j)$ is a morphism in $\mathcal{C}$ such that for any morphism $f: j \to k$ in J we have $D(f) \circ P_j = P_k$

A morphism $(U, P) \to (U', P')$ of Cones is a morphism $f: U \to U'$ in $\mathcal{C}$ such that $P'_j \circ f = P_j$ for all $j \in J$

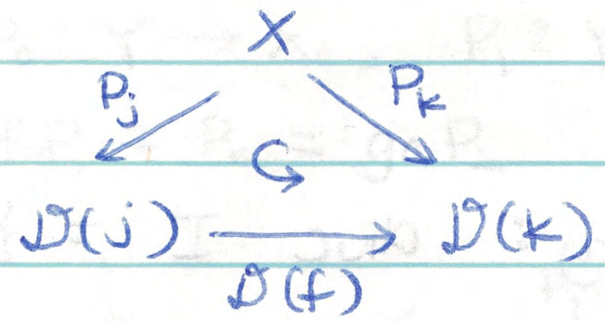
$\mathcal{D}$:



$\lim \mathcal{D}$

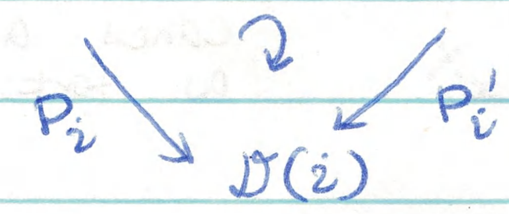


+ universal property



$\forall\ j \xrightarrow{f} k$ morphism of J

- a cone over $\mathcal{D}$ with apex X



- morphism of cones $(X, P) \to (X', P')$

The category of cocones over a diagram $\mathcal{D}$ is define analogously, using injection morphisms $c_j : \mathcal{D}(j) \longrightarrow U$ such that $c_j = c_j \circ \mathcal{D}(f)$ and a morphism of cocones satisfying $f \circ c_j = c_j'$.

Def ≃

The limit of a diagram $\mathcal{D}: J \to \mathcal{C}$ in a category $\mathcal{C}$ (if exists), is the terminal object in the category $\mathrm{Cone}_{\mathcal{D}}$, denoted by $\varprojlim_J \mathcal{D} = \varprojlim_{i \in I} \mathcal{D}(i)$

Likewise, the colimit of $\mathcal{D}$ is the initial object in the category $\mathrm{Cocone}_{\mathcal{D}}$

Examples

① Products

Indexing category: disc $I = \{ \overset{1_i}{\circlearrowright} *_i \}_{i \in I}$

. . . . . .

② Equalizers

$I = \{ 0 \rightrightarrows 1 \}$

cones are known as forks

③ Sequential limits

$I = \{ 0 \to 1 \to 2 \to \dots \}$

④ Terminal object

$I = \phi$ (empty category)

$\mathcal{D}: I \to \mathcal{C}$ (empty functor)

⑤ Pullbacks

$I = \left\{ \begin{array}{ccc} & & 1 \\ & & \downarrow \\ 2 & \to & 0 \end{array} \right\}$

$\mathcal{D}: I \to \mathcal{C}$

X_1

$\downarrow f$ in $\mathcal{C}$

$X_2 \xrightarrow[g]{} X_0$

Unfolding the above definitions, one can notice some redundancy. Specifically the definition of a cone involves three morphisms

Products in Group

Free product - coproduct

(direct) Cartesian product - product

semi direct product - homotopy quotient

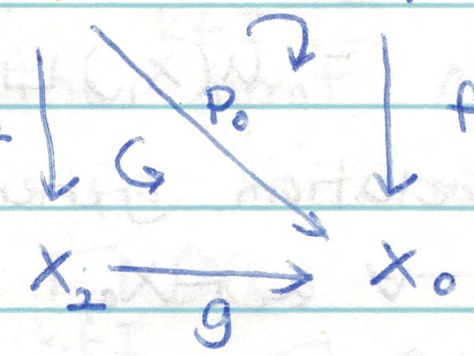
fiber product - pullback

amalgamated product - pushout

$P_0 : Y \to X_0$, $P_1 : Y \to X_1$ and $P_2 : Y \to X_2$ such that (homotopy colimit)

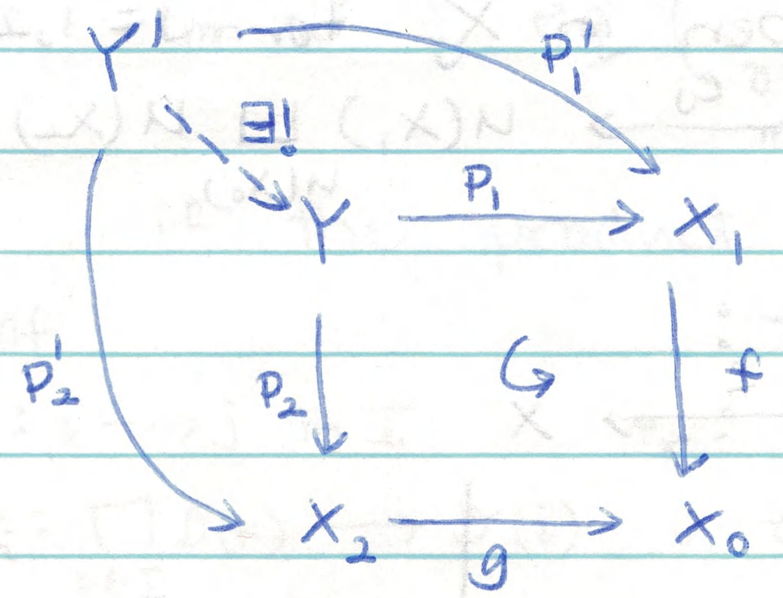
$f \circ P_1 = P_0 = g \circ P_2$

$\lim_{I} D =$



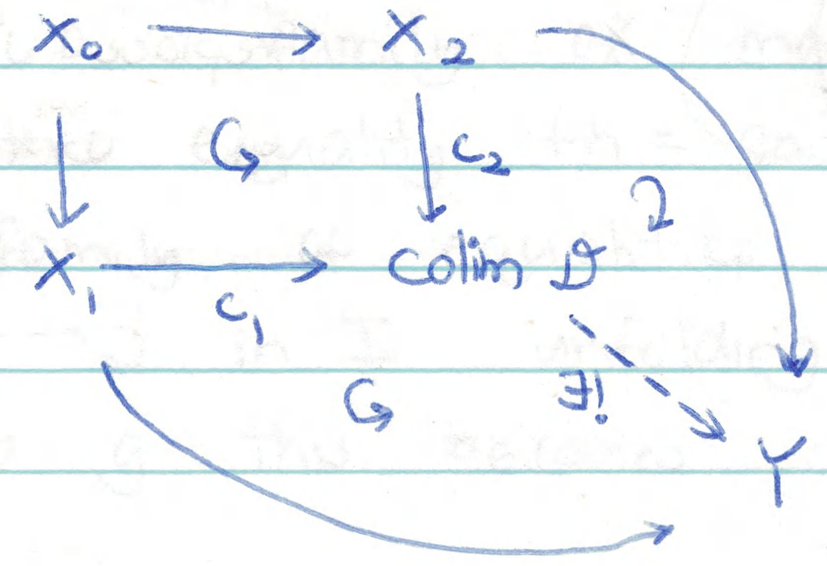
This tells us that the data of P_0 is redundant and it is suffices to consider P_1 and P_2.

Universal property



(6) Pushout (pullback in opposite category)

$I = \{ 2 \leftarrow 0 \to 1 \}$



$\operatorname{colim} D = X_1 \underset{X_0}{\sqcup} X_2$

$\sqcup$: \sqcup $\xrightarrow{f}$: \buildre f \over \to

$\overrightarrow{f}$: \mathop{\to}\limits_{-f}

For example in Top

$$U(X_1 \underset{X_0}{\sqcup} X_2) = U(X_1) \underset{U(X_0)}{\sqcup} U(X_2)$$

$$= U(X_1) \sqcup U(X_2) / \sim$$

$\sim$: equivalance relation generated by

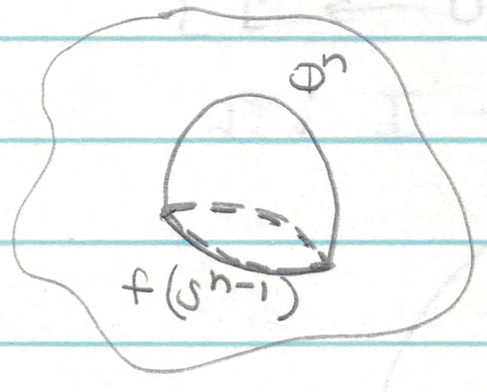
$f(x_0) \sim g(x_0) \quad \forall \ x_0 \in X_0$

$V \subseteq U(X_1) \underset{U(X_0)}{\sqcup} U(X_2)$ is open if and only if $c_j^{-1}(V)$ is open in X_j for $j = 1,2$.

$$U(X_j) \xhookrightarrow{c_j} U(X_1) \underset{U(X_0)}{\sqcup} U(X_2)$$

Handle attachment:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X \sqcup D^n \end{array}$$



X

replace $S^{n-1} \hookrightarrow D^n$

by $S^{n-1} \times \partial I \hookrightarrow D^n \times I$

Expressing limits via equalizers and products.

Proposition

For any diagram $\mathcal{D}: I \to \mathcal{C}$, the equalizer of $\prod_{k \in I} \mathcal{D}(k) \underset{g}{\overset{f}{\rightrightarrows}} \prod_{r: i \to j} \mathcal{D}(j)$, if it exists, is the limit of $\mathcal{D}$. Here the components of f and g are

$$f_r = p_j : \prod_{k \in I} \mathcal{D}(k) \longrightarrow \mathcal{D}(j) \quad \text{and}$$

$$g_r = \mathcal{D}(r) \circ p_i : \prod_{k \in I} \mathcal{D}(k) \longrightarrow \mathcal{D}(j).$$

i.e. $$\lim \mathcal{D} \cong \mathrm{eq}\left(\prod_{r: i \to j \text{ in } I} \mathcal{D}(j) \underset{g}{\overset{f}{\rightrightarrows}} \prod_{k \in I} \mathcal{D}(k) \right),$$

if the product and equalizer exist.

proof

$r: i \to j$ in I $\qquad f_r : \prod_{k \in I} \mathcal{D}(k) \xrightarrow{p_j} \mathcal{D}(j) \qquad f_r = p_j$

$p_i : \prod_{k \in I} \mathcal{D}(k) \longrightarrow \mathcal{D}(i)$ in $\mathcal{C}$ and $\mathcal{D}(r) : \mathcal{D}(i) \to \mathcal{D}(j)$

$$g_r : \prod_{k \in I} \mathcal{D}(k) \xrightarrow{p_i} \mathcal{D}(i) \xrightarrow{\mathcal{D}(r)} \mathcal{D}(j)$$

$$g_r = \mathcal{D}(r) \circ p_i$$

morphism $X \to E$ can be identify with morphism $h: X \to \prod_{k \in I} \mathcal{D}(k)$ such that $fh = gh$ and here h is a family of morphism $h_k : X \to \mathcal{D}(k)$ thus the equality $fh = gh$ can be decompose into family of equalities $(fh)_r = (gh)_r$ for each $r: i \to j$ in I. unfolding the definitions of f and g this becomes $p_j h = \mathcal{D}(r) p_i h$

Decomposing h into its components, we have $h_j = D(\gamma) h_i$. Therefore a morphism $X \to E$ is the same as a family of morphisms $h_k : X \to D(k)$ such that $h_j = D(\gamma) h_i$ for each $\gamma : i \to j$, precisely what a cone over D is. This proves that $E = eq\left(\prod_{k \in I} D(k) \rightrightarrows \prod_{\gamma : i \to j} D(j)\right)$ with its projection maps satisfies the universal property of a limit.

<u>Or</u>

Define $\lim D \underset{\beta}{\overset{\alpha}{\rightleftarrows}} eq\Big(\underbrace{\prod_{k \in I} D(k) \overset{f}{\underset{g}{\rightrightarrows}} \prod_{\gamma : i \to j} D(j)}_{E}\Big)$

α is induced by

$$\lim D \xrightarrow{\alpha'} \prod_{k \in I} D(k) \overset{f}{\underset{g}{\rightrightarrows}} \prod_{\gamma : i \to j} D(j)$$

$$\alpha'(k) : \lim D \xrightarrow{P_k} D(k)$$

$$\lim D \searrow \prod_{\gamma : i \to j} D(j)$$

$f \circ \alpha' = g \circ \alpha'$

$\|$

$P_j = D(\gamma) \circ P_j$ (by the def^n of cone)

Thus the data of $\alpha : \lim D \to eq$ is equ to a cone over D. β is define in the same fashion and $\alpha \circ \beta = id_{eq}$

$\beta \circ \alpha = id_{\lim D}$

Corollary

In any variety of algebras (ie. Group, Ab, mod_R, Vect_k, Alg_k, but not Field) we can compute $\lim D$ by computing $\lim U \circ D$.
i.e. The limit of underlying sets, and equipping it with the induced algebraic operations.

By applying the above result to $\mathcal{C}^{op}$, we get analogous discription of colimits.

Proposition

For any diagram $D : I \to \mathcal{C}$, the coequalizer E of $\coprod_{r: i \to j} D(j) \xrightarrow[g]{f} \coprod_{k \in I} D(k)$, if it exists, is the colimit of D. Here the components of f and g are given by

$f_r = c_i : D(i) \longrightarrow \coprod_{k \in I} D(k)$ and

$g_r = c_j \circ D(r) : D(i) \longrightarrow D(j) \longrightarrow \coprod_{k \in I} D(k)$.

The injection maps are given by the injection maps for the coproduct that are further decomposed with the canonical injection maps to the coequalizer.

Example

In Set, $\lim D = \{ x \in \prod_{k \in I} D(k) : D(r)(x_i) = x_j \}$.

where $r: i \to j$ is an arbitrary morphism in I. The projection maps are obtain by restricting the projection maps of $\prod_{k \in I} \mathcal{D}(k)$.

$$\operatorname{Colim} \mathcal{D} = \coprod_{k \in I} \mathcal{D}(k) \Big/ \langle \mathcal{D}(r)(x_i) = x_j \mid r: i \to j \rangle,$$

where $\langle - | - \rangle$ denotes the equivalance relation generated by the given set of pairs. The injection maps are obtained from the injection maps of $\coprod_{k \in I} \mathcal{D}(k)$ by composing with the quotient map.

Complete and Cocomplete Categories

Def^n

A category $\mathcal{C}$ is (co)complete if any small diagram $\mathcal{D}: I \to \mathcal{C}$ has a (co)limit

Examples

① Set is (co)complete

$$\lim \mathcal{D} = \left\{ x \in \prod_{i \in I} \mathcal{D}(i) \;\middle|\; \forall\, r: i \to j : \mathcal{D}(r)(x_i) = x_j \right\}$$

$$\operatorname{Colim} \mathcal{D} = \coprod_{i \in I} \mathcal{D}(i) \Big/ \left\langle \iota_i(v) = \iota_j(\mathcal{D}(r)(v)) \;\middle|\; r: i \to j,\ v \in \mathcal{D}(i) \right\rangle$$

(monads) variety of algebra: Any (algebraic) structure that defined wing only algebric operations, not inequalities.

Field: $0 \neq 1$

$\forall x \neq 0 \; \exists y: xy = 1$

② P-poset e_P: Objects $= P$

$$mor(x,y) = \begin{cases} *, & x \leq y \\ \phi, & \text{otherwise} \end{cases}$$

e_P is complete $\Leftrightarrow$ e_P is cocomplete

$\Leftrightarrow$ P is complete

i.e. P admits suprema of arbitrary subsets.

$(\mathbb{R}, \leq)$ is complete $\rightsquigarrow$ $e_{\mathbb{R}}$ is (co)complete

$(\mathbb{Q}, \leq)$ is not complete $\rightsquigarrow$ $e_{\mathbb{Q}}$ is not (co)complete

③ Group (Ring, CRing, $Vect_K$, Mod_R, Alg_R, is (co)complete Monoid, Gset, Lattice)

Limits: Take the limit of underlying sets and equip it with indexwise operation

Colimits: Present all objects in a diagram wing generators and relations, and take the disjoint union of all sets of generators and rel.

Example: $G_1 = \langle x, y : xy^2 = 1 \rangle$

$G_2 = \langle x, y : xy^2 = 1 \rangle$

$G_1 * G_2 = \langle x, y, a, b : xy^2 = 1, ab^2 = 1 \rangle$

$G\text{-}Set = Set_G$ $\quad e_G := Fun(BG, e)$

$BG \in Cat$

$Ob = \{*\}$

$mor(*,*) = \{G\}$

In e_G

$Ob: \quad BG \xrightarrow{F} e$

$* \longmapsto F(*) = X$

$* \xrightarrow{g} * \longmapsto F(*) \longrightarrow F(*)$

(Thm $\exists\ G \xrightarrow[\text{homo}^{\text{ism}}]{gp} Aut\ F$

$G \curvearrowright F(*)$ - some object of e)

$mor: \quad F \Rightarrow G$ (natural transformation)

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ F(*) & & G(*) \end{array}$$

naturality

$$\begin{array}{ccc} X & \xrightarrow{F(g)} & X \\ \alpha \downarrow & \circlearrowright & \downarrow \alpha \\ X' & \xrightarrow[G(g)]{} & X' \end{array}$$

④ Top is (co) complete

Limits: Compute the limit of underlying sets and equip it with the topology generated by a subbase whose elements are preimages of open subsets under projection maps. (Initial topology)

Colimits: Compute the colimit of underlying sets and equip it with a topology whose open subsets are those subsets whose preimages under arbitrary injection maps are open. (Final topology)

(5) Field is not (co)complete

Ban_1 is (co)complete

Ban is finitely (co) complete, but does not admit infinite (co)products, so it is neither complete nor cocomplete.

The above examples may create an impression that if a category is either complete or cocomplete, then it is both complete and cocomplete. This is not true, but counterexamples are rather convoluted. The reason for this is that requiring the category to be accessible, which ensure that one of the properties implies both of them.

Preservation of limits and colimits by functors.

Defs

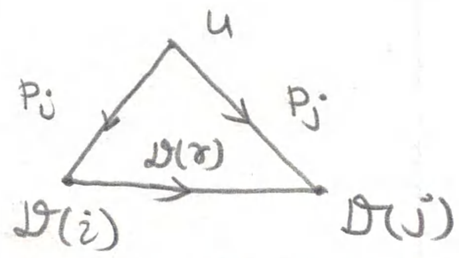
(i) suppose $D: I \to e$ is a diagram that admit a (co)limit. we say that a functor $F: e \to E$

Cone $\mathcal{D}: I \longrightarrow \mathcal{C}$ (diagram)

(U, P_i) $U \in \mathcal{C}$

$P_i : U \longrightarrow \mathcal{D}(i)$

$\mathcal{D}(\gamma) P_i = P_j \quad \forall\ \gamma : i \to j$



preserve this (co)limit if applying F to the components of the (co)limit (co)cone over $\mathcal{D}$ produces a (co)limit (co)cone over $F(\mathcal{D})$

i.e. If $\forall\ (U, P_i)$ a terminal (limit cone) cone over $\mathcal{D}$, $(F(U), F(P_i))$ is a terminal cone over $F(\mathcal{D})$.

(ii) Let $\mathcal{C}, \mathcal{E}$ be two complete categories. $F: \mathcal{C} \to \mathcal{E}$ is (co)continuous if it preserve (co)limits of small diagrams in $\mathcal{C}$.

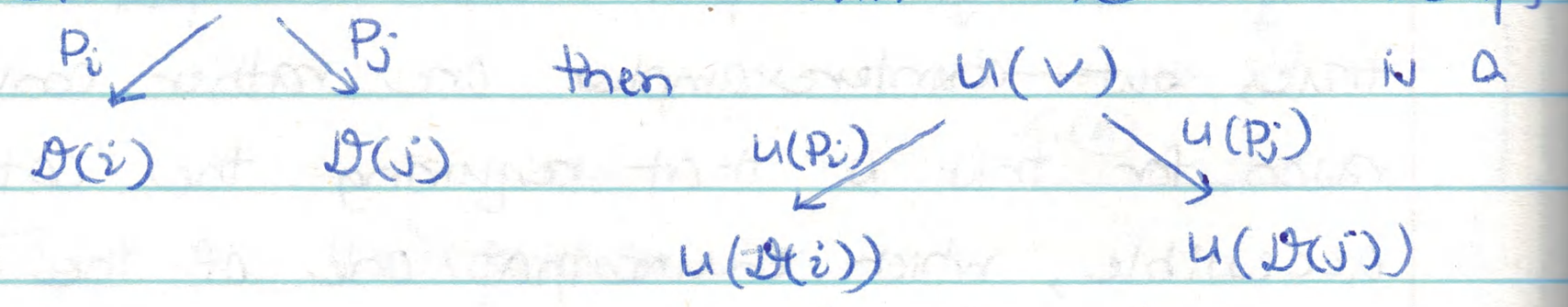
$F(\lim \mathcal{D}) \cong \lim (F \circ \mathcal{D}) \quad \forall\ \mathcal{D}$

Example

① U : Group $\xrightarrow{\text{forgetful}}$ Set is continuous.

i.e.

If V is a limit cone in Group,

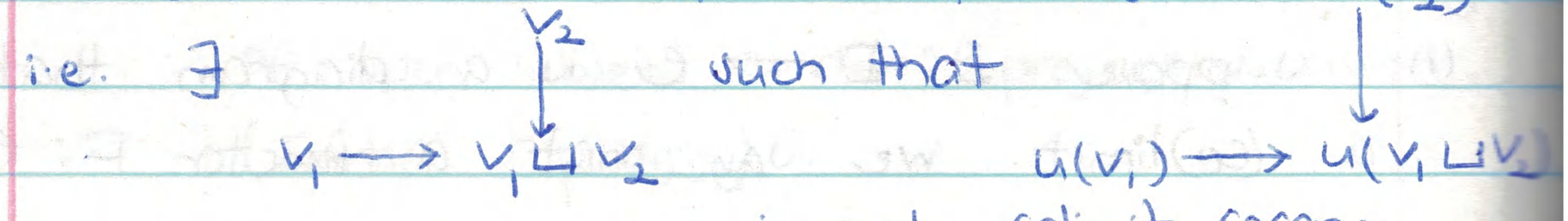


is a

limit cone in Set. This follows from the construction of limits in Group (in general any algebraic variety).

However U is not cocontinuous.

i.e. $\exists$ V_2 such that



is not colimit cocone (coproduct)

$F \cong Hom(-, X)$ $A \mapsto F(A)$ (rep. object)

$A \xrightarrow{f} B \mapsto F(A) \xrightarrow{f} F(B)$

In $Vect_{\mathbb{K}}$

$V_1 \sqcup V_2 = V_1 \oplus V_2$ and $U(V_1 \oplus V_2) = U(V_1) \times U(V_2)$

$\neq U(V_1) \sqcup U(V_2)$

$V_2 \downarrow$ $U(V_2) \downarrow$

$V_1 \longrightarrow V_1 \oplus V_2$ $U(V_1) \longrightarrow U(V_1 \oplus V_2)$

Thus $U: Vect_{\mathbb{K}} \longrightarrow Set$ is also not cocontinuous.

(2) $U: Top \longrightarrow Set$ is continuous and cocontinuous.

Representable Functors

Def^n

Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$.

a) the represented functor of X is a functor

$Hom(-, X): \mathcal{C}^{op} \longrightarrow Set$

$A \mapsto Hom(A, X)$

$A \xrightarrow{f} B \mapsto Hom(B, X) \longrightarrow Hom(A, X)$

$B \xrightarrow{g} X \mapsto A \xrightarrow{gf} X$

A functor $F: \mathcal{C}^{op} \longrightarrow Set$ is representable if it is naturally isomorphic to a functor $Hom(-, X)$ for some representing object $X \in \mathcal{C}$, which is unique up to a unique isomorphism.

b) Dually, we have the corepresented functor of X, namely $Hom(X, -): \mathcal{C} \longrightarrow Set$, and we say

that a functor $F: \mathcal{C} \to Set$ is corepresentable if it is isomorphic to $Hom(X, _)$ for some corepresenting object $X \in \mathcal{C}$.

Example

① Free Group

$$F: Group \longrightarrow Set$$

$$G \longmapsto U(G) \times U(G)$$

$$G \xrightarrow{\varphi} G' \longmapsto U(G) \times U(G) \xrightarrow{\varphi \times \varphi} U(G') \times$$

Claim: F is corepresentable

i.e. $F \cong mor(X, _)$ for some $X \in Group$

The representing object is the free group on two elements $X = \mathbb{Z} * \mathbb{Z}$

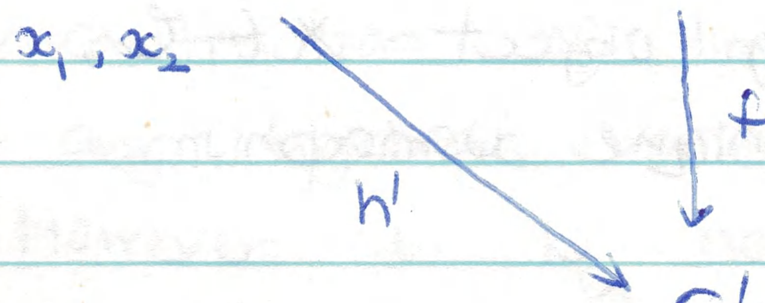
$$X \xrightarrow{h} G$$

$$(x_1, x_2) \xrightarrow{F(h)} (h(x_1), h(x_2))$$

$$\in U(X) \times U(X) \qquad \in U(G) \times U(G)$$

$$X \xrightarrow{h} G \qquad f(h(x_1)) = h'(x_1)$$

$$f(h(x_2)) = h'(x_2)$$



$$F(G) \qquad U(G) \times U(G)$$

$F(f)$ $\qquad$ $f \times f$

$$F(G') \qquad U(G') \times U(G')$$

$\otimes$: "otimes"

bilinear: linear in each argument

eg: inner product
quadratic form

(2) Tensor products ($Vect_{\mathbb{K}}$ or Mod_R)

Let $V, W \in Vect_{\mathbb{K}}$. Define

$F: Vect_{\mathbb{K}} \longrightarrow Set$

$A \longmapsto F(A) = \{ b : V \times W \xrightarrow{b} A$ is $\mathbb{K}$-bilinear $\}$

$A \xrightarrow{f} A' \longmapsto F(A) \xrightarrow{F(f)} F(A')$

$b \longmapsto b'$ such that

$F(f)(b)(v,w)$

$= f(b(v,w))$

$= b'(v,w)$

Bilinear

$B: V \times W \longrightarrow A$ such that

(i) $\forall v \in V$, $W \longrightarrow A$

$w \longmapsto B(v,w)$ is linear

(ii) $\forall w \in W$, $V \longrightarrow A$

$v \longmapsto B(v,w)$ is linear

Using the continuity of F, we can show that it is corepresentable. Its corepresenting object is is known as the "tensor product of V and W over $\mathbb{K}$" and, is denoted by $V \otimes_{\mathbb{K}} W$. Elements of $V \otimes W$ are known as tensors.

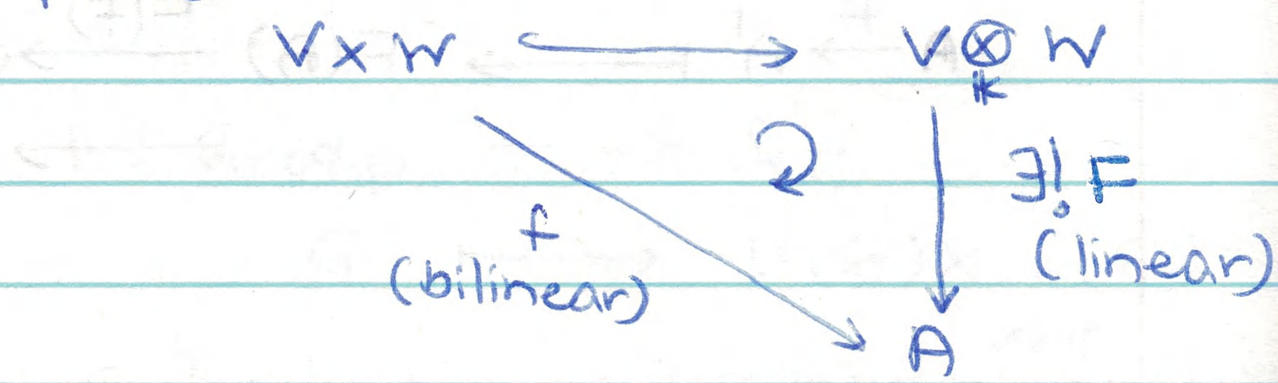
Further $F(A) \cong Hom(V \otimes W, A)$

i.e. $\{ V \times W \xrightarrow{\mathbb{K}\text{-bilinear}} A \} \cong \{ V \otimes W \xrightarrow{\mathbb{K}\text{-linear}} A \}$

In particular, the identity map $V \otimes W \to V \otimes W$ corresponds to a bilinear map $V \times W \to V \otimes W$, $(v, w) \mapsto v \otimes w$

which is known as the 'universal bilinear map'.

Universal property



a) Properties of tensors

$\forall\, v \in V$ and $w \in W$

$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

$(\lambda v) \otimes w = v \otimes (\lambda w) = \lambda (v \otimes w) \quad \forall\, \lambda \in k.$

b) Elements of the form $v \otimes w$ (elementary tensors) span $V \otimes W$

c) If $\{e_i\}_{i \in I}$ is a basis for V and $\{f_j\}_{j \in J}$ is a basis for W, then $\{e_i \otimes f_j\}_{I \times J}$ is a basis of $V \otimes W$. In particular

$$\dim (V \otimes W) = \dim V \cdot \dim W$$

$$\dim (V \oplus W) = \dim V + \dim W$$

$V \times W \xrightarrow{\text{bilinear}} A \quad \cong \quad V \otimes W \xrightarrow{\text{linear}} A$

$B \longmapsto f$

$f(v \otimes w) = B(v, w)$

knowing the value of f on all elementary tensor define f uniquely

$B \longleftarrow f$

$B(v, w) = f(v \otimes w)$

Free

(3). Loop space

$\Omega : \text{Top} \longrightarrow \text{Set}$

$X \longmapsto$ set of all loops in X

$= \{ S^1 \xrightarrow{ctu} X \}$

$= \text{Hom}(S^1, X)$

$x \xrightarrow{f} Y \longmapsto \text{Hom}(S^1, X) \to \text{Hom}(S^1, Y)$

$S^1 \xrightarrow{g} X \longmapsto fg$

$f \downarrow$

Y

Remark

A representable functor $e^{op} \longrightarrow$ Set always preserve limits. This is an immediate consequence of the universal property of limits in e^{op} (colimits in e) and limits in e. For instance, the binary (co)products we have bijections

$\text{Hom}(X, A \times B) \longrightarrow \text{Hom}(X, A) \times \text{Hom}(X, B)$ and

$\text{Hom}(A \sqcup B, X) \longrightarrow \text{Hom}(A, X) \times \text{Hom}(B, X)$

In Set, $(Y^X)^A = Y^{A \times X}$

$\{A \to Y^X\} \cong \{A \times X \to Y\}$

Typical Application

(i) Construct some functor $F: \mathcal{C} \to Set$

(ii) prove that F is continuous
i.e. Preserve limits (usually trivial)

(iii) Invoke an abstract existance theorem to show that F is corepresentable.
i.e. F is naturally isomorphic to some
$Mor(X, _)$ $X \in \mathcal{C}$

(iv) The object X is unique upto a unique isomorphism. e.g. $V \otimes_K W$

Freyd's Representability Theorem

Let $F: \mathcal{C} \to Set$ be a continuous functor. If it satisfy the solution set condition:
there exist a set $\Phi \subseteq \mathcal{C}$ so that for any $c \in \mathcal{C}$ and any $x \in F(c)$, there is an $s \in \Phi$, an element $y \in F(s)$, and a morphism $s \xrightarrow{f} c$ so that $F(f)(y) = x$.
then F is representable.

④ Function space

Let $X, Y \in Top$ and consider the functor

$F: Top^{op} \longrightarrow Set$

$A \longmapsto C(A \times X, Y)$

$A \to A' \longmapsto C(A' \times X, Y) \longrightarrow C(A \times X, Y)$

Lattice : Finitely generated free abelian group with an inner product

$\mathbb{Z}^n \cong A$

$A \times A \longrightarrow \mathbb{R}$ this gives $A \hookrightarrow \mathbb{R}^n$

$$A \times X \xrightarrow{(f, 1_X)} A' \times X \xrightarrow{g} Y$$

F preserve limits if and only if the functor

$- \times X : \mathrm{Top} \longrightarrow \mathrm{Top}$

$A \longmapsto A \times X$ preserve colimits, equivalently coequalizers. The latter condition holds iff X is core-compact:

For any open $U \in \tau_X$, we have $U = \bigcup_{V \ll U} V$. i.e. any open cover of U admit a finite subcover of V.

If X is Hausdorff, it is core-compact iff it is locally compact. The representing object is usually denoted by Y^X. By definition

$$\{ A \xrightarrow{\text{continuous}} Y^X \} \cong \{ A \times X \xrightarrow{\text{continuous}} Y \}.$$

In particular $Y^X = \{ X \xrightarrow{\text{continuous}} Y \}$.

$\mathrm{Hom}(A, Y^X) \cong \mathrm{Hom}(A \times X, Y)$: representability

the topology on Y^X is generated by a subbasis consisting of sets $O_{U,V} = \{ X \xrightarrow{f} Y : U \ll f^{-1}(V) \}$, where $U \subseteq X$, $V \subseteq Y$ are open. If X is Hausdorff, this topology coincides with the compact-open topology.

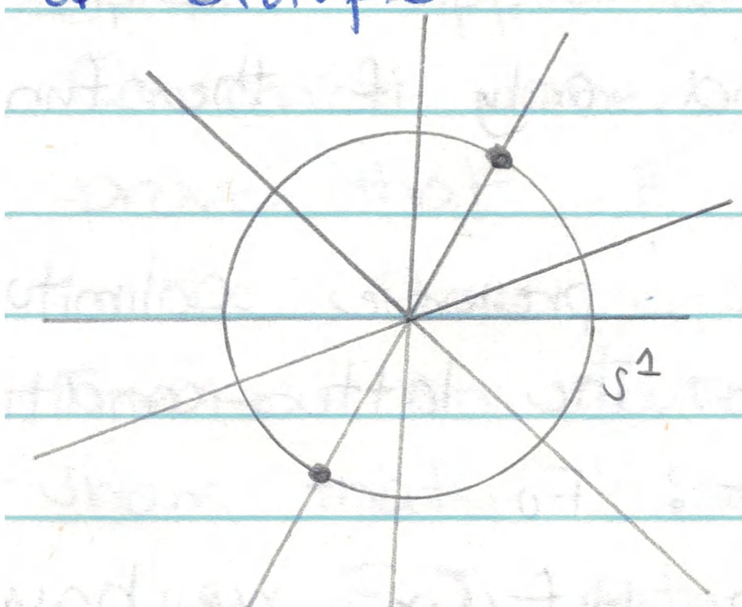
(5) Grassmannian

$V \in \mathrm{Vect}_{\mathbb{R}}$ (any vector space also works)

$$\mathrm{Gr}_n(V) = \{ W \subseteq V : \dim(W) = n \}$$

$0 \leqslant n \leqslant \dim(V)$

For example: $V = \mathbb{R}^2$ $\quad n=1$



$Gr_1(\mathbb{R}^2) = \{ w \subseteq \mathbb{R}^2 : \dim(w) = 1 \}$

$= S^1 / x \sim -x$ $\quad$ ← set

$z \longmapsto z^2$

$\simeq S^1$ $\quad$ quotient map.

$Gr_0(V) = \{0\}$

$Gr_1(V) = \mathbb{PR}^{\dim V - 1}$

$Gr_{\dim(V)}(V) = \{V\}$

How can we equip $Gr_n(V)$ with a (suitable) structure of a

topological space?

smooth manifold?

algebraic variety?

Answer: Representable functor!

Philosophy / Thesis

To define spaces, space like structures we represented functors. e.g. Y^X

To define algebraic structures use coprepresentable functors e.g. $X \otimes_{\mathbb{K}} Y$

Lets define the representable functor of

$Gr_n(V)$: Top

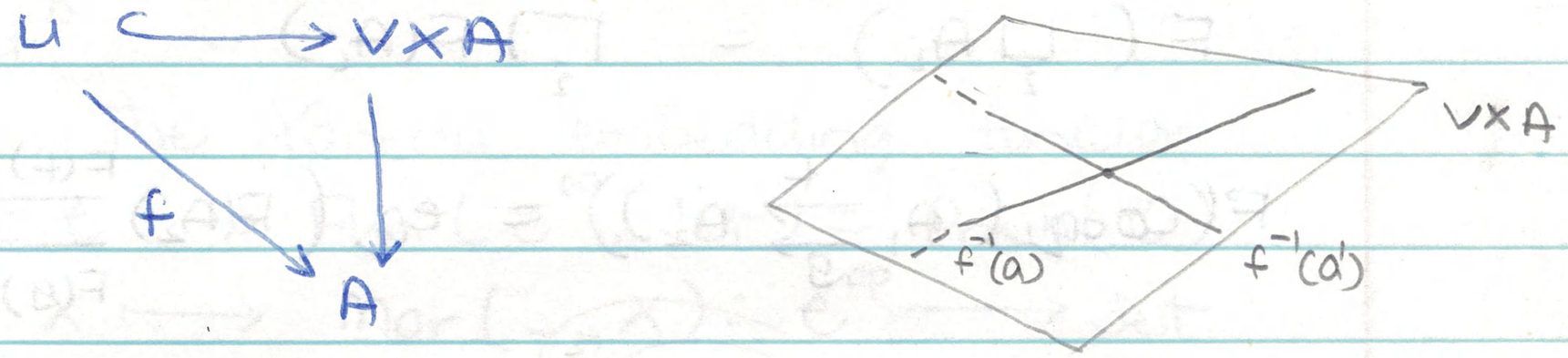
what is a continuous map

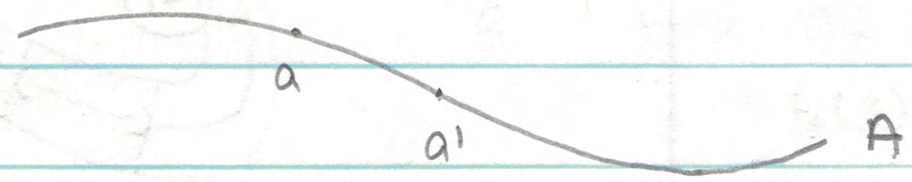
$$A \longrightarrow Gr_n(V) \quad ?$$

(arb. top. space)

$\forall a \in A$

$f^{-1}(a) = W \subseteq V$ is required to be a vector subspace of dim n.





$U(A) \longrightarrow Gr_n(V)$ is continuous if locally on $A = \bigcup_{i \in I} B_i$ (open cover) $\exists B_i \xrightarrow[g]{\text{ctn}} GL(V)$ automorphism of V

such that

$$g \circ u\big|_{f^{-1}(B_i)} = \mathbb{R}^n \times B_i$$

B_i B_i

$\mathbb{R}^n \subseteq V$ (fixed n-dim subspace)

$n|m$ iff any prime that divides n also divides m (with multiplicity)

(2) Cayley's theorm is an example of Yoneda embedding

"Any group is isomorphic to a subgroup of a permutation group"

(3) Mat_R

objects: positive integers

$\mathrm{Mor}(m,n)$: $n \times m$ matrices whose entries are elements of ring R

Yoneda embedding in this context says "every row operation on matrices with n rows is defined by left multiplication by some $n \times n$ elementary matrix"

Let $X, Y \in \mathrm{Top}$

(i) $\mathrm{Mor}(\{*\}, X) \cong \mathrm{Mor}(\{*\}, Y)$

$\Leftrightarrow$ same cardinality

(ii) $\mathrm{Mor}(I, X) \cong \mathrm{Mor}(I, Y)$

$\Leftrightarrow$ same path space

(iii) $\mathrm{Mor}(S^1, X) \cong \mathrm{Mor}(S^1, Y)$

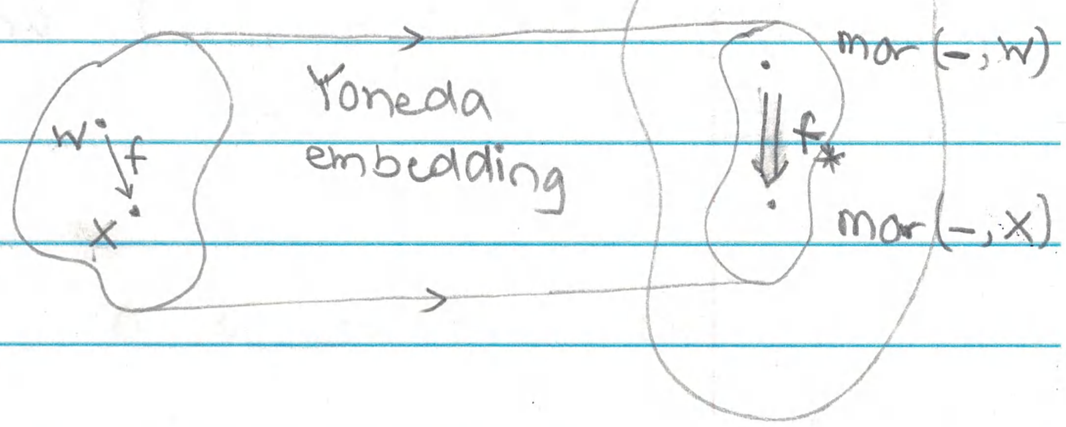
$\Leftrightarrow$ same free loop space

If $\mathrm{Mor}(-, X) \cong \mathrm{Mor}(-, Y)$, what do we know about X and Y?

$Y : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) = \mathrm{Set}^{\mathcal{C}^{op}}$

$X \longmapsto \mathrm{Mor}(-, X)$

$f \longmapsto f_*$ (pullback)



Presheves on $\mathcal{C}$

Does Y preserve that X shares with other objects in $\mathcal{C}$?

$\mathrm{Mor}(w, x) \longrightarrow \mathrm{Mor}(\mathrm{Mor}(-, w), \mathrm{Mor}(-, x))$

$f \longmapsto f_*$

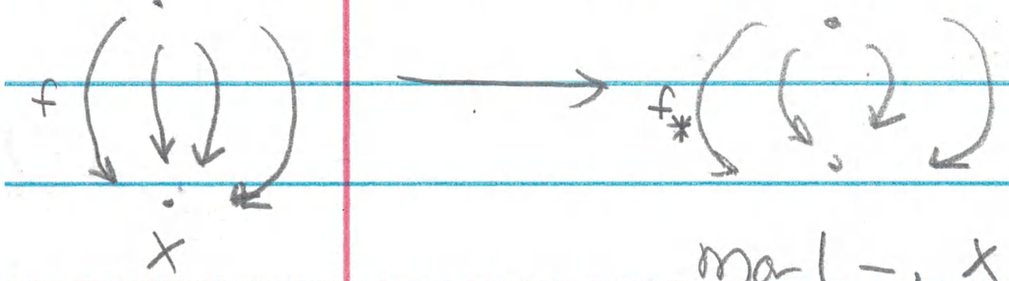
Is this map a bijection?

$\mathcal{C}^{op} \longrightarrow$ Set

$A \xrightarrow{g} B$

$\mathrm{Mor}(-, w) \Rightarrow \mathrm{Mor}(-, x)$

$$\begin{array}{ccc} \mathrm{Mor}(A, w) & \xleftarrow{g_*} & \mathrm{Mor}(B, w) \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ \mathrm{Mor}(A, x) & \xleftarrow{g_*} & \mathrm{Mor}(B, x) \end{array}$$



$\mathrm{Mor}(-, w) \longrightarrow \mathrm{Mor}(-, x)$

X and Y are isomorphic if and only if their represented functors $\mathrm{Mor}(-, X)$ and $\mathrm{Mor}(-, Y)$ are isomorphic.

In other words, there is a bijection

$$\mathrm{Mor}(X, Y) \cong \mathrm{Mor}(\mathrm{Mor}(-, X), \mathrm{Mor}(-, Y))$$

In fact the Yoneda lemma says this is true for any functor $F \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$.

(not only for $\mathrm{Mor}(-, Y)$)

Thm

$$\mathrm{Mor}(\mathrm{Mor}(-, X), F) \cong F(X)$$

$$\{\mathrm{Mor}(-, X) \overset{\alpha}{\Rightarrow} F\} \longrightarrow \{F(X)\}$$

$$\alpha_X : \mathrm{Mor}(X, X) \Rightarrow F(X) \longmapsto \alpha_X(1_X)$$

$$1_X \longmapsto \alpha_X(1_X)$$

$$\beta_w : \mathrm{Mor}(w, X) \to F(w) \longleftarrow C$$

$$w \xrightarrow{g} X \longmapsto F(g)(c)$$

Example

① Poset P

objects : elements of P

$$\mathrm{Mor}(a, b) = \begin{cases} \{*\}, & \text{if } a \leq b \\ \phi, & \text{otherwise} \end{cases}$$

i.e $a \to b$ iff $a \leq b$

Yoneda lemma says

$a \leq b$ if and only if, for any $c \in P$ we have $c \leq a \Rightarrow c \leq b$

(i) In $(\mathbb{R}, \leq)$, this says any real number is uniquely determine by its "Dedekind cuts"

(ii) In $(\mathbb{N}, \leq)$, $n \leq m$ this says if $n | m$

$$\mathrm{Top}^{op} \xrightarrow{F} \mathrm{Set}$$

$$A \longmapsto \{A \xrightarrow{cts} Gr_n(V)\} = \mathrm{Hom}(A, Gr_n(V))$$

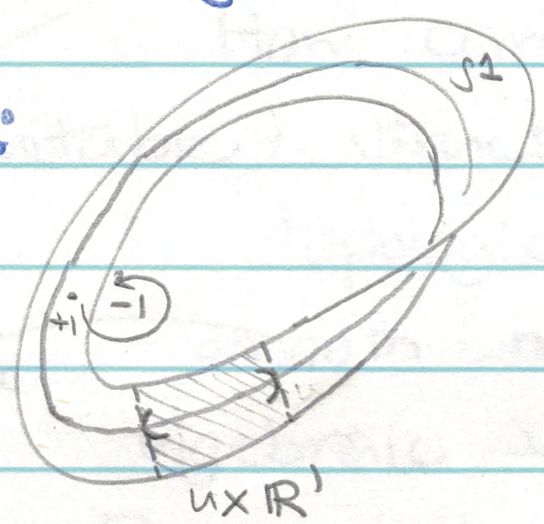
$$A \xrightarrow[h]{cts} A' \longmapsto \{A' \xrightarrow[g]{cts} Gr_n(V)\} \longrightarrow \{A \xrightarrow[goh]{} Gr_n(V)\}$$
(continuous)

It remains to show that F preserve limits.
(equivalently products and equalizers)

$$F\left(\coprod_i A_i\right) = \prod_i F(A_i)$$

$$F(\mathrm{Coeq}(A_1 \underset{g}{\overset{f}{\rightrightarrows}} A_2)) = \mathrm{eq}\left(F(A_2) \underset{F(g)}{\overset{F(f)}{\rightrightarrows}} F(A_1)\right)$$

Example :



$A = S^1$

$A \longrightarrow Gr_1(\mathbb{R}^2)$

(möbius bundel)

möbius

$S^1 \setminus \{p\} \longleftrightarrow$

Trivial bundel

The Yoneda Lemma

Suppose we have a representable functor $F: \mathcal{C}^{op} \to Set$. How can we construct its representing object? To answer this question, we first investigate how $\mathcal{C}$ embeds into $Fun(\mathcal{C}^{op}, Set)$.

Def^n

The Yoneda embedding functor

$$Y: \mathcal{C} \longrightarrow Fun(\mathcal{C}^{op}, Set) = Presh(\mathcal{C})$$

$$X \longmapsto Mor(-, X): \mathcal{C}^{op} \longrightarrow Set$$

$$A \longmapsto Mor(A, X)$$

$$A \xrightarrow{f} A' \longmapsto Mor(A', X) \to Mor(A, X)$$

$$g \longmapsto gf = f_*(g)$$

$$X \xrightarrow{f} X' \longmapsto \alpha: Mor(-, X) \Rightarrow Mor(-, X')$$

(natural transformation)

$$\alpha_A: Mor(A, X) \xrightarrow{f \circ -} Mor(A, X')$$

(we have to show the naturality)

Observation

A functor $F: \mathcal{C}^{op} \to Set$ is representable if and only if there is $X \in \mathcal{C}$ such that $F \cong Y(X) = Mor(-, X)$.

Proposition

$Y: \mathcal{C} \longrightarrow Fun_{\text{representable}}(\mathcal{C}^{op}, Set)$ is an equivalance of categories.

Philosophy

non-representable functors are "generalized objects" glued from objects in $\mathcal{C}$

starting from an arbitrary category $\mathcal{C}$, one can construct a category of "generalized objects glued from objects of $\mathcal{C}$". Formally, such a generalized object G is specified by defining the abstract set of morphisms $X \to G$ $\forall X \in \mathcal{C}$ and specifying how map $X \to G$ can be compose with $X' \to X$ in $\mathcal{C}$ to give a map $X' \to G$.

Example

① Simplices ($\mathcal{C}$)

Objects: Finite, nonempty, linearly ordered $\{0 < 1 < 2 < \dots < n\}$, $n \geq 0$

Morphisms: nondecreasing functions

Then $Fun(\mathcal{C}^{op}, Set)$ is known as Set (category of simplicial sets), which should be thought as a bunch of simplices glued together

The Yoneda embedding sends a simplex to the corresponding simplicial set consisting of that simplex alone.

(2) Generalized smooth manifolds (smooth spaces)

e ob: $\{\mathbb{R}^n : n \geq 0\}$ (finite dimensional real vector spaces)

$\mathrm{Mor}(\mathbb{R}^n, \mathbb{R}^m) : C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ (smooth functions)

$\mathrm{Fun}(e^{op}, \mathrm{Set})$ is the category of "generalized manifolds" or "smooth spaces." This contains the category of smooth manifolds as a full subcategory. The Yoneda embedding sends $\mathbb{R}^n$ to itself now considered as a generalized manifold.

If X is a generalized manifold, we know smooth functions $\mathbb{R}^n \longrightarrow X \ \forall n \geq 0$

- Any $\mathbb{R}^k$ is a smooth space and we know $\mathbb{R}^n \longrightarrow \mathbb{R}^k$
- $\underline{C}^\infty(\mathbb{R}^n, \mathbb{R}^m) = \mathrm{Fun}(e^{op}, \mathrm{Set})$

  $\underline{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)(\mathbb{R}^k) : \mathbb{R}^k \longrightarrow (\mathbb{R}^n \longrightarrow \mathbb{R}^m)$

  $= \mathbb{R}^n \times \mathbb{R}^k \xrightarrow{\text{smooth}} \mathbb{R}^m$

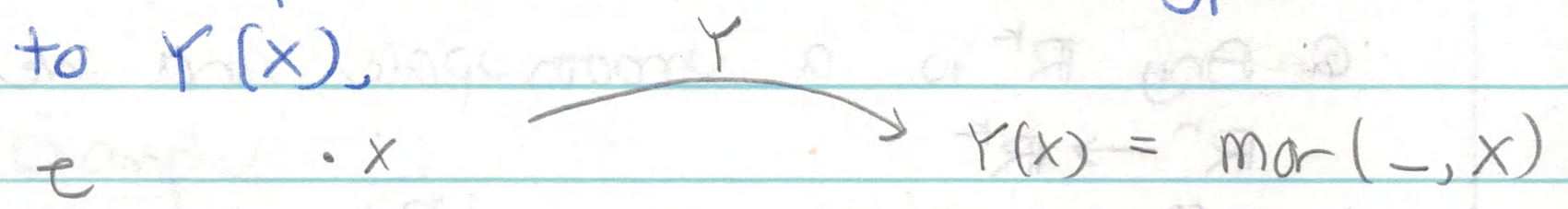
In $k=0$, points in $\underline{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ are smooth functions $\mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\forall\, \alpha : \mathrm{mor}(-,X) \Rightarrow \mathrm{mor}(-,Y)\ \ \exists!\ X \xrightarrow{f} Y : \alpha = f_*$ (pullback)
Every natural transformation between representable functors arise from a morphism between their representing object.

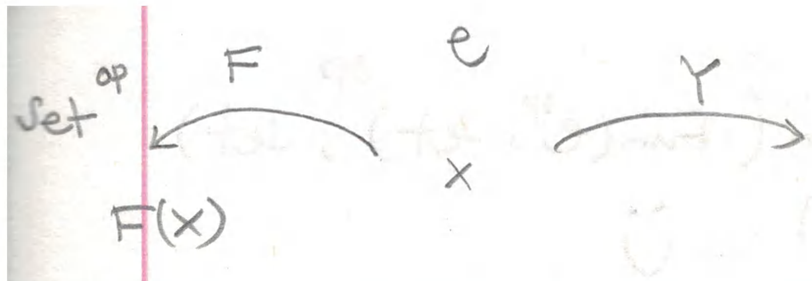
(3) In algebraic geometry $e = \mathrm{CRing}^{op}$
$= \mathrm{Aff}\mathcal{Sch}$
(affine schemes)

$\mathrm{Fun}(\mathrm{CRing}, \mathcal{Set})$ is the category of "generalized schemes" (Grothendieck functor of points formalism). This contains the category of schemes, Affine shemes, $\mathcal{C}^\infty(\mathbb{R}^n)$, manifold and algebraic spaces as full subcategories. The embedding sends an affine scheme to itself, considered as a generalized scheme.

Consider an object $X \in e$ and its incarnation $Y(X) \in \mathrm{Fun}(e^{op}, \mathcal{Set})$ as a generalized object. Given another object $W \in e$, we can come up with two different types from W to $Y(X)$,



namely $Y(X)(W) = \mathrm{mor}(W, X)$ and a $Y(X) \longrightarrow Y(W)$. A priori, these two maps may be different, but the Yoneda lemma shows that they are in fact exactly the same.



Lemma (The Yoneda lemma)

$\forall X \in e \quad \forall F: e^{op} \longrightarrow Set,$

$Mor(Y(X), F) \cong F(X)$

This isomorphism of sets is natural in both X and F

morally $X \longrightarrow F$

Here each natural transformation $\alpha: Mor(-, X) \Rightarrow F$ get maps to the element $\alpha_X(1_X) \in F(X)$.

Sketch of the proof:

A natural transformation $\alpha: Y(X) \Rightarrow F$ has components $Mor(W, X) \longrightarrow F(W)$ for any $W \in e$. In particular taking $W = X$ yeldo a function $\alpha_X: Mor(X, X) \longrightarrow F(X)$, which we can evaluate on 1_X to obtain an element of $F(X)$.

Vice versa, an element $p \in F(X)$ yeilds a natural transformation $Mor(-, X) \Rightarrow F$ with components $Mor(W, X) \longrightarrow F(W)$ for any $W \in e$ that sends a map $h: W \longrightarrow X$ to the element $F(h)(p) \in F(W)$.

By inspection, the two natural transformations constructed above are mutually inverse, which completes the proof.

$Fun(\mathcal{C}^{op}, Set) \longrightarrow Fun(Fun(\mathcal{C}^{op}, Set)^{op}, Set)$
is also fully faithful ☺

Proposition

$Y : \mathcal{C} \longrightarrow Fun(\mathcal{C}^{op}, Set)$ is fully faithful.

proof

$$\begin{aligned} Mor(X, Y) &\cong Mor(Mor(_, X), Mor(_, Y)) \\ &= Mor(Y(X), Y(Y)) \\ &= Y(Y)(X) \end{aligned}$$

(Consequence of Yoneda lemma)

However this functor is not essentially surjective.

Remark

This category $Fun(\mathcal{C}^{op}, Set)$ is always complete and cocomplete for any $\mathcal{C}$.
All small limits and colimits exists.

$\mathcal{G} \curvearrowright M$ that is not free, the quotien space $M/\mathcal{G}$ in general not in $\mathcal{C}$, but in $Fun(\mathcal{C}^{op}, Set)$.

Proposition

Colimit preserving functors $Fun(\mathcal{C}^{op}, Set) \xrightarrow{Y} \longrightarrow \mathcal{D}$ are equivalent to functors $\mathcal{C} \longrightarrow \mathcal{D}$.

i.e. Y is the universal cocompletion functor.

Proposition

$$\mathcal{D}: I \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$$

$$\lim \mathcal{D}(X) = \lim_{i \in I} \mathcal{D}(i)(X)$$

$$\mathrm{colim}\, \mathcal{D}(X) = \mathrm{colim}_{i \in I} \mathcal{D}(i)(X)$$

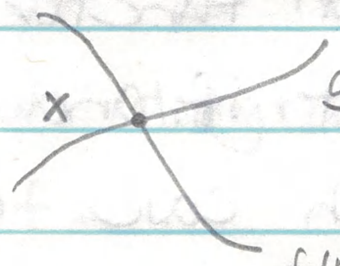
limits and colimits in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ are computed pointwise (indexwise).

Example

$$\mathbb{R} \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{R}^2$$

$$\mathbb{R} \xrightarrow{Y} \mathrm{Fun}(\mathbb{R}^{op}, \mathrm{Set})$$

(smooth spaces)



(derived geometry)

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{R} \\ \downarrow & & \downarrow g \\ \mathbb{R} & \xrightarrow[f]{} & \mathbb{R}^2 \end{array}$$

the intersection of f and g can be computed as an equalizer.

Proposition

If $F: \mathcal{C}^{op} \to \mathrm{Set}$ is representable and X, Y are representing objects of F, then $X \cong Y$.

proof:

Since Y is a functor if $X \cong Y$, then $\mathrm{Mor}(-, X) \cong \mathrm{Mor}(-, Y)$ (functors preserve isomorphism).

On the other hand Y is fully faithful functor.
Thus $Y(X) \cong Y(Y)$ implies $X \cong Y$.

Suppose $h: Y(X) \longrightarrow Y(Y)$ is an isomorphism

Since Y is fully faithful $\exists!\ X \underset{}{\overset{f}{\rightleftarrows}} Y$ s.t. $Y(f) = h$ and $Y(g) = h^{-1}$

Then $1_{Y(X)} = h^{-1} \circ h = Y(g) \circ Y(f)$

$= Y(fg) = Y(1_X)$

Since Y is faithful $fg = 1_X$ and similarly $gf = 1$

Existance of representing Objects.

In this section we construct a certain diagram $D: I \to \mathcal{C}$ from a functor $F: \mathcal{C}^{op} \to$ Set such that colim D exist if and only if F is representable, in which case colim D is a representing object of F.

Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. Consider the slice category of $\mathcal{C}$ over X.

$\mathcal{C}/X$ objects : $W \xrightarrow{f} X$

mor (f, g): $W \xrightarrow{f} X$, $V \xrightarrow{g} X$, $h: W \to V$ with $gh = f$

Consider the diagram $D: \mathcal{C}/X \to \mathcal{C}$ defined by $D(W \xrightarrow{f} X) = W$

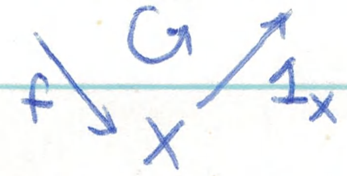
$D\left(\begin{array}{ccc} W & \longrightarrow & X \\ & \searrow \quad \nearrow & \\ & V & \end{array}\right) = W \longrightarrow V$

Then $\mathcal{D}$ has a colimit, $\operatorname{colim}\mathcal{D} \cong X$. In particular X can be recover from $\mathcal{D}$.

For: $\mathcal{C}/X$ has a terminal object, namely

$$X \xrightarrow{1_X} X.$$

For all $W \xrightarrow{f} X$



Thus $\operatorname{colim}\mathcal{D} = \mathcal{D}\left(X \xrightarrow{1_X} X\right)$

$= X.$

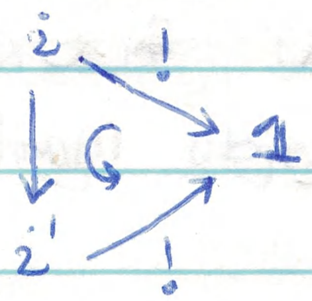
Lemma

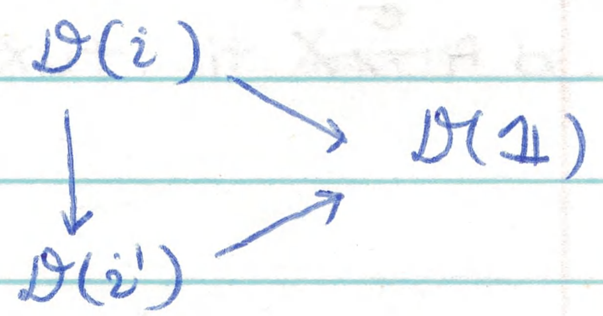
If I has a terminal object $\mathbb{1}$ and $\mathcal{D}: I \to \mathcal{C}$ is a diagram, then $\operatorname{colim}\mathcal{D} = \mathcal{D}(\mathbb{1})$

proof:

$\forall i \in I \quad \{\mathcal{D}(i) \xrightarrow{c_i} \mathcal{D}(\mathbb{1})\}$ is a cocone (c_i : injection map)

over $\mathcal{D}(\mathbb{1})$, and a $\exists!\ i \longrightarrow \mathbb{1}$





universal property

For any cocone $\{\mathcal{D}(i) \longrightarrow W\}_{i \in I}$ there is a unique $\mathcal{D}(\mathbb{1}) \longrightarrow W$.

now take $i = \mathbb{1}$ in $\mathcal{C}$, for this unique morphism. Example: initial object

$0 \to 1 \to 2 \to \cdots$: sequantial colimit

$0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots$: sequantial limit

Defⁿ

The category of elements El(F) of a functor $F: e^{op} \to Set$ defines as follows:

Objects : (X, e) , $X \in e$ and $e \in F(X)$

$mor((X,e),(X',e')) : X \xrightarrow{f} X'$ in e such that

$$F(f)(e') = e$$

$$X \xrightarrow{f} X'$$

$$F(X') \xrightarrow{F(f)} F(X)$$
$$e' \longmapsto e$$

Observation

Let $F: e^{op} \longrightarrow Set$

$$A \longmapsto mor(A, X) \text{ , } X \in e$$

representable functor.

$El(F) =$ Objects : $(A, e \in mor(A, X))$

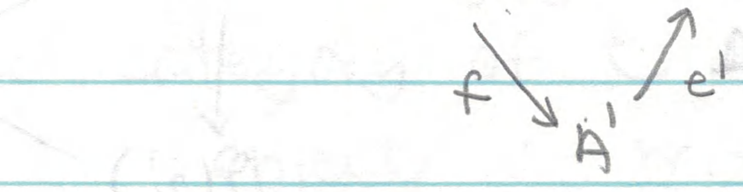
$mor((A, e \in mor(A,X)), (A', e' \in mor(A', X)))$

$: A \xrightarrow{f} A'$ in e such that

$$F(f)(e') = e$$

$A \xrightarrow{e} X$

$A \xrightarrow{e} X$ in e



$$mor(A', X) \xrightarrow{F(f)} mor(A, X)$$
$$e' \longmapsto e' \circ f$$

Here $El(mor(-, X)) = e/X$

Defⁿ

diagram in category El(F)

The Grothendieck construction $\int F$ of a functor $F: e^{op} \longrightarrow Set$ is a functor $\int F: El(F) \longrightarrow e$ that sends $(X, e) \longmapsto X$ and

$$(X, e) \longrightarrow (X', e') \longmapsto X \longrightarrow X'.$$

Observation

If $F: e^{op} \longrightarrow Set$

$A \longmapsto mor(A, X)$, then

$\int F: El(F) \longrightarrow e$

$e/X \longrightarrow e$

$w \xrightarrow{f} X \longmapsto w$

$w \xrightarrow{f} X \longmapsto w \longrightarrow v$

$\downarrow g \quad \nearrow g$

v

is the forgetful functor

moreover $colim(\int F) \overset{Lemma}{=} (\int F)(X \xrightarrow{1_X} X)$

$= X$ - the representing object of F

Therefor, for a given $F: e^{op} \to Set$

$\int F: El(F) \to e$, if F is representable, $colim(\int F)$ is the representing object. Vice versa if $colim(\int F)$ exist, it must be the representing object.

Theorem

If e has arbitrary colimits then the presheaf of sets $F: e^{op} \to Set$ preserve limits, and F is representable if and only if $colim \int F$ exists. In this case, $colim \int F$ is a representing object of F and the cocone over $colim \int F$ encodes precisely an isomorphism $F \longrightarrow Y(colim(\int F))$.

Example

① Group $\qquad U: \text{Group} \xrightarrow{\text{forgetful}} \text{Set}$

$G \in \text{Group} \qquad U(G) \in \text{Set}$

$El(U): \quad (G, p \in U(G))$

$\qquad Mor((G,p),(H,p'))$

$\qquad\quad G \xrightarrow{\varphi} H \text{ such that } \varphi(p) = p'$

$\int U : El(U) \longrightarrow \text{Group}$

$\qquad (G,p) \longmapsto G$

$(G,p) \xrightarrow{\varphi} (H,p') \longmapsto G \xrightarrow{\varphi} H$

Locally compact Hausdorff abelian topological group

LCHA

Any object in this category carries an essentially unique natural Haar measure:

σ-algebra : Borel σ-algebra generated by compact sets

Haar measure : unique upto a positive multiplicative constants, countably additive, left translation invariant and regular (both outer and inner)

multiplicative

Character : continuous group homomorphism to $\mathbb{T}$.

(an element of the dual group)

(i) $\chi : \mathbb{R} \longrightarrow \mathbb{T}$

$s \longmapsto e^{i\alpha s}$, $\alpha \in \mathbb{R}$

$\hat{\mathbb{R}} = \mathrm{Hom}(\mathbb{R}, \mathbb{T})$

$= (\mathbb{R}, +)$

$\mu \longmapsto \left(\alpha \longmapsto \int \chi \, d\mu \right)$, μ-measure on $\mathbb{R}$

$\downarrow$ R-N derivative

$f\,ds \longmapsto \left(\alpha \longmapsto \int \chi \, d\mu \right)$

$f \longmapsto \left(\alpha \longmapsto \int_{\mathbb{R}} e^{i\alpha s} f(s)\, ds \right)$, $e^{i\alpha s} \in \hat{\mathbb{R}}$

: Fourier transformation

(ii) $\chi : \mathbb{T} \longrightarrow \mathbb{T}$

$z \longmapsto z^n$

$\hat{\mathbb{T}} = \mathrm{Hom}(\mathbb{T}, \mathbb{T})$

$= (\mathbb{Z}, +)$

$\mu \longmapsto \left(n \longmapsto \int z^n \, d\mu \right)$, μ-measure on $\mathbb{T}$.

$\downarrow$ R-N derivative

$f \longmapsto \left(n \longmapsto \int (\cos nt + i \sin nt) f(t)\, dt \right)$

$f : \mathbb{T} \longrightarrow \mathbb{C}$ (periodic function)

: Fourier series

(iii) $\chi : \mathbb{Z} \longrightarrow \mathbb{T}$

$n \longmapsto q^n$

μ: measure on $\mathbb{Z}$ such that

$d\mu = f(n)\, dn$

$f : \mathbb{Z} \longrightarrow \mathbb{C}$

$f \longmapsto \left(q \underset{\in \mathbb{T}}{\longmapsto} \sum_{n \in \mathbb{Z}} q^n f(n) = \sum_{n \in \mathbb{Z}} (\cos nt + i \sin nt) f(n) \right)$

: Fourier series

Hom - Tensor adjunction

$$\mathrm{Hom}_S(X \otimes_R Y, Z) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_S(Y, Z))$$

Adjunction

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. An adjunction between $\mathcal{C}$ and $\mathcal{D}$ is a triple (L, R, Ψ), where $L: \mathcal{C} \to \mathcal{D}$ is the lef adjoint functor, $R: \mathcal{D} \to \mathcal{C}$ is the right adjoint functor, and $\Psi : \mathrm{mor}(L_, _) \longrightarrow \mathrm{mor}(_, R_)$ is a (pair natural isomorphism of functor $\mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$.

$$(X, Y) \longmapsto \mathrm{mor}_{\mathcal{C}}(X, RY) \cong \mathrm{mor}_{\mathcal{D}}(LX, Y).$$

$\Psi_{X,Y}$ natural one to one correspondance: $LX \xrightarrow{f} Y$ and $X \xrightarrow{g} RY$ are adjoint to g.

Schematically we write

$L: \mathcal{C} \rightleftarrows \mathcal{D} : R$ or $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$.

Remark

We can unfold the definition of Ψ as follows:

For any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ we have an isomorphism of sets. $\Psi_{X,Y} : \mathrm{mor}(LX, Y) \to \mathrm{mor}(X, RY)$

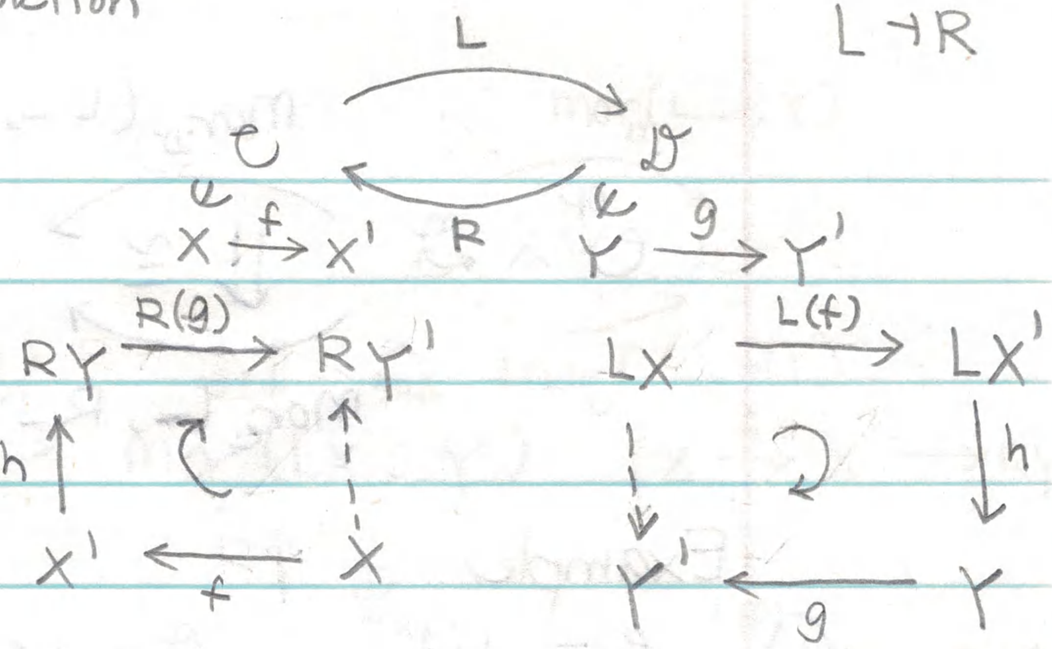
For any morphism $X \xrightarrow{f} X'$ in $\mathcal{C}$ and $Y \xrightarrow{g} Y'$ in $\mathcal{D}$ the following diagram commutes

$$\begin{array}{ccc} \mathrm{mor}_{\mathcal{D}}(LX', Y) & \xrightarrow{g \circ - \circ L(f)} & \mathrm{mor}_{\mathcal{D}}(LX, Y') \\ \downarrow \Psi_{X',Y} & \circlearrowright & \downarrow \Psi_{X,Y'} \\ \mathrm{mor}_{\mathcal{C}}(X', RY) & \xrightarrow{R(g) \circ - \circ f} & \mathrm{mor}_{\mathcal{C}}(X, RY') \end{array}$$

This is in fact a combination of two naturality

Hom-product adjuction

$L \dashv R$



In otherwords, maps $\mathrm{mor}(L-,-)$ are reformulation of maps $\mathrm{mor}(-,R-)$.

$$\mathrm{mor}_{\mathcal{D}}(LX,Y) \overset{\text{natural iso}}{\cong} \mathrm{mor}_{\mathcal{C}}(X,RY)$$, natural in both X and Y.

what are functors that are naturally isomorphic?

Fixed Y, $\mathcal{C}^{op} \xrightarrow{\mathrm{mor}(L-,Y)} \mathrm{Set}$ $\quad \Phi_{-,Y} = \alpha_Y \quad$ $\mathcal{C}^{op} \xrightarrow{\mathrm{mor}(-,RY)} \mathrm{Set}$

$X \longmapsto \mathrm{mor}_{\mathcal{D}}(LX,Y)$ $\quad \Rightarrow \quad$ $X \longmapsto \mathrm{mor}_{\mathcal{C}}(X,RY)$

Fixed X, $\mathcal{D} \xrightarrow{\mathrm{mor}(LX,-)} \mathrm{Set}$ $\quad \Phi_{X,-} = \beta_X \quad$ $\mathcal{D} \xrightarrow{\mathrm{mor}(X,R-)} \mathrm{Set}$

$Y \longmapsto \mathrm{mor}_{\mathcal{D}}(LX,Y)$ $\quad \Rightarrow \quad$ $Y \longmapsto \mathrm{mor}_{\mathcal{C}}(X,RY)$

thus there are two natural isomorphisms occurs in here.

naturality in X

$X \xrightarrow{f} X' \to RY$

$$\begin{array}{ccc} \mathrm{mor}_{\mathcal{D}}(LX',Y) & \xrightarrow{-\circ L(f)} & \mathrm{mor}_{\mathcal{D}}(LX,Y) \\ \Phi_{X',Y}\downarrow & \circlearrowright & \downarrow\Phi_{X,Y} \\ \mathrm{mor}_{\mathcal{C}}(X',RY) & \xrightarrow{-\circ f} & \mathrm{mor}_{\mathcal{C}}(X,RY) \end{array}$$

$LX \xrightarrow{L(f)} LX' \to Y$

$LX \to Y \xrightarrow{g} Y'$

naturality in Y

$X \to RY \xrightarrow{R(g)} RY'$

$$\begin{array}{ccc} \mathrm{mor}_{\mathcal{D}}(LX,Y) & \xrightarrow{g\circ -} & \mathrm{mor}_{\mathcal{D}}(LX,Y') \\ \Phi_{X,Y}\downarrow & \circlearrowright & \downarrow\Phi_{X,Y'} \\ \mathrm{mor}_{\mathcal{C}}(X,RY) & \xrightarrow{R(g)\circ -} & \mathrm{mor}_{\mathcal{C}}(X,RY') \end{array}$$

square!

free

forgetful

$\mathcal{C}^{op} \times \mathcal{D}$ $\Downarrow \cong$ Set

$Mor_{\mathcal{D}}(L-,-)$

$Mor_{\mathcal{C}}(-,R-)$

Example

① $\mathcal{C} = Set$ $\mathcal{D} = Vect_{\mathbb{K}}$

$L: Set \longrightarrow Vect_{\mathbb{K}}$ (free functor)

$S \longmapsto \mathbb{K}[S]$ the vector space over $\mathbb{K}$ with basis S

(free vector space span by S)

$S \xrightarrow{f} S' \longmapsto \mathbb{K}[S] \longrightarrow \mathbb{K}[S']$

linear map whose restriction to basis element is precisely f.

(linear map generated by f)

$R: Vect_{\mathbb{K}} \longrightarrow Set$ (forgetful functor)

Let $X \in Set$ and $Y \in Vect_{\mathbb{K}}$

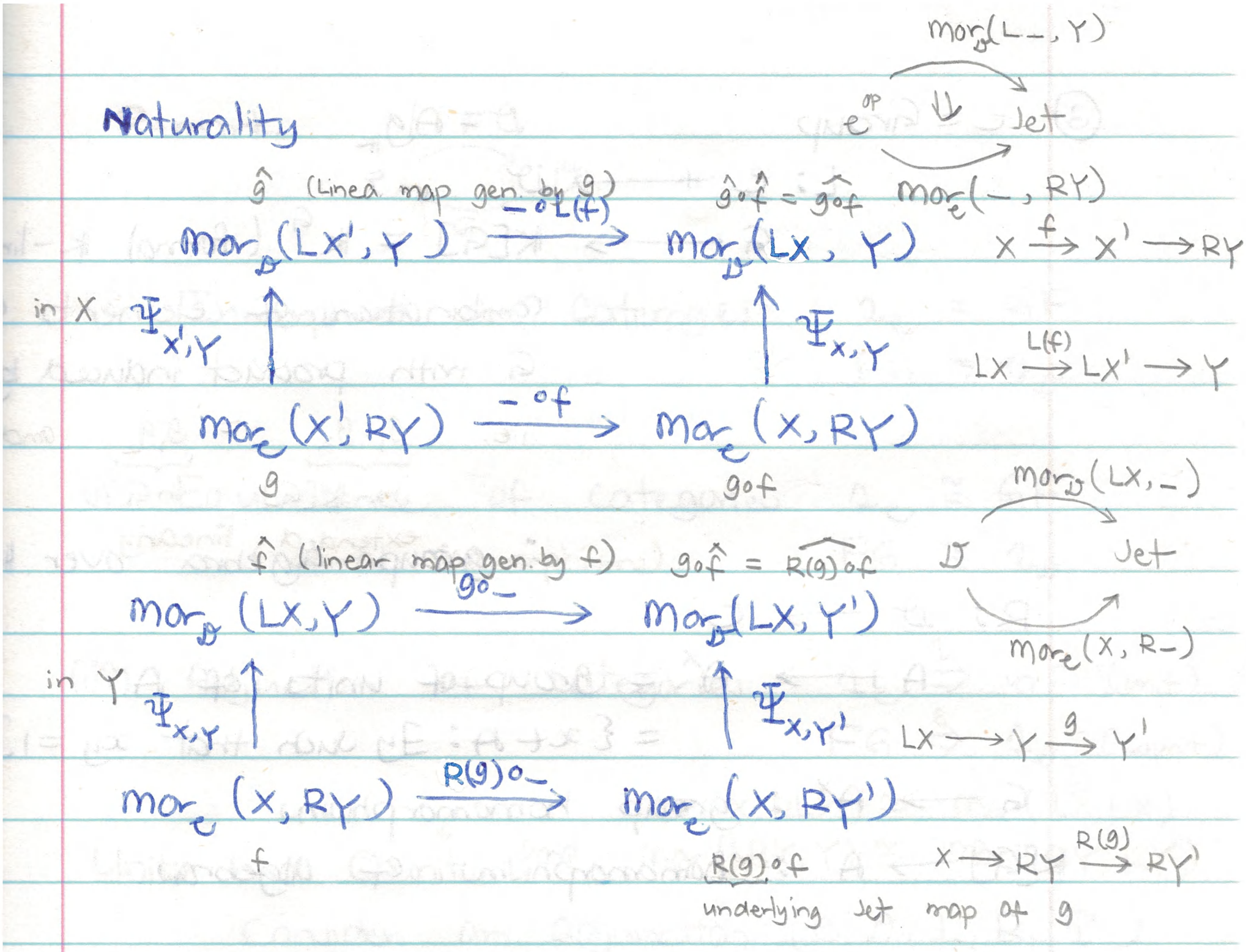
$\mathcal{F}_{X,Y} : Mor_{\mathcal{C}}(X, RY) \longrightarrow Mor_{\mathcal{D}}(LX, Y)$

$X \xrightarrow{f} RY \longmapsto LX \longrightarrow Y$

function Linear map

The isomorphism $\mathcal{F}_{X,Y}$ sends a function of sets to the linear map generated by that function. It is a bijection because a linear map is uniquely determined by its values on a set of basis elements.

Naturality



② $\mathcal{C} = \mathbf{Set}$ $\quad \mathcal{D} = \mathbf{CRing}$ $\quad (\mathbf{CAlg}_{\mathbb{K}})$

$L: \mathcal{C} \longrightarrow \mathcal{D}$

$S \longmapsto L(S) = \mathbb{Z}[S]$ (polynomial with S variables) or $\mathbb{K}[S]$

$R: \mathcal{D} \longrightarrow \mathcal{C}$ (forgetful)

$S \longrightarrow R(A)$ (functions)

$\mathbb{Z}[S] \longrightarrow A$ (ring homomorphisms)

Similarly this example works with $\mathcal{D} = \mathbf{Group}$, $\mathbf{Mod}_R$, ... or any other variety of algebras.

③ $e = \text{Group}$ $\quad \mathcal{D} = \mathbb{K}\text{Alg}$

$L: e \longrightarrow \mathcal{D}$

$G \longmapsto \mathbb{K}[G] = \mathbb{K}^G$ (formal) $\mathbb{K}$-linear combinations of elements of G with product induced by G

i.e. $\underbrace{g_1 \cdot g_2}_{\text{in } \mathbb{K}[G]} = \underbrace{g_1 g_2}_{\text{in } G}$ and

$\therefore$ group algebra extended linearly over $\mathbb{K}$.

$R: \mathcal{D} \longrightarrow e$

$A \longmapsto A^\times$ = group of units of A

$= \{x \in A : \exists y \text{ such that } xy = 1\}$

$G \longrightarrow A^\times$: group homomorphisms

$\mathbb{K}[G] \longrightarrow A$: homomorphisms of algebras.

④ $e = \text{Vect}_{\mathbb{K}}$ $\quad \mathcal{D} = \text{CAlg}_{\mathbb{K}}$

$L: \text{Vect}_{\mathbb{K}} \longrightarrow \text{CAlg}_{\mathbb{K}}$

$V \longmapsto S(V)$ - symmetric algebra over V

(polynomials in $\dim V$ variables with coefficients in $\mathbb{K}$)

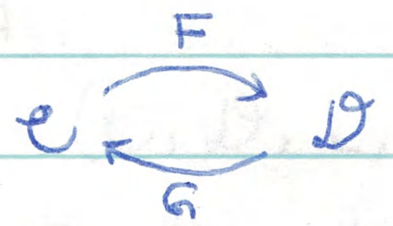
$R: \text{CAlg}_{\mathbb{K}} \longrightarrow \text{Vect}_{\mathbb{K}}$ (forgetful functor)

$\Psi : \{S(V) \xrightarrow[\text{of algebras}]{\text{homomorphism}} A\} \longrightarrow \{V \xrightarrow[\text{map}]{\text{linear}} R(A)\}$

This is an isomorphism by universal property of symmetric algebras.

$$Mor_{\mathcal{D}}(LX, Y) \cong Mor_{\mathcal{C}}(X, RY)$$

Remark



(i) Isomorphism of Categories $1_{\mathcal{C}} = GF$
$FG = 1_{\mathcal{D}}$

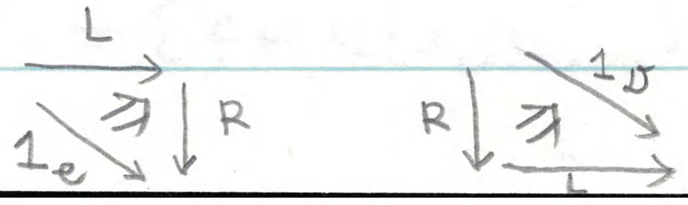
(ii) Equivalance of categories $1_{\mathcal{C}} \cong GF$
(natural isomorphism) $FG \cong 1_{\mathcal{D}}$

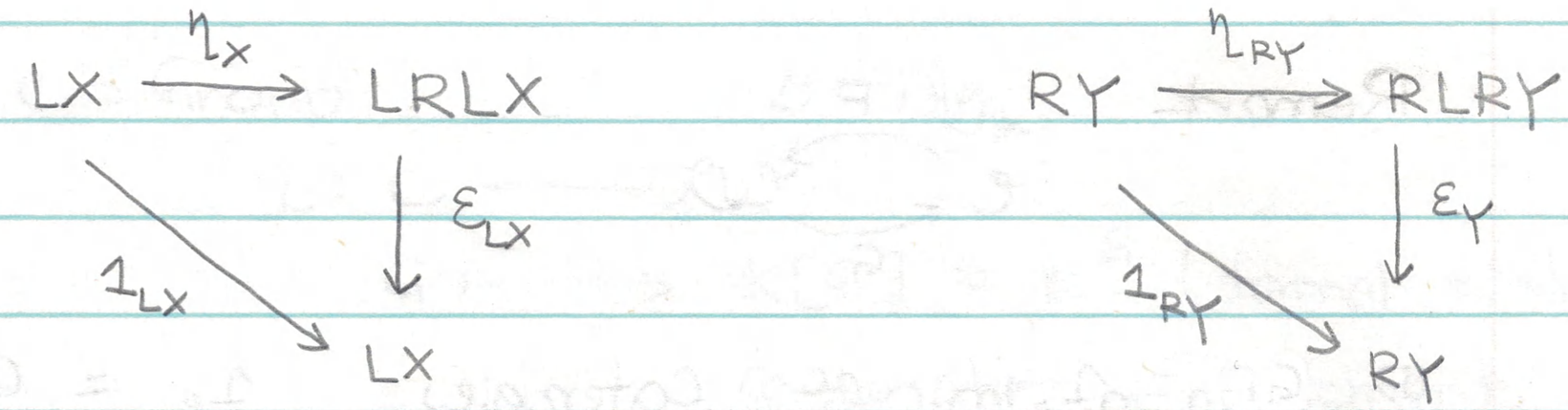
(iii) Adjuction of categories $1_{\mathcal{C}} \overset{\eta}{\Rightarrow} GF$ (unit)
$FG \overset{\varepsilon}{\Rightarrow} 1_{\mathcal{D}}$ (counit)

Units and Counits and $Mor(X, RLX) \cong Mor(LX, LX)$
$Mor(LRY, Y) \cong Mor(RY, RY)$

Consider an adjunction $(\mathcal{C}, \mathcal{D}, L, R, \Phi)$.
The unit of an object $X \in \mathcal{C}$ is the morphism $X \xrightarrow{\eta_X} RLX$ that is adjoint to $LX \xrightarrow{1_{LX}} LX$.
Dually the counit an object $Y \in \mathcal{D}$ is the morphism $LRY \xrightarrow{\varepsilon_Y} Y$ that is adjoint to $RY \xrightarrow{1_{RY}} RY$

The naturality of Φ implies that the unit maps can be assembled into a natural transformation $1_{\mathcal{C}} \Rightarrow RL$, whereas for counits we get a natural transformation $LR \Rightarrow 1_{\mathcal{D}}$.





— triangle identities

Example



unit : $c \in Set$ $\quad c \longmapsto U(F(c))$

$\alpha \longmapsto \alpha$ (a a k-linear combination)

The precise way to define 'finite formal sum' is a finitely supported function $\phi: C \to k$ corresponding to the expression $\sum_{\alpha \in C} \phi(\alpha)\alpha$.

i.e. $F(C) = k^C$

$\alpha \in C \longmapsto \chi_\alpha \in U(F(C))$

(The indicator function of α)

counit : $V \in Vect_k$ $\quad V \longleftarrow F(U(V))$

evaluation of this linear combination in V $\longleftarrow$ $k_1\alpha_1 + k_2\alpha_2 + \dots + k_n\alpha_n$

$\alpha_n \in U(V)$

Proposition

Suppose $(\mathcal{C}, \mathcal{D}, L, R, \Psi)$ is an adjunction. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ such that on objects $X \in \mathcal{C}$ the unit $X \longrightarrow RLX$ is an isomorphism. Similarly $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ on objects $Y \in \mathcal{D}$ such that the counit map $LRY \to Y$ is an isomorphism. Denote by L' and R' the restriction of functors L and R on these subcategories. We have $L'(\mathcal{C}') \subseteq \mathcal{D}'$ and $R'(\mathcal{D}') \subseteq \mathcal{C}'$. Also the restricted adjunction $(\mathcal{C}', \mathcal{D}', L', R', \Psi)$ is an equivalance of categories.

now unit and counit maps are isomorphisms

Example (Nullstellensatz)

For a fixed $n \in \mathbb{N}$ and algebraically closed field $\mathbb{K}$, consider the sets of elements in the polynomial ring $\mathbb{K}[x_1, x_2, \ldots, x_n]$ and in the vector space $\mathbb{K}^n$. There are contravariant functors between poset categories

$$P(\mathbb{K}[x_1, x_2, \ldots, x_n])^{op} \underset{I}{\overset{V}{\rightleftarrows}} P(\mathbb{K}^n)$$

$S \in$ $\quad$ $T \in$

$$V(S) = \{(t_1, t_2, \ldots, t_n) \in \mathbb{K}^n : f(t_1, t_2, \ldots, t_n) = 0 \ \forall f \in S\}$$

$$I(T) = \{f \in \mathbb{K}[x_1, x_2, \ldots, x_n] : f(t_1, t_2, \ldots, t_n) = 0 \ \forall (t_1, t_2, \ldots, t_n) \in T\}$$

and these define mutual right adjoints because $T \subseteq V(S)$ if and only if $S \subseteq I(T)$.

A set of points $T \subseteq \mathbb{K}^n$ defines a fixed point $T = V(I(T))$ if and only if T is closed in the Zariski topology on $\mathbb{K}^n = \operatorname{Spec}(\mathbb{K}[x_1, x_2, \ldots, x_n])$

$$\operatorname{Spec} : \mathbf{CRing}^{op} \longrightarrow \mathbf{Top}$$

$$R \longmapsto \operatorname{Spec}(R) : \text{set of prime ideals of } R$$

The topology on $\operatorname{Spec}(R)$ is the Zariski topology i.e. $V_I \subseteq \operatorname{Spec}(R)$ is closed if it is the set of all prime ideals containing a fixed ideal $I \subseteq R$

$$R \xrightarrow{\varphi} S \longmapsto \operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$$

$$P \longmapsto \varphi^{-1}(P)$$

and this function φ^{-1} is continuous wrt Zariski topology

A set of polynomials $S \subseteq \mathbb{K}[x_1, x_2, \ldots, x_n]$ defines a fixed point $S = I(V(S))$ if and only if it defines a radical ideal of $\mathbb{K}[x_1, x_2, \ldots, x_n]$. Then by proposition the poset of Zariski closed subsets of $\mathbb{K}^n$ is isomorphic to the opposite of the poset of radical ideals of $\mathbb{K}[x_1, x_2, \ldots, x_n]$.

$\mathbb{k}$ - algebraically closed field

$\mathrm{CAlg}_{\mathbb{k}}$

Objects : Commutative algebras over $\mathbb{k}$ equip with finite family of generators

Morphisms : homomorphisms of algebras (may not preserve generators)

$\mathrm{AffVar}_{\mathbb{k}}$

Objects : affine varities over $\mathbb{k}$

(n, V) ie. $V \subseteq \mathbb{k}^n$ for some $n \geq 0$ such that there are polynomials $P_1, P_2, \ldots, P_m$ over $\mathbb{k}$ satisfying $V = \{ z \in \mathbb{k}^n : P_i(z) = 0 \quad \forall i \}$

Morphisms : Regular maps

$(n, V) \longrightarrow (m, V')$ e. $V \subseteq \mathbb{k}^n \xrightarrow{f} V' \subseteq \mathbb{k}^m$ $\exists q_1, q_2, \ldots, q_m$ (may not unique) polynomials of n variables $f(z) = (q_1(z), q_2(z), \ldots, q_m(z))$

$\mathrm{CAlg}_{\mathbb{k}}^{op} \xrightarrow{\mathrm{Spec}} \mathrm{AffVar}_{\mathbb{k}}$

$A \longmapsto V \subseteq \mathbb{k}^n$

$\{x_1, x_2, \ldots, x_n\}$ - generators

$\{ z \in \mathbb{k}^n : P(z) = 0, P \in \mathbb{k}[x_1, x_2, \ldots, x_n] \}$ - evaluation of P at A in zero

$A' \xrightarrow{f} A \longmapsto V \longrightarrow V'$

$\{x_i'\}$ $\{x_j\}$ $f(x_i') = q_i(x_1, x_2, \ldots, x_n) \rightsquigarrow$ regular function

$\mathcal{O}$

$\mathrm{Aff\,Var}_{\Bbbk} \longrightarrow \mathrm{CAlg}_{\Bbbk}^{op}$

$V \subseteq \Bbbk^n \longmapsto \{ V \xrightarrow{\text{regular maps}} \Bbbk \}$

generators = coordinate projections

$V \subseteq \Bbbk^n \longrightarrow V' \subseteq \Bbbk^m \longmapsto \mathcal{O}(V') \longrightarrow \mathcal{O}(V)$

$V' \to \Bbbk \longmapsto V \to \Bbbk$

$V \searrow V' \nearrow$

A commutative algebra is called reduced if it has no nilpotent elements.

i.e $x \neq 0$ and $\exists n > 0$ such that $x^n = 0$.

Theorem (Hilbert's nullstellensatz)

There is a equivalance of categories

$\mathrm{CAlg}_{\Bbbk}^{\text{reduced}} \rightleftarrows \mathrm{Aff\,Var}_{\Bbbk}$

radical ideals

Propaition

Right adjoints preserve limits, and dually left adjoints preserve colimits.

Internal Category

Examples

(i) TopGrp is Grp internal to Top

(ii) LieGrp is Grp internal to Man

(iii) A small category is category internal to Set

Def^n (1)

Let $\mathcal{C}$ be a category with all finite limits. A category internal to $\mathcal{C}$ consists of two objects $C_0, C_1 \in \mathcal{C}$ together with

- source and target morphism $C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$
- identity assigning morphism $C_0 \xrightarrow{1_-} C_1$
- Composition morphism $C_1 \underset{C_0}{\times} C_1 \xrightarrow{m} C_1$

satisfying the usual categorical laws

(i) source and target of identity morphism:

$$\begin{array}{ccc} C_0 & \xrightarrow{1_-} & C_1 \\ & \searrow^{1_{C_0}} & \downarrow{\scriptstyle s} \\ & & C_0 \end{array} \qquad \begin{array}{ccc} C_0 & \xrightarrow{1_-} & C_1 \\ & \searrow^{1_{C_0}} & \downarrow{\scriptstyle t} \\ & & C_0 \end{array}$$

(ii) source and target of composite morphism:

$$\begin{array}{ccc} C_1 \underset{C_0}{\times} C_1 & \xrightarrow{m} & C_1 \\ \downarrow{\scriptstyle Pr_1} & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow{s} & C_0 \end{array} \qquad \begin{array}{ccc} C_1 \underset{C_0}{\times} C_1 & \xrightarrow{m} & C_1 \\ \downarrow{\scriptstyle Pr_2} & & \downarrow{\scriptstyle t} \\ C_1 & \xrightarrow{t} & C_0 \end{array}$$

Here the pullback $C_1 \underset{C_0}{\times} C_1$ is defined via the commutative square

$$\begin{array}{ccc} C_1 \underset{C_0}{\times} C_1 & \xrightarrow{pr_2} & C_1 \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow[t]{} & C_0 \end{array}$$

(iii) associativity for composition of morphisms

$$\begin{array}{ccc} C_1 \underset{C_0}{\times} C_1 \underset{C_0}{\times} C_1 & \xrightarrow{m \times 1_{C_1}} & C_1 \underset{C_0}{\times} C_1 \\ \downarrow{\scriptstyle 1_{C_1} \times m} & & \downarrow{\scriptstyle m} \\ C_1 \underset{C_0}{\times} C_1 & \xrightarrow{m} & C_1 \end{array}$$

(iv) unitality laws for composition of morphisms

$$\begin{array}{ccccc} C_0 \underset{C_0}{\times} C_1 & \xrightarrow{1_- \times 1_{C_1}} & C_1 \underset{C_0}{\times} C_1 & \xleftarrow{1_{C_1} \times 1_-} & C_1 \underset{C_0}{\times} C_0 \\ & {\scriptstyle pr_2} \searrow & \downarrow{\scriptstyle m} & \swarrow {\scriptstyle pr_1} & \\ & & C_1 & & \end{array}$$

There are equivalent definitions of this definition which are easy to generalize for higher categories.

Equivalent definition (2)

A $\mathcal{C}$ internal category is a functor $X : \Delta^{OP} \longrightarrow \mathcal{C}$ that satisfies the legal condition: $\forall\ m, n \geq 0$

$$X_{m+n} \xrightarrow{\cong} X_m \underset{X_0}{\times} X_n$$

i.e. $X_n \cong X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1$ (spying condition)

To derive the last definition, take

$C_0 = X_0$, $C_1 = X_1$ and $C_1 \underset{C_0}{\times} C_1 = X_2$ and so on.

$$\begin{array}{ccc} X_{m+n} & \xrightarrow{X_c} & X_m \\ {\scriptstyle X_k}\downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array} \quad : \text{legal condition}$$

where $X_{m+n} \xrightarrow{X_c} X_m$, $c : [m] \longrightarrow [m+n]$

$\{0, \ldots, m\} \hookrightarrow \{0, \ldots, \underbrace{n, \ldots, n+m}\}$

$X_{m+n} \xrightarrow{X_k} X_n$, $k : [n] \longrightarrow [m+n]$

$\{0, \ldots, n\} \hookrightarrow \{0, \ldots, \underbrace{n}, \ldots\}$

$X_1 \xrightarrow{s,t} X_0$

$s, t : [0] \longrightarrow [1]$

$0 \overset{s}{\longmapsto} 0$

$0 \overset{t}{\longmapsto} 1$

$X_0 \xrightarrow{1-} X_1$

$[1] \longrightarrow 0$

(identity assigning morphism)

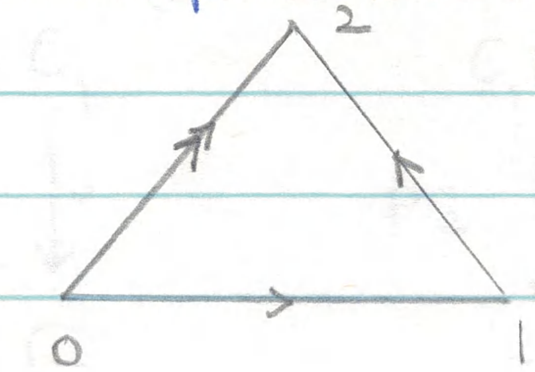
$X_2 \xrightarrow{m} X_1$

$[1] \longrightarrow [2]$

$0 \longmapsto 0$

$1 \longmapsto 2$

multiplication map (composition)



$[1] \longrightarrow [2]$

$0 \longmapsto 1$

$1 \longmapsto 1$

gives the map that maps a compatible pair of morphisms to the identity morphism of the common object.

Segal space $X : X_{m+n} \xrightarrow{\cong} X_m \underset{X_0}{\times} X_n \quad \forall n, m$

is a model for ∞-categories.

$$\underset{①}{\text{simplicial}} \xrightarrow[\text{nerve}]{} \underset{②}{}$$

$$C_0 = X_0, \; C_1 = X_1, \ldots$$

Segal Theorem (nerve theorem)

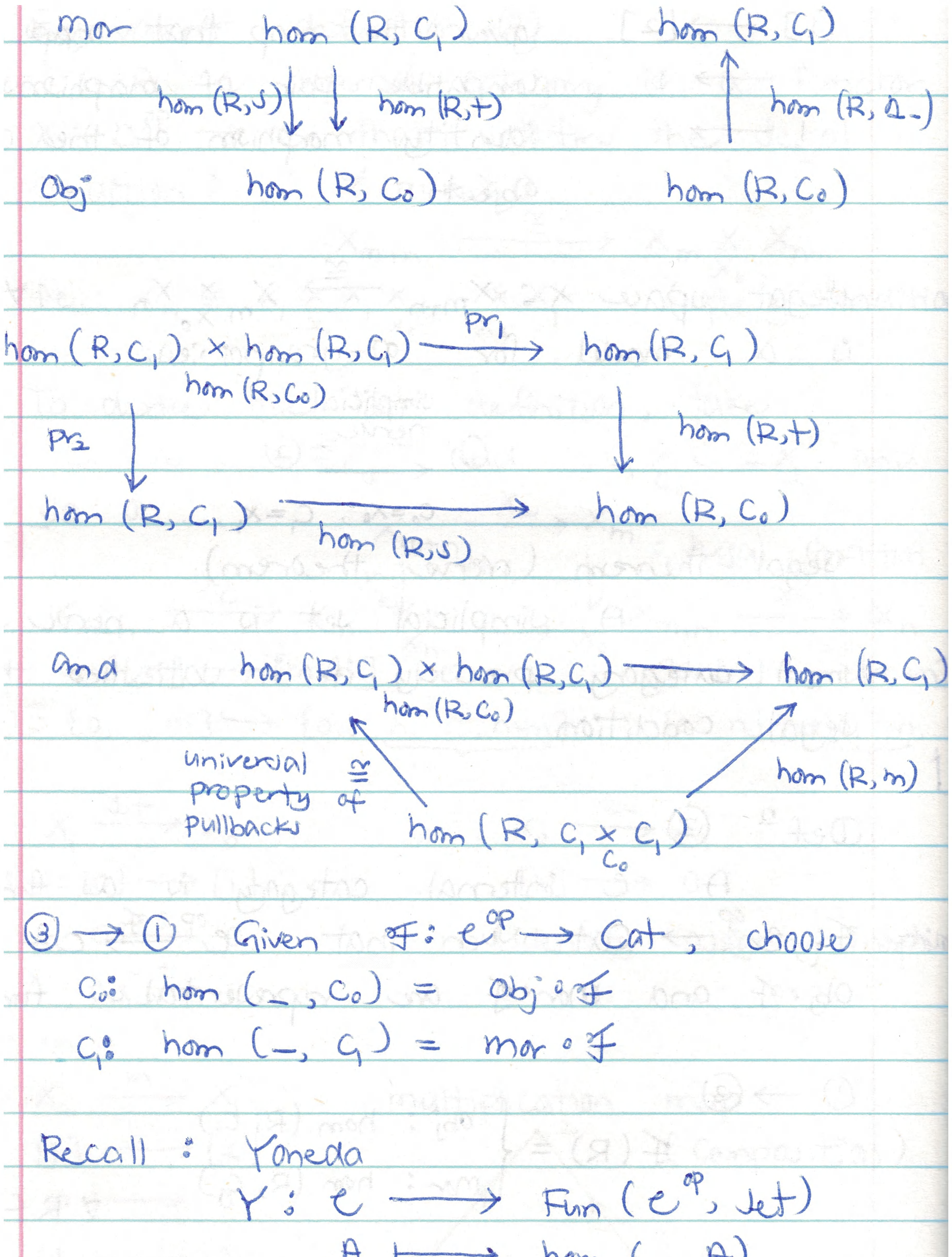
A simplicial set is a nerve of a small category precisely if it satisfies the Segal condition

Def^n ③

A ∞ internal category is a functor $\mathcal{F} : \mathcal{C}^{op} \to Cat$ such that $\mathcal{C}^{op} \xrightarrow{\mathcal{F}} Cat \underset{\text{mor}}{\overset{\text{obj}}{\rightrightarrows}} Set$ $Obj \circ \mathcal{F}$ and $mor \circ \mathcal{F}$ are representable functors.

① $\longrightarrow$ ③

$$\mathcal{F}(R) = \begin{cases} obj : \hom(R, C_0) \\ mor : \hom(R, C_1) \end{cases} \quad \forall R \in \mathcal{C}$$



③ → ① Given $\mathcal{F}: \mathcal{C}^{op} \to Cat$, choose

C_0: $\hom(-, C_0) = Obj \circ \mathcal{F}$

C_1: $\hom(-, C_1) = mor \circ \mathcal{F}$

Recall: Yoneda

$Y: \mathcal{C} \longrightarrow Fun(\mathcal{C}^{op}, Set)$

$A \longmapsto \hom(-, A)$

Y is fully faithful and ess im (Y) = representable presheaves

C_1 are given by $\hom(-, C_1) = \mathrm{mor} \circ F$

$s \downarrow\downarrow t$ — $s \circ F \downarrow \quad \downarrow t \circ F$

C_0 — $\hom(-, C_0) = \mathrm{obj} \circ F$

where $\mathrm{Cat} \underset{s}{\overset{t}{\rightrightarrows}} \mathrm{Set}$ (mor over, obj under) are two natural transformations.

Identity assigning morphism is given by the natural transformation

$$\begin{array}{ccc} C_1 & & \\ {\scriptstyle 1_-}\uparrow & & \\ C_0 & \xrightarrow{\mathcal{C}^{op} \xrightarrow{F} \mathrm{Cat}} & \mathrm{Set} \ (\mathrm{mor} \text{ over}, \ \mathrm{obj} \text{ under}) \end{array}$$

1_- : $\mathrm{mor} \Rightarrow \mathrm{obj}$, $\mathrm{mor} \circ F \Rightarrow \mathrm{obj} \circ F$

i.e. For any $\mathcal{C} \xrightarrow{G} \mathcal{D}$ in Cat,

$$\begin{array}{ccc} 1_x \ \mathrm{mor}\, \mathcal{C} & \xrightarrow{\mathrm{mor}\, G} & \mathrm{mor}\, \mathcal{D} \\ {\scriptstyle 1_-}\uparrow & \# & \uparrow{\scriptstyle 1_-} \\ x \in \mathrm{obj}\, \mathcal{C} & \xrightarrow[\mathrm{obj}\, G]{} & \mathrm{obj}\, \mathcal{D} \end{array}$$

$G(1_x) = 1_{G(x)}$, $G(x)$.

Multiplication (composition)

$$\begin{array}{ccc} \hom(-, C_1) \times \hom(-, C_1) & \xrightarrow{pr_1} & \hom(-, C_1) \\ {\scriptstyle pr_2}\downarrow \ {\scriptstyle \hom(-, C_0)} & \# & \downarrow{\scriptstyle t \circ F} \\ \hom(-, C_1) & \xrightarrow[s \circ F]{} & \hom(-, C_0) \end{array}$$

$C_1 \underset{C_0}{\times} C_1$ exist in $\mathcal{C}$ and by the universal property of pullbacks we have $\hom(-, C_1 \underset{C_0}{\times} C_1) \cong \hom(-, C_1) \underset{\hom(-, C_0)}{\times} \hom(-, C_1)$.

$$\text{nor} \quad \hom(-, C_1 \times_{C_0} C_1) = \text{mor} \times \text{mor}$$

$$\downarrow m$$

$$\hom(-, C_1) = \text{mor} \circ \mathcal{F}$$

is given by $\mathcal{C}^{op} \to Set$ such that

$$\forall\ x \xrightarrow{f} y \text{ in } \mathcal{C}$$

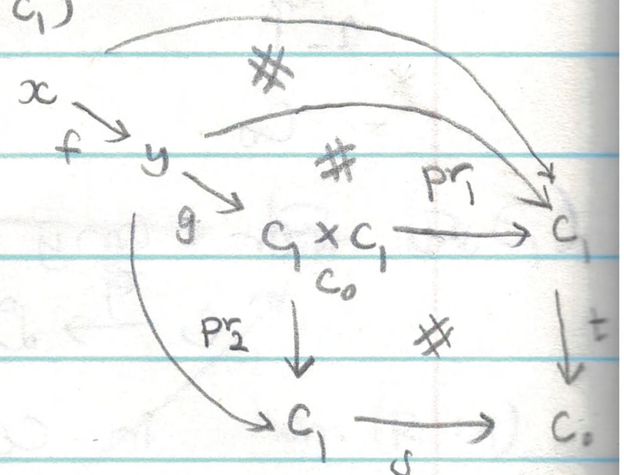
$$\hom(y, C_1 \times_{C_0} C_1) \xrightarrow{-f} \hom(x, C_1 \times_{C_0} C_1)$$

$$\downarrow pr_1 - \qquad \# \qquad \downarrow pr_1 -$$

$$\text{mor} \circ \mathcal{F}(y) \longrightarrow \text{mor} \circ \mathcal{F}(x)$$

$$= \hom(y, C_1) \xrightarrow{-f} = \hom(x, C_1)$$

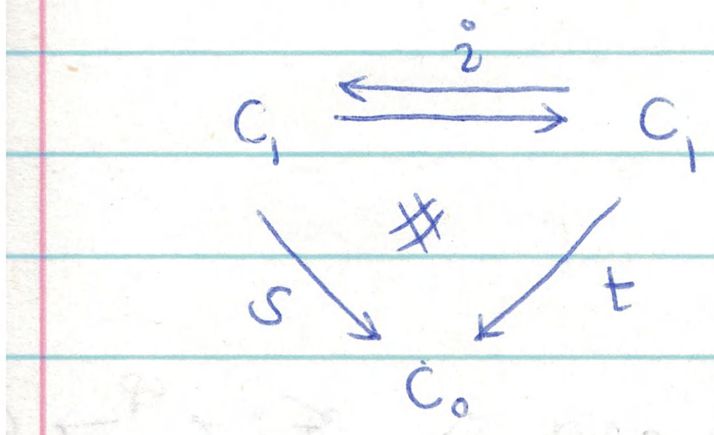
$\therefore$



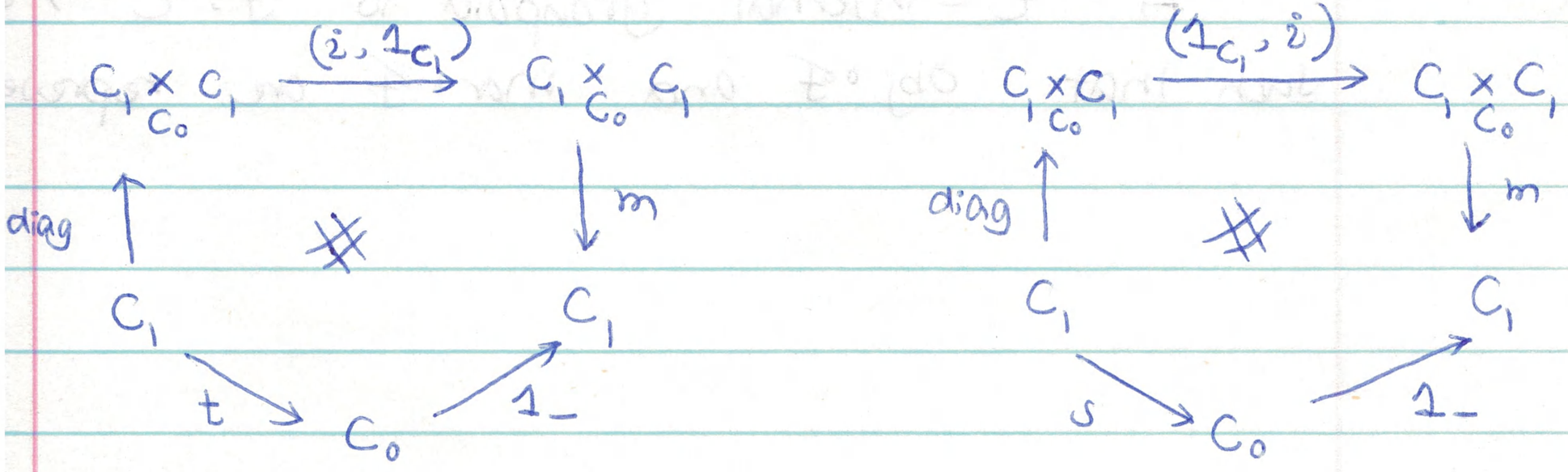
$$t(pr_1(gf)) = s(pr_2(gf))$$

Internal groupoid

A category internal to $\mathcal{C}$ is an internal groupoid if there is an inversion map $C_1 \xrightarrow{i} C_1$ such that



and



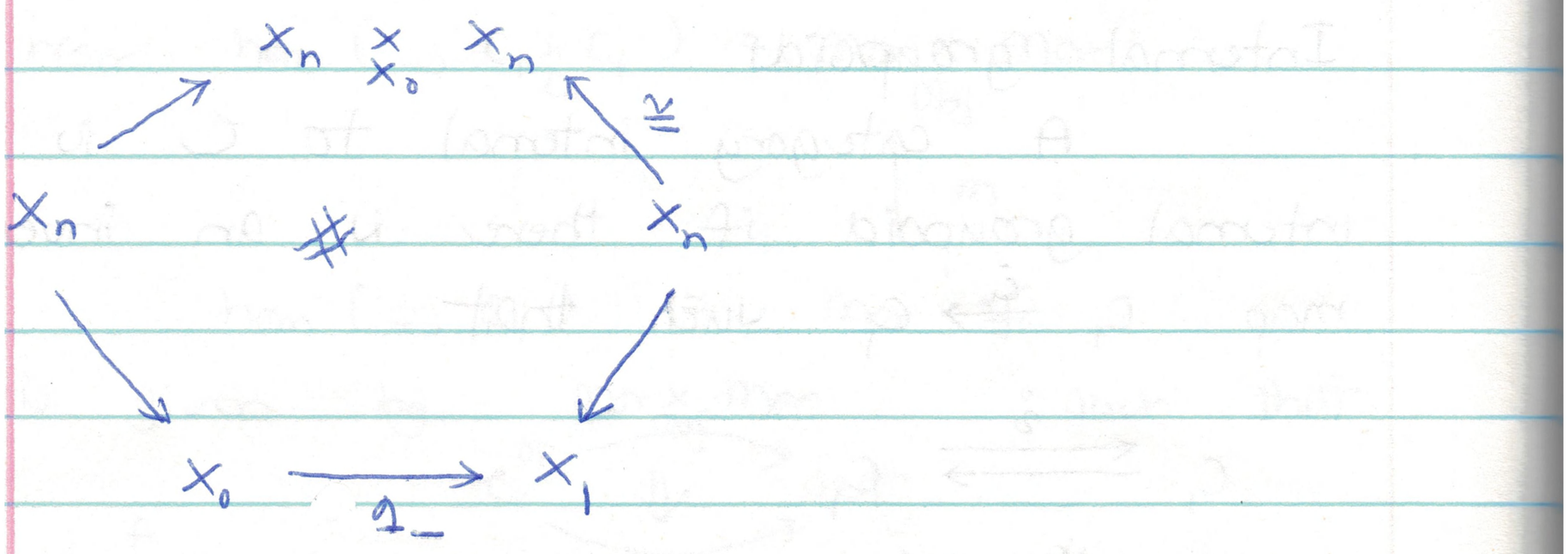
Segal type definition of internal groupoids

Δ_{dihedral} : objects - Δ (simplices)

morphisms - generated by morphisms of Δ and reversal maps.

Def^{n}

$\mathcal{C}$ - internal groupoid is a functor $\Delta^{op}_{\text{dihedral}} \xrightarrow{X} \mathcal{C}$ that satisfies the segal condition +



Def^2 (3)

A $\mathcal{C}$-internal groupoid is $\mathcal{F}: \mathcal{C}^{op} \to$ Grpd such that Obj$\circ\mathcal{F}$ and Mor$\circ\mathcal{F}$ are representable