Topology

Dmitri Pavlov

Contents

1 Preface

These notes offer an elementary introduction to topology. Why bother writing a new text when so many exist already? Two main features distinguish this text from all others known to the author:

- The selection of material is governed by its applications outside of topology proper. In particular, we cover topics such as K-theory and sheaf cohomology, which are typically omitted from the more traditional expositions.
- We do not hesitate to use modern machinery when it enhances clarity and simplifies the exposition. In particular, the following modern tools are used.
  - Simplicial sets are used because they provide the most rapidly accessible introduction to homology, cohomology, and fundamental groups. In particular, computations can be made once basic definitions are given, unlike for singular homology. Additionally, we can omit the rather intricate subtleties of general topology, such as the fact that typical categories of topological spaces (e.g., compactly generated spaces) are neither locally presentable nor locally cartesian closed, which becomes troublesome when performing the small object argument or constructing the space of sections of a bundle, etc.
  - Homotopy limits and homotopy colimits are already omnipresent in classical treatments in their specific incarnations, such as constructions with mapping path spaces, mapping telescopes, etc. We give a systematic treatment, which simplifies the presentation and makes it easier to organize the acquired knowledge. Additionally, it allows for a simplified treatment of topics such as generalized homology theories.
  - Model categories make it easier to systematically treat the numerous derived constructions occurring in topology, such as the homotopy (co)limits mentioned above, derived mapping spaces, homological algebra constructions, etc. In particular, they also eliminate many repetitive technical arguments with resolutions. Additionally, they make it easy to set up higher algebra in spaces and spectra.

Elementary homotopy theory

By “elementary” homotopy theory we mean the part of homotopy theory that can be treated with reasonable clarity and concision without using the machinery of model category (which could be referred to as “abstract” homotopy theory). This includes, in particular:

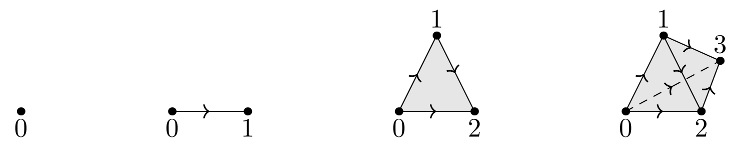
- Elementary theory of simplicial sets and operations on them;
- Elementary invariants of simplicial sets, such as homology, cohomology, and the fundamental group(oid);
- Elementary theory of Kan complexes, simplicial weak equivalences, and dervied mapping spaces.
- Cup product in cohomology and Poincaré duality.

2 Supplementary literature

Probably the most readable source is a set of notes by Joyal and Tierney [NSHT]. A 1999 book by Goerss and Jardine [SHTgj] is the only modern book on simplicial homotopy theory. Both of these sources require that one is familiar with elementary category theory, see Definition 12.1, Definition 13.1, §18.0. Four classical expositions appeared between 1967 and 1971: [CFHT, SOAT, SHTc, SAT]. These may be more difficult to read due to their age.

3 Simplices

The goal of this section is to formalize the following pictures:



Such objects are known as simplices. An n-dimensional simplex has $n+1$ vertices, typically numbered from 0 to n. We only record the combinatorial information about simplices, which in this case amounts to recording the set of vertices and their ordering. In the above picture, the ordering is indicated by drawing an arrow from a vertex with a smaller number to a vertex with a larger number. The numbers of vertices can be reconstructed from the arrows, starting with the lowest numbers: the vertex 0 only has outgoing arrows and no incoming arrows; the vertex 1 has a single incoming arrow from the vertex 0, all other arrows are outgoing; and so on up to the vertex n, which has no outgoing arrows. Accordingly, we do not record the numbers of vertices below, but only their ordering.

Recall that a (totally) *ordered set* is a pair $(S, \leq)$, where S is a set and $\leq$ is a binary relation on S (i.e., a subset of $S \times S$) such that $a \leq b$ and $b \leq a$ implies $a = b$, $a \leq b$ and $b \leq c$ implies $a \leq c$, and $a \leq b$ or $b \leq a$ is true for any a and b. We define $a < b$ to mean $a \leq b$ and $a \neq b$. A *morphism of ordered sets* $f\colon A = (S, \leq_A) \to B = (T, \leq_B)$ is an *order-preserving* (alias *nondecreasing*) map of sets $g\colon S \to T$, i.e., $a \leq b$ implies $f(a) \leq f(b)$. Any finite ordered set is isomorphic to an initial segment of natural numbers, i.e., $\{0 < 1 < \cdots < n-1\}$. Thus, its elements can be compared by comparing their numbers.

Definition 3.1. A *simplex* is a finite nonempty ordered set, whose elements are known as *vertices*. A *morphism of simplices* (alias *map of simplices*) is a morphism of ordered sets. Used in 3.0*, 3.0*, 3.0*, 3.1*, 3.1*, 3.1*, 3.2, 3.2, 3.2, 3.2, 3.2, 3.2, 3.2, 3.3, 3.3, 3.3, 3.3, 3.3, 3.3*, 3.3*, 3.4, 3.5, 3.6, 3.6, 3.8, 3.9, 4.1*, 4.1*, 4.1*, 4.4, 5.0*, 5.1, 5.3, 5.3, 5.3, 5.3*, 5.3*, 5.6, 5.7, 5.8, 5.8, 5.8, 5.8, 5.8, 5.8, 5.9*, 6.1, 6.1, 6.2, 6.2, 6.2, 6.3, 6.4, 6.5, 6.5, 6.6, 7.1, 7.3, 7.3, 7.5, 7.5*, 7.5*, 7.6, 7.6, 8.1, 9.2, 9.2, 9.2, 9.2, 9.3*, 9.3*, 9.5, 9.9, 9.9, 10.1, 10.1, 11.4, 11.6*, 12.0*, 12.5, 12.5, 12.5, 12.5, 13.7, 13.7, 13.7, 13.7.

Up to an isomorphism of simplices (defined below), the only simplices are $\{0 < 1 < 2 < \cdots < n\}$ for all $n \geq 0$. We often stress this fact by using the bold letter $\mathbf{n}$ for such a simplex, and by abuse of notation also for any simplex whose underlying set has $n+1$ elements.

Remark 3.2. One may wonder why we defined simplices as finite nonempty ordered sets instead of simply saying that a simplex is a set $\{0, 1, \ldots, n\}$ and a map of simplices is a nondecreasing map of sets. The principal reason for the above definition is that we want to be able to remove a vertex or several vertices from a simplex and obtain a new simplex. For instance, removing vertices 2 and 4 from the simplex $\{0, 1, 2, 3, 4, 5\}$ yields the simplex $\{0, 1, 3, 5\}$. The naive definition would force us to renumber the vertices of this simplex as $\{0, 1, 2, 3\}$. Such renumberings would in general be quite difficult to keep track of. However, we only really need the relative ordering of vertices and not their numbers, which motivates the above definition.

Exercise 3.3. Prove the following properties of maps of simplices.

- The identity map $\mathrm{id}_{\mathbf{m}}\colon \mathbf{m} \to \mathbf{m}$ is a morphism of simplices.
- If $f\colon \mathbf{l} \to \mathbf{m}$ and $g\colon \mathbf{m} \to \mathbf{n}$ are morphisms of simplices, then their composition $g \circ f\colon \mathbf{l} \to \mathbf{n}$ is also a morphism of simplices.
- The associativity property is satisfied: $h \circ (g \circ f) = (h \circ g) \circ f$ for any morphisms of simplices $f\colon \mathbf{k} \to \mathbf{l}$, $g\colon \mathbf{l} \to \mathbf{m}$, $h\colon \mathbf{m} \to \mathbf{n}$.
- The unitality property is satisfied: $\mathrm{id}_{\mathbf{l}} \circ f = f \circ \mathrm{id}_{\mathbf{k}} = f$ for any map of simplices $f\colon \mathbf{k} \to \mathbf{l}$.

Used in 12.5.

These properties imply that the composition of finitely many morphisms of simplices does not depend on the order of composition and is again a morphism of simplices, so we can simply denote it by $f_n \circ \cdots \circ f_1$.

Definition 3.4. An *isomorphism of simplices* is a morphism of simplices $f\colon \mathbf{m} \to \mathbf{n}$ for which there is a morphism $g\colon \mathbf{n} \to \mathbf{m}$ such that $g \circ f = \mathrm{id}_{\mathbf{m}}$ and $f \circ g = \mathrm{id}_{\mathbf{n}}$. Used in 3.1*, 3.5, 11.4.

Exercise 3.5. Show that a map of simplices is an isomorphism of simplices if and only if its underlying map of sets is a bijection.

Definition 3.6. If $\mathbf{m} = (V, \leq)$ is a simplex, we set $\mathsf{U}(\mathbf{m}) = V$ and refer to it as the *underlying set of a simplex* $\mathbf{m}$. Likewise, if $f\colon \mathbf{m} \to \mathbf{n}$ is a morphism of simplices from $\mathbf{m} = (V_\mathbf{m}, \leq_\mathbf{m})$ to $\mathbf{n} = (V_\mathbf{n}, \leq_\mathbf{n})$, then we set $\mathsf{U}(f)\colon \mathsf{U}(\mathbf{m}) \to \mathsf{U}(\mathbf{n})$ to the underlying map of sets $V_\mathbf{m} \to V_\mathbf{n}$ and refer to it as the *underlying map of a morphism of simplices* f. Used in 13.4.

From the definition of U we immediately see that $\mathsf{U}(g \circ f) = \mathsf{U}(g) \circ \mathsf{U}(f)$ and $\mathsf{U}(\mathrm{id}_\mathbf{m}) = \mathrm{id}_{\mathsf{U}(\mathbf{m})}$.

Definition 3.7. The *dimension of a simplex* $\mathbf{m}$ is denoted $\dim \mathbf{m}$ and is defined to be $\#\mathsf{U}(\mathbf{m}) - 1$, where $\#$ denotes cardinality.

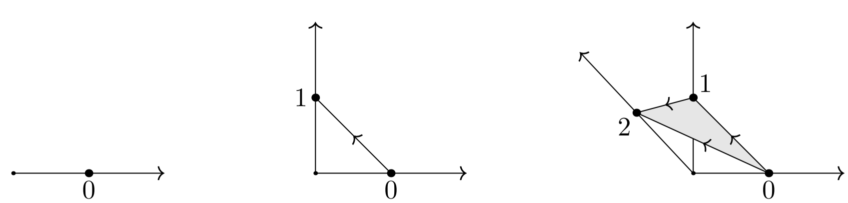
Remark 3.8. It may be unclear why one would want an ordering on the set of vertices of a simplex. After all, the geometric pictures do not seem to indicate the existence of such an ordering. Indeed, one could drop the data of an ordering altogether, obtaining *symmetric simplices*, which give rise to *symmetric simplicial sets*. The homotopy theory of symmetric simplicial sets is equivalent (in the sense defined later) to the homotopy theory of simplicial sets, so from an abstract point of view there is no difference between the two notions. However, there is a substantial practical difference, which is manifested in the fact that for any simplex $\mathbf{m}$ there is exactly one isomorphism $\mathbf{m} \to \mathbf{m}$, namely, $\mathrm{id}_\mathbf{m}$, whereas if $\mathbf{m}$ was a symmetric simplex, any permutation of $\mathsf{U}(\mathbf{m})$ would give such an isomorphism. Taken together, such isomorphisms would form a symmetric group of order $\dim \mathbf{m} + 1$, a nontrivial group. Having a trivial group of automorphisms makes the exposition considerably simpler, which is why we do not use symmetric simplices. The idea of using ordered simplices was introduced by Eilenberg in 1943 [SHTe]. His paper discusses the historical context of this definition.

Remark 3.9. If we allow the empty ordered set as a simplex, we get *augmented simplices*. These give rise to *augmented simplicial sets*, which are an important ingredient in many constructions, but their homotopy theory is not equivalent to that of simplicial sets.

4 Geometric realization of simplices

Definition 4.1. The *geometric realization of a simplex* $\mathbf{m}$ is the set $|\mathbf{m}| = \{x \in \mathbf{R}_{\geq 0}^{\mathsf{U}(\mathbf{m})} \mid \sum_{s \in \mathsf{U}(\mathbf{m})} x_s = 1\}$. The *geometric realization of a map of simplices* $f\colon \mathbf{m} \to \mathbf{n}$ is the map of sets $|\mathbf{m}| \to |\mathbf{n}|$ that sends $x \in |\mathbf{m}|$ to $y \in |\mathbf{n}|$ such that $y_t = \sum_{s \in \mathsf{U}(\mathbf{m}) : f(s) = t} x_s$. Used in 5.0*, 5.4*, 9.3*, 13.5.

Observe that $\sum_{t \in \mathsf{U}(\mathbf{n})} y_t = \sum_{t \in \mathsf{U}(\mathbf{n})} \sum_{s \in \mathsf{U}(\mathbf{m}) : f(s) = t} x_s = \sum_{s \in \mathsf{U}(\mathbf{m})} x_s = 1$, so the above formula indeed defines a map $|\mathbf{m}| \to |\mathbf{n}|$. We examine the low-dimensional cases of simplices $\mathbf{m} = \{0 < 1 < \cdots < m\}$ for $m \leq 2$. The set $|\mathbf{0}| = \{1\} \subset \mathbf{R}^1$ is a point. In particular, maps $\mathbf{0} \to \mathbf{m}$ pick some vertex of $\mathbf{m}$ and their geometric realization is a map $|\mathbf{0}| \to |\mathbf{m}|$, i.e., a point in $|\mathbf{m}|$, which we refer to as a *geometric vertex* of $|\mathbf{m}|$. We have $|\mathbf{1}| = \{(x, 1-x) \mid x \in [0,1]\}$ and the two vertices of $|\mathbf{1}|$ are $\mathrm{e}_0 = (1,0)$ and $\mathrm{e}_1 = (0,1)$. Finally, $|\mathbf{2}| = \{(x, y, 1-x-y) \mid x, y, x+y \in [0,1]\}$ and the vertices are $\mathrm{e}_0 = (1,0,0)$, $\mathrm{e}_1 = (0,1,0)$, $\mathrm{e}_2 = (0,0,1)$. We capture these observations in the following pictures:



Thus, the geometric idea behind the definition of geometric realization is clear by now: an m-dimensional simplex with vertices $\{0 < 1 < 2 < \cdots < m\}$ is realized as a subset of $\mathbf{R}^{m+1}$. Any vertex $i \in \mathsf{U}(\mathbf{m})$ is realized by the ith unit vector e_i (the ith coordinate is 1 and the others are 0). Furthermore, any point in $|\mathbf{m}|$ is a unique convex combination of vertices. (A convex combination is a linear combination with nonnegative coefficients that sum to 1.) Given a morphism of simplices $f\colon \mathbf{m} \to \mathbf{n}$, it gives rise to a unique linear map $\mathbf{R}^f\colon \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ that sends the unit vector $\mathrm{e}_i \in \mathbf{R}^{m+1}$ corresponding to a vertex $i \in \mathsf{U}(\mathbf{m})$ to the unit vector $\mathrm{e}_{f(i)} \in \mathbf{R}^{n+1}$. We have $\mathbf{R}^f(|\mathbf{m}|) \subset |\mathbf{n}|$ and the restriction of $\mathbf{R}^f$ to $|\mathbf{m}|$ and $|\mathbf{n}|$ is precisely $|f|$.

Remark 4.2. We have $|\mathrm{id}_{\mathbf{m}}| = \mathrm{id}_{|\mathbf{m}|}$ and $|g \circ f| = |g| \circ |f|$ for any $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{n} \to \mathbf{p}$. For the latter relation, observe that evaluating both sides on some $x \in |\mathbf{m}|$ and taking the uth component ($u \in \mathsf{U}(\mathbf{p})$) yields

$$\sum_{s \in \mathsf{U}(\mathbf{m}) : g(f(s)) = u} x_s = \sum_{t \in \mathsf{U}(\mathbf{n}) : g(t) = u} \quad \sum_{s \in \mathsf{U}(\mathbf{m}) : f(s) = t} x_s,$$

which holds because s runs over identical sets in both cases. Used in 13.5.

Remark 4.3. Depending on the situation at hand, one may want to equip $|\mathbf{m}|$ with a structure of a topological space, smooth manifold, etc. In algebraic geometry $\mathbf{R}_{\geq 0}$ does not make sense, so one uses instead $\mathbf{A}^1$, the affine line. Used in 13.5.

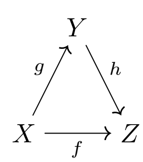
Exercise 4.4. Suppose $f\colon \mathbf{m} \to \mathbf{n}$ is a map of simplices. Show that if $\mathsf{U}(f)$ is injective respectively surjective if and only if $|f|$ is. Used in 5.4*.

5 Maps of simplices

We now examine in more detail the notion of a map of simplices and its geometric realization.

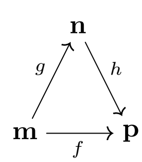
Definition 5.1. We say that a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ is injective respectively surjective if $\mathsf{U}(f)$ is. The *relative dimension* of f is defined as $\dim f = \dim \mathbf{m} - \dim \mathbf{n}$ and the *relative codimension* of f is defined as $\operatorname{codim} f = -\dim f = \dim \mathbf{n} - \dim \mathbf{m}$. A *face inclusion* is an injective map of simplices of relative codimension 1. A *degenerate map* is a surjective map of simplices of relative dimension 1. Used in 5.1, 5.1, 5.3*, 5.4, 5.4*, 5.5, 5.5, 5.6, 5.6, 5.7, 5.7, 5.7*, 5.7*, 5.8, 5.8, 6.3, 6.3, 6.4, 8.2, 8.3.

Definition 5.2. A *factorization* of a map (of objects of any type) $f\colon X \to Z$ is a triple (Y, g, h), where Y is an object of the same type as X and Z and $g\colon X \to Y$ and $h\colon Y \to Z$ are maps such that $f = h \circ g$.



Used in 5.3.

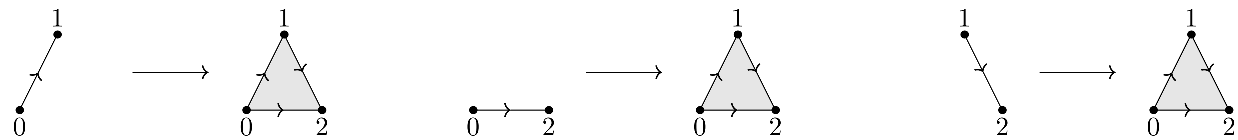
Lemma 5.3. Any map of simplices $f\colon \mathbf{m} \to \mathbf{p}$ admits a unique factorization $(\mathbf{n}, g, h)$, where $\mathbf{n}$ is a simplex and $g\colon \mathbf{m} \to \mathbf{n}$ and $h\colon \mathbf{n} \to \mathbf{p}$ are maps of simplices such that $f = h \circ g$, g is injective and h is surjective.



Proof. For existence, construct a simplex $\mathbf{n}$ by setting its underlying set to the image of $\mathsf{U}(f)$ and equipping it with the ordering induced from $\mathbf{p}$. The map $g\colon \mathbf{m} \to \mathbf{n}$ is obtained by restricting the codomain of $f\colon \mathbf{m} \to \mathbf{p}$ to $\mathbf{n}$. The map $h\colon \mathbf{n} \to \mathbf{p}$ is the inclusion map. By construction, $f = h \circ g$, the map g is surjective, and h is injective. ∎

For a fixed $\mathbf{n}$, injective maps $f\colon \mathbf{m} \to \mathbf{n}$ can be identified with nonempty subsets of $\mathsf{U}(\mathbf{n})$. Indeed, the image of $\mathsf{U}(f)$ is a nonempty subset of $\mathsf{U}(\mathbf{n})$. Different injective maps yield different subsets, and any nonempty subset of $\mathsf{U}(\mathbf{n})$ can be equipped with the induced order thereby giving rise to an injective map of simplices. Thus, an n-dimensional simplex admits exactly $2^{n+1} - 1$ injective maps into it, with $\binom{n+1}{k+1}$ injective maps with domain of dimension k. For instance, a 0-simplex $\mathbf{0}$ has a single injective map with image $\{0\}$ (itself), a 1-simplex $\mathbf{1} = \{0 < 1\}$ has images $\{0\}$, $\{1\}$, and $\{0 < 1\}$, a 2-simplex $\mathbf{2} = \{0 < 1 < 2\}$

has images $\{0\}$, $\{1\}$, $\{2\}$, $\{0<1\}$, $\{0<2\}$, $\{1<2\}$, and $\{0<1<2\}$. Here are the three face inclusions for $\mathbf{2}$:



0-simplices can be identified with vertices, depicted by dots in our pictures. 1-simplices are given by pairs of vertices $v_0 < v_1$ and are depicted by arrows. 2-simplices are specified by a triple vertices $v_0 < v_1 < v_2$ and are depicted by shaded triangles. We have no good way to depict simplices of dimension 3 and higher, so this information must be inferred from the context.

Exercise 5.4. Prove that an injective map of simplices of codimension d can be presented as a composition of d face inclusions. Is such a presentation unique? Used in 5.9*.

By Exercise 4.4, the geometric realization of a surjective map of simplices is also surjective. The easiest examples are given by maps $\mathbf{m}\to\mathbf{0}$ that send all vertices of $\mathbf{m}$ to the only vertex of $\mathbf{0}$. The next easiest example are given by two maps $f,g\colon\mathbf{2}\to\mathbf{1}$ that send $0\mapsto 0$, $2\mapsto 1$, and $1\mapsto 0$ respectively $1\mapsto 1$. The maps $|f|$, $|g|$ can be depicted as the following horizontal projection maps:

Exercise 5.5. Prove that any surjective map of simplices $f\colon\mathbf{m}\to\mathbf{n}$ can be presented as a composition of degenerate maps. Is such a presentation unique? Used in 5.9*.

Exercise 5.6. Suppose $f\colon\mathbf{m}\to\mathbf{n}$ is a map of simplices. Prove that if f is surjective respectively injective, then there is a map of simplices $g\colon\mathbf{n}\to\mathbf{m}$ such that $f\circ g=\mathrm{id}_{\mathbf{n}}$ respectively $g\circ f=\mathrm{id}_{\mathbf{m}}$. Used in 8.4*.

Remark 5.7. One may question the desirability of having degenerate maps in the first place. Indeed, one can allow only injective maps of simplices as morphisms, which give rise to *semisimplicial sets*. The homotopy theory of semisimplicial sets is equivalent (in the sense defined later) to the homotopy theory of simplicial sets, so from an abstract point of view there is no difference between the two notions. However, there is a substantial practical difference, which is manifested in the fact that the model structure (to be defined later) on semisimplicial sets is not right proper, and a semisimplicial set that is not weakly contractible in this model structure must have infinitely many simplices, which makes computations difficult.

We finish this section by introducing notation for face inclusions and degenerate maps and establishing some identities between them.

Definition 5.8. Suppose $\mathbf{m}$ is a simplex and $i\in\mathsf{U}(\mathbf{m})$ is a vertex of $\mathbf{m}$. Denote by $\mathrm{d}^{\mathbf{m},i}\colon\mathbf{m}\setminus i\to\mathbf{m}$ the face inclusion obtained by removing the vertex i from $\mathbf{m}$, the resulting simplex denoted $\mathbf{m}\setminus i$. Denote by $\mathrm{s}^{\mathbf{m},i}\colon\mathbf{m}\sqcup i\to\mathbf{m}$ the degenerate map obtained by repeating the vertex i in $\mathbf{m}$, the resulting simplex denoted $\mathbf{m}\sqcup i$. We denote these maps by d^i and s^i if $\mathbf{m}$ can be inferred from the context.

Exercise 5.9. Verify the following *simplicial identities* by expanding the definitions of d^i and s^j:

$$\mathrm{d}^j\mathrm{d}^i=\mathrm{d}^i\mathrm{d}^{j-1}\qquad(i<j)$$

$$\mathrm{s}^j\mathrm{s}^i=\mathrm{s}^i\mathrm{s}^{j+1}\qquad(i\le j)$$

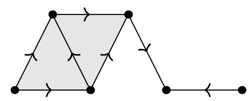
$$\mathrm{s}^j\mathrm{d}^i=\begin{cases}\mathrm{d}^i\mathrm{s}^{j-1}, & i<j\\ \mathrm{id}, & i=j\text{ or }i=j+1\\ \mathrm{d}^{i-1}\mathrm{s}^j, & i>j+1.\end{cases}$$

Used in 5.9*, 5.9*, 11.6*, 11.6*.

The significance of these identities lies in the fact that any map of simplices can be presented as a composition of maps of the form d^i and s^j, as shown in Exercise 5.4 and Exercise 5.5. One can show that the simplicial identities generate all possible equalities between formal compositions of maps d^i and s^j, i.e., if the compositions of two different chains of such maps are equal, then we can transform one chain into another by applying some sequence of simplicial identities. This fact can be used to give a very different definition of a simplicial set (and simplicial maps): a simplicial set is given by a sequence of sets X_n for all integer $n \geq 0$ together with maps $\mathrm{d}_i\colon X_n \to X_{n-1}$ for all $0 \leq i \leq n$ ($n > 0$) and $\mathrm{s}_j\colon X_n \to X_{n+1}$ for all $0 \leq j \leq n$ ($n \geq 0$) such that the above simplicial identities are satisfied, with the order of composition reversed and superscripts replaced by subscripts. his definition was in fact commonly used during the early period of development of simplicial aspects of topology, which hindered its development by making it less accessible to newcomers.

6 Simplicial sets

Our goal in this section is to formalize pictures like this:



Namely, we have a bunch of simplices that may overlap only if their intersection is again a simplex, and the ordering of vertices is respected, i.e., we cannot glue edges with opposite orientations. We only record the combinatorial discrete data that shows how simplices stick together to each other, not their spatial position or orientation. This is achieved by recording for any dimension $n \geq 0$ the set of simplices of this dimension and for any map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ and any simplex s of dimension n we specify a simplex of dimension m that maps to s via the map f. (This will be made more precise below.)

Definition 6.1. A *simplicial set* X is specified as follows. For any simplex $\mathbf{m}$ we specify a set of *m-simplices* of X, denoted by $X_\mathbf{m}$. For any map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ we specify a *simplicial structure map* $X_f\colon X_\mathbf{n} \to X_\mathbf{m}$. We require that the following functoriality property is satisfied: $X_{\mathrm{id}_\mathbf{m}} = \mathrm{id}_{X_\mathbf{m}}$ for any simplex $\mathbf{m}$ and $X_{g\circ f} = X_f \circ X_g$ for any maps of simplices $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{n} \to \mathbf{p}$. Used in 1.0*, 3.9, 5.9*, 5.9*, 6.2, 6.4, 6.4, 6.4, 6.4, 6.4, 6.4, 6.5, 7.2, 7.3, 7.5, 7.5, 7.5*, 8.0*, 8.1, 8.1, 8.1, 8.3, 8.3, 8.4, 9.3*, 9.4*, 9.5, 9.8, 9.9, 10.0*, 10.4, 11.2, 11.3, 11.5, 12.0*, 12.6, 12.6, 13.7, 13.7, 15.0*, 17.1.

Remark 6.2. Morally (and shown to be rigorous in Lemma 7.5), one should think of $X_\mathbf{m}$ as the set of all "maps" $\mathbf{m} \to X$ (later formalized as $\Delta^\mathbf{m} \to X$, where $\Delta^\mathbf{m}$ is the simplicial set associated to $\mathbf{m}$), i.e., all possible ways to embed the simplex $\mathbf{m}$ into X. We emphasize that we do *not* remember that the elements of $X_\mathbf{m}$ are maps of any kind, for us $X_\mathbf{m}$ is an abstract set and essentially the only information we have about $X_\mathbf{m}$ is its cardinality. Given an element of $X_\mathbf{n}$ and a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$, we can consider the "composition" $\mathbf{m} \to \mathbf{n} \to X$ (later formalized as the composition $\Delta^\mathbf{m} \to \Delta^\mathbf{n} \to X$), which is an element of $X_\mathbf{n}$. Thus a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ yields a map of sets $X_f\colon X_\mathbf{n} \to X_\mathbf{m}$. This interpretation also makes obvious the properties of X_f that we imposed in the above definition. Again, X_f is an (abstract) map of sets, we do not remember that is comes from the composition with f. This point of view is made completely rigorous in Lemma 7.5 and also in Definition 6.5. Used in 6.6, 7.5.

Remark 6.3. If $f\colon \mathbf{m} \to \mathbf{n}$ is a degenerate map of simplices, we refer to X_f as a *degeneration map*. Likewise for face inclusions, which yield *face map*. Used in 6.4.

Remark 6.4. Simplicial sets were defined in 1949 by Eilenberg and Zilber in [SSCSH, §8], where they are called *complete semi-simplicial complexes*. This terminology is no longer in use, but the word "complex" survives in many derivative names of constructions involving simplicial sets, such as "Kan complex" and "function complex". Here "complete" refers to the presence of degeneration maps, i.e., a *semi-simplicial complex* is defined just like a simplicial set, but requiring all morphisms of simplices to be face inclusions. "Semi" refers to the fact that two different n-simplices can have the same $(n+1)$-tuple of vertices, i.e., an (ordered) simplicial complex can be defined as a semi-simplicial complex X such that the vertex map

$X_{\mathbf{n}} \to X^{n+1}$ is injective for all $n \geq 0$. However, the following definition is much more common: a (locally) ordered *simplicial complex* is a triple (V, S), where V is a set and $S \subset (2^V \setminus \{\emptyset\}) \times 2^{V \times V}$ is a set of pairs $(\sigma, \leq)$ (known as simplices) consisting of a nonempty subset σ of V equipped with a total ordering $\leq$ such that for any $\sigma' \subset \sigma$ the induced pair $(\sigma', \leq')$ belongs to S and any subset of V is the first component of at most one pair in S. Given an ordered simplicial complex (V, S), we construct a simplicial set X by setting $X_{\mathbf{m}}$ to the set of maps $h\colon \mathsf{U}(\mathbf{m}) \to V$ for which there is a $(\sigma, \leq) \in S$ such that $h(\mathbf{m}) \subset \sigma$ and h is order-preserving. The simplicial structure maps X_f are defined using precomposition. The advantages of simplicial sets became clear soon after their introduction. In his review of Kan's 1957 paper [KanCSS] John C. Moore (of Borel–Moore homology, Eilenberg–Moore spectral sequence, and the Milnor–Moore Hopf algebra paper) writes "In recent years it has become evident that for most purposes in homotopy theory it is more convenient to use semi-simplicial complexes instead of topological spaces." Used in 6.4, 6.4, 6.4.

Definition 6.5. (The Yoneda embedding.) Given a simplex $\mathbf{p}$, we define a simplicial set $\Delta^{\mathbf{p}}$ as follows: $(\Delta^{\mathbf{p}})_{\mathbf{m}}$ is the set of morphisms of simplices $\mathbf{m} \to \mathbf{p}$ and $(\Delta^{\mathbf{p}})_f\colon (\Delta^{\mathbf{p}})_{\mathbf{n}} \to (\Delta^{\mathbf{p}})_{\mathbf{m}}$ for a morphism of simplices $f\colon \mathbf{m} \to \mathbf{n}$ is a map that sends an element $g \in (\Delta^{\mathbf{p}})_{\mathbf{n}}$ (i.e., a morphism $g\colon \mathbf{n} \to \mathbf{p}$) to the element $g \circ f \in (\Delta^{\mathbf{p}})_{\mathbf{m}}$. Used in 6.2, 13.6, 13.6.

Remark 6.6. In this definition one can see the ideology of Remark 6.2 applied quite literally: $(\Delta^{\mathbf{p}})_{\mathbf{m}}$ was *defined* as the set of maps $\mathbf{m} \to \mathbf{p}$. The functoriality property follows immediately from the associativity and unitality properties for maps of simplices.

Exercise 6.7. Compute the cardinality of $(\Delta^{\mathbf{p}})_{\mathbf{m}}$.

Exercise 6.8. Show that $\Delta^{g \circ f} = \Delta^g \circ \Delta^f$ and $\Delta^{\mathrm{id}_{\mathbf{m}}} = \mathrm{id}_{\Delta^{\mathbf{m}}}$.

7 Simplicial maps

Definition 7.1. A *map of simplicial sets* (alias *morphism of simplicial sets* or *simplicial map*) $f\colon X \to Y$ is a family of maps of sets $f_{\mathbf{m}}\colon X_{\mathbf{m}} \to Y_{\mathbf{m}}$ (indexed by a simplex $\mathbf{m}$) such that the following *naturality property* is satisfied: for any map of simplices $g\colon \mathbf{m} \to \mathbf{n}$ the following diagram commutes:

$$\begin{array}{ccc} X_{\mathbf{m}} & \xleftarrow{X_g} & X_{\mathbf{n}} \\ {\scriptstyle f_{\mathbf{m}}}\downarrow & & \downarrow{\scriptstyle f_{\mathbf{n}}} \\ Y_{\mathbf{m}} & \xleftarrow{Y_g} & Y_{\mathbf{n}}. \end{array}$$

Used in 5.9*, 7.2, 7.2, 7.3, 7.4*, 7.5, 7.5, 7.5*, 7.6, 11.6*, 11.6*, 12.6, 12.6, 12.6, 13.7, 14.1, 28.2, 28.3, 28.5, 28.7.

Definition 7.2. The set of all morphisms of simplicial sets $X \to Y$ is known as the hom-set and is denoted by $\hom(X, Y)$. If X is a simplicial set and $f\colon Y \to Z$ is a simplicial map, then $\hom(X, f)\colon \hom(X, Y) \to \hom(X, Z)$ $(g \mapsto f \circ g)$ and $\hom(f, X)\colon \hom(Z, X) \to \hom(Y, X)$ $(g \mapsto g \circ f)$ denote the maps of sets induced by composing a given element of the hom-set with the morphism f.

Warning 7.3. Simplicial maps should not be confused with maps of simplices. The former are between simplicial sets, the latter are between simplices.

Definition 7.4. The *identity map of a simplicial set* X is the map $\mathrm{id}_X\colon X \to X$ such that $(\mathrm{id}_X)_{\mathbf{m}} = \mathrm{id}_{X_{\mathbf{m}}}$. The *composition of simplicial maps* $f\colon X \to Y$ and $g\colon Y \to Z$ is the map $g \circ f\colon X \to Z$ such that $(g \circ f)_{\mathbf{m}} = g_{\mathbf{m}} \circ f_{\mathbf{m}}$. Used in 12.6.

The *associativity and unitality properties* are satisfied for compositions and identity maps: $\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X = f$ and $(g \circ f) \circ e = g \circ (f \circ e)$ for all simplicial maps $e\colon W \to X$, $f\colon X \to Y$, $g\colon Y \to Z$.

Lemma 7.5. (The Yoneda lemma.) Consider a simplicial set X and a simplex $\mathbf{m}$. The canonical map of sets

$$\hom(\Delta^{\mathbf{m}}, X) \to X_{\mathbf{m}}$$

that sends a map of simplicial sets $f\colon \Delta^{\mathbf{m}} \to X$ to $f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}}) \in X_{\mathbf{m}}$ is an isomorphism. In other words, elements of $X_{\mathbf{m}}$, can be canonically identified with maps of simplicial sets $\Delta^{\mathbf{m}} \to X$, which yields a formal justification of Remark 6.2. Under this correspondence, the simplicial structure map $X_f\colon X_{\mathbf{n}} \to X_{\mathbf{m}}$ for a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ is identified with the map

$$\hom(\Delta^{\mathbf{n}}, X) \to \hom(\Delta^{\mathbf{m}}, X)$$

given by composing with the simplicial map $\Delta^f\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$. Used in 6.2, 6.2, 8.0*, 8.1, 8.2.

Proof. For injectivity, suppose that $f, g\colon \Delta^{\mathbf{m}} \to X$ are such that $f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}}) = g_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}})$. Then for any element h of $(\Delta^{\mathbf{m}})_{\mathbf{k}}$ (i.e., a map of simplices $h\colon \mathbf{k} \to \mathbf{m}$) we have $h = (\Delta^{\mathbf{m}})_h(\mathrm{id}_{\mathbf{m}})$, so $f_{\mathbf{k}}(h) = f_{\mathbf{k}}((\Delta^{\mathbf{m}})_h(\mathrm{id}_{\mathbf{m}}))$, which by the naturality property (taking $\Delta^{\mathbf{m}}$ for X and X for Y) equals $X_h(f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}}))$, which by the assumption on f and g equals $X_h(g_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}})) = g_{\mathbf{k}}((\Delta^{\mathbf{m}})_h(\mathrm{id}_{\mathbf{m}})) = g_{\mathbf{k}}(h)$. Thus $f_{\mathbf{k}}(h) = g_{\mathbf{k}}(h)$ for all h, so $f = g$.

For surjectivity, suppose that $a \in X_{\mathbf{m}}$. We construct a map of simplicial sets $f\colon \Delta^{\mathbf{m}} \to X$ by setting $f_{\mathbf{k}}(h) = X_h(a) \in X_{\mathbf{k}}$ for any $h\colon \mathbf{k} \to \mathbf{m}$. The naturality property is satisfied because for any $g\colon \mathbf{p} \to \mathbf{q}$ and $b \in (\Delta^{\mathbf{m}})_{\mathbf{q}}$ (i.e., $b\colon \mathbf{q} \to \mathbf{m}$) we have $f_{\mathbf{p}}((\Delta^{\mathbf{m}})_g(b)) = f_{\mathbf{p}}(b \circ g) = X_{b\circ g}(a)$ and $X_g(f_{\mathbf{q}}(b)) = X_g(X_b(a)) = X_{b\circ g}(a)$. We have $f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}}) = X_{\mathrm{id}_{\mathbf{m}}}(a) = a$, so the image of f is indeed a.

For the last claim about simplicial structure maps, observe that the following diagram commutes for any map of simplices $f\colon \mathbf{m} \to \mathbf{n}$:

$$\begin{array}{ccc} \hom(\Delta^{\mathbf{n}}, X) & \xrightarrow{\hom(\Delta^f, X)} & \hom(\Delta^{\mathbf{m}}, X) \\ {\scriptstyle f\mapsto f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}})}\downarrow & & \downarrow{\scriptstyle f\mapsto f_{\mathbf{n}}(\mathrm{id}_{\mathbf{n}})} \\ X_{\mathbf{n}} & \xrightarrow{X_f} & X_{\mathbf{m}}. \end{array}$$

We already established that the vertical maps are isomorphisms, which proves the claim. ∎

Example 7.6. Any set S gives rise to a *discrete simplicial set* $\operatorname{dis} S$ such that $(\operatorname{dis} S)_{\mathbf{m}} = S$ for any simplex $\mathbf{m}$ and $(\operatorname{dis} S)_f = \mathrm{id}_S$ for any map of simplices f. A discrete simplicial set can be visualized as a bunch of isolated points indexed by the elements of S. In particular, if S is a singleton, then $\operatorname{dis} S$ is isomorphic to $\Delta^{\mathbf{0}}$. A map of sets $f\colon S \to T$ yields a map of simplicial sets $\operatorname{dis} f\colon \operatorname{dis} S \to \operatorname{dis} T$. We have $\operatorname{dis} \mathrm{id}_{\mathbf{m}} = \mathrm{id}_{\operatorname{dis} \mathbf{m}}$ and $\operatorname{dis}(g \circ f) = \operatorname{dis} g \circ \operatorname{dis} f$. Used in 13.8, 13.8.

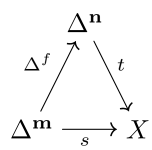
8 Simplices of a simplicial set

Recall that a *simplex of a simplicial set* X is an element of $X_{\mathbf{n}}$ for some simplex $\mathbf{n}$, or, equivalently by Lemma 7.5, a simplicial map $\Delta^{\mathbf{n}} \to X$. If $\mathbf{n}$ is fixed, we talk about $\mathbf{n}$-simplices of X. We can also use a natural number n instead of a simplex $\mathbf{n}$, taking $\mathbf{n} = \{0 < 1 < 2 < \cdots < n\}$.

Warning 8.1. Simplices should not be confused with simplices of a simplicial set. The latter "live" in a given simplicial set X, whereas the former are "disembodied abstract simplices" and are homeless. A simplex $\mathbf{n}$ in the former sense yields a simplicial set $\Delta^{\mathbf{n}}$ and the Yoneda lemma tells us that maps $\Delta^{\mathbf{n}} \to X$ can be identified with $\mathbf{n}$-simplices of X. When we say "n-simplex" or "$\mathbf{n}$-simplex" we always use the latter meaning.

Remark 8.2. The Yoneda lemma allows us mostly to ignore the distinction between a simplex $\mathbf{m}$ and its associated simplicial set $\Delta^{\mathbf{m}}$. For instance, we could talk about degenerate maps $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$ (instead of $\mathbf{m} \to \mathbf{n}$), and likewise for face maps, etc. We likewise blur the distinction between $X_{\mathbf{m}}$ and $\hom(\Delta^{\mathbf{m}}, X)$ as well as X_f and $\hom(\Delta^f, X)$.

Definition 8.3. An $\mathbf{m}$-simplex $s\colon \Delta^{\mathbf{m}} \to X$ is *degenerate* if there is a surjective map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ that is not an isomorphism and an $\mathbf{n}$-simplex $t\colon \Delta^{\mathbf{n}} \to X$ such that $t \circ \Delta^f = s$.



Used in 9.3*, 10.0*, 10.0*, 11.5, 11.5, 14.1*.

Proposition 8.4. (Eilenberg and Zilber [SSCSH, 8.3].) Every $\mathbf{m}$-simplex is a unique degeneration of a unique nondegenerate simplex. In other words, for any simplicial set X and for any simplex $s\colon \Delta^{\mathbf{m}} \to X$ there is a surjective map of simplices $f\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$ and a nondegenerate simplex $t\colon \Delta^{\mathbf{n}} \to X$ such that $s = t \circ f$. The pair (f, t) is unique up to a unique isomorphism: if $(f'\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}'}, t'\colon \Delta^{\mathbf{n}'} \to X)$ is another such pair, then there is a (necessarily unique) isomorphism $h\colon \Delta^{\mathbf{n}} \to \Delta^{\mathbf{n}'}$ such that $h \circ f = f'$ and $t' \circ h = t$.

Proof. We claim that the pair (f, t) for which $\dim \operatorname{codom} f$ is as small as possible is a pair for which t is nondegenerate. Indeed, if t is itself degenerate via some pair (g, u), then the pair $(g \circ f, u)$ would have the same properties and $\dim \operatorname{codom} g \circ f < \dim \operatorname{codom} f$.

Suppose now that (f', t') is another pair with the same properties. By Exercise 5.6, the surjective map f respectively f' has a section $g\colon \Delta^{\mathbf{n}} \to \Delta^{\mathbf{m}}$ respectively $g'\colon \Delta^{\mathbf{n}'} \to \Delta^{\mathbf{m}}$, i.e., $f \circ g = \mathrm{id}_{\Delta^{\mathbf{n}}}$ and $f' \circ g' = \mathrm{id}_{\Delta^{\mathbf{n}'}}$. We claim that $h = f' \circ g\colon \Delta^{\mathbf{n}} \to \Delta^{\mathbf{n}'}$ is the desired isomorphism. Indeed, $t \circ f = s = t' \circ f'$, so $(t' \circ f') \circ g = t \circ f \circ g = t$, i.e., $t' \circ h = t$. Likewise, $(f' \circ g) \circ f = f' \circ (g \circ f) = f'$, i.e., $h \circ f = f'$. The isomorphism h is unique because there is at most one isomorphism between any two simplices. ∎

9 Examples of simplicial sets in mathematics

9.1. ***Singular simplicial sets***

Remark 9.2. In this section, a *geometric space* and a *map of geometric spaces* refers to one of the following:

- metric spaces and contractive maps;
- metric spaces and continuous maps;
- topological spaces and continuous maps;
- uniform spaces and uniform maps;
- smooth manifolds and smooth maps;
- convex spaces and affine maps;
- real analytic spaces and real analytic maps.

In fact, what we need is that $|\mathbf{m}|$ is a geometric space for any simplex $\mathbf{m}$, $|f|$ is a map of geometric spaces for any map of simplices $f\colon \mathbf{m}\to\mathbf{n}$, for any maps of simplices f and g we have $|g\circ f| = |g|\circ|f|$, and for any simplex $\mathbf{m}$ we have $|\mathrm{id}_{\mathbf{m}}| = \mathrm{id}_{|\mathbf{m}|}$. Used in 9.2, 9.2, 9.3, 9.3, 9.3, 9.3*, 13.5, 13.9, 13.9, 13.9.

Definition 9.3. The *singular simplicial set* $\operatorname{Sing} M$ of a geometric space M is defined by setting $(\operatorname{Sing} M)_{\mathbf{m}}$ to the set of maps of geometric spaces $|\mathbf{m}|\to M$ and $(\operatorname{Sing} M)_f\colon (\operatorname{Sing} M)_{\mathbf{n}}\to(\operatorname{Sing} M)_{\mathbf{m}}$ to the map of sets that sends $a\colon|\mathbf{n}|\to M$ to $a\circ f\colon|\mathbf{m}|\to M$. Given a map of geometric spaces $r\colon M\to N$, the induced map $\operatorname{Sing} r\colon \operatorname{Sing} M\to\operatorname{Sing} N$ is defined by setting $(\operatorname{Sing} r)_{\mathbf{m}}\colon(\operatorname{Sing} M)_{\mathbf{m}}\to(\operatorname{Sing} N)_{\mathbf{m}}$ to the map of sets that sends $a\colon|\mathbf{m}|\to M$ to $r\circ a\colon|\mathbf{m}|\to N$. Used in 9.3*, 9.3*, 13.9.

The idea behind the singular simplicial set is that we "probe" a geometric space M by mapping all possible geometric realizations of simplices into it, and record the resulting information in a simplicial set.

Singular simplicial sets are important in theoretical considerations, but direct computations with them are impractical due to the huge number of simplices involved. For instance, if $M = \{(x,y)\mid x^2+y^2\}$, i.e., a circle, then $\operatorname{Sing} M$ has uncountably many simplices in every dimension, e.g., one vertex for every point, and even more higher-dimensional simplices. On the other hand, the simplicial circle of Example 10.4 has a single nondegenerate simplex in dimensions 0 and 1 and is much easier to work with in practice.

9.4. ***Nerves and Vietoris simplicial sets***

In this section we introduce two classical constructions of simplicial sets: nerves and Vietoris complexes. Both were introduced in 1927, the former by Paul Alexandroff [Approx, §13] and the latter by Leopold Vietoris [ZH].

The input data to both of these constructions is a triple (X,Y,R), where X and Y are sets and $R\subset X\times Y$ is a relation from X to Y.

In typical applications X is the underlying set of some space, whereas Y is a cover of that space, i.e., a family of subspaces of that space, whose union is X. We define $(x,y)\in R$ if $x\in y$, i.e., a point x belongs to the element y of the cover.

Definition 9.5. (Paul Alexandroff, 1927.) The *nerve* of (X,Y,R) is the simplicial set $\mathrm{N}(X,Y,R)$ defined by setting $\mathrm{N}(X,Y,R)_{\mathbf{m}}$ to the set of maps $f\colon \mathsf{U}(\mathbf{m})\to Y$ for which there is an element $x\in X$ such that $(x,y)\in R$ for any $y\in\operatorname{im} f$. For a map of simplices $f\colon\mathbf{m}\to\mathbf{n}$ the structure map

$$\mathrm{N}(X,Y,R)_f\colon \mathrm{N}(X,Y,R)_{\mathbf{n}}\to\mathrm{N}(X,Y,R)_{\mathbf{m}}$$

sends $f\colon\mathsf{U}(\mathbf{n})\to Y$ to $f\circ\mathsf{U}(f)$. Used in 9.7*.

Definition 9.6. The *Vietoris complex* $\mathrm{V}(X,Y,R)$ of (X,Y,R) is defined as $\mathrm{N}(Y,X,R^{\mathsf{op}})$, where R^{op} is the image of R under the isomorphism $X\times Y\to Y\times X$. Used in 9.6*, 9.7, 9.7*.

Thus, an $\mathbf{m}$-simplex of the Vietoris complex is a family of elements of X indexed by the vertices of $\mathbf{m}$, for which there is an element $y\in Y$ such that $(x,y)\in R$ for all x in the family.

Remark 9.7. If the set Y is ordered, we may also consider the ordered variant of $\mathrm{N}(X, Y, R)$, which requires the map f to be order-preserving. This also applies to the Vietoris complex.

It is useful to interpret the above definitions when X is the underlying set of some metric or topological space and Y is an open cover of that space. An $\mathbf{m}$-simplex of the nerve is a family of $\dim \mathbf{m}+1$ elements of the open cover that have a nonempty intersection. An $\mathbf{m}$-simplex of the Vietoris complex is a family of $\dim \mathbf{m}+1$ points in X that together form a subset of some element of the open cover.

A remarkable theorem due to Dowker [HGR] shows that the simplicial sets $\mathrm{V}(X, Y, R)$ and $\mathrm{N}(X, Y, R)$ are weakly equivalent (to be defined later).

9.8. ***Classifying simplicial sets of groups and monoids***

We now move on to a very different example, one that does not have any obvious geometric underpinnings at all.

Recall that a *monoid* is a set equipped with an associative operation. Formally, a monoid is a triple $(S, \cdot, 1)$, where $S \in \mathsf{Set}$, $\cdot\colon S \times S \to S$ is a binary operation, $1 \in S$ is the identity element, and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ and $1 \cdot x = x \cdot 1 = x$ for all $x, y, z \in S$. In particular, any group is a monoid. Other simple examples of monoids include $(\mathbf{N}, +, 0)$ (natural numbers with addition and zero) and $(\mathbf{N}, \cdot, 1)$ (natural numbers with multiplication and one). The multiplication operation need not be commutative. For instance, given a set X we can consider the noncommutative monoid $(S^S, \circ, \mathrm{id}_S)$, whose elements are maps $S \to S$, multiplication is given by the composition of maps, and the identity element is given by the identity map.

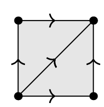
Definition 9.9. The *classifying simplicial set of a monoid* M is the simplicial set $\mathrm{B}M$ that sends a simplex $\mathbf{m}$ to $\mathsf{U}(M)^{\dim \mathbf{m}-1}$ and a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ to the map $\mathsf{U}(M)^{\dim \mathbf{n}-1} \to \mathsf{U}(M)^{\dim \mathbf{m}-1}$ whose ith component $(0 \le i < \dim \mathbf{m})$ is the product of components with indices in $[f(i), f(i+1))$. Used in 13.10.

Many important invariants of groups and other algebraic structures are defined in terms of $\mathrm{B}M$. For instance, the homology of a group G is defined as the homology of $\mathrm{B}M$. Likewise for cohomology of groups.

10 Generators and relations for simplicial sets

An n-dimensional simplex has $2^{n+1}-1$ nondegenerate simplices and the number of its k-simplices (degenerate or not) grows exponentially with k. Spelling out the details of such constructions is cumbersome, especially if more than one simplex is involved. In this section we introduce a mechanism that allows us to specify simplicial sets by listing their nondegenerate simplices and how they glue together.

To illustrate this idea, consider the following picture:



Specifying the simplices of such a simplicial set directly would be cumbersome and error-prone. What we would like to say instead is that the above simplicial set is obtained by gluing two 2-simplices along the diagonal 1-simplex, which happens to be the 1st face of both 2-simplices (i.e., the face opposite to the middle vertex). The following definition formalizes this idea.

Definition 10.1. A *system of generators and relations for a simplicial set* is specified as follows. For any simplex $\mathbf{m}$ we specify a set of *generating* $\mathbf{m}$*-simplices* $G_{\mathbf{m}}$. For any maps of simplices $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{m} \to \mathbf{p}$ we specify a subset $R_{f,g} \subset G_{\mathbf{n}} \times G_{\mathbf{p}}$.

The subset $R_{f,g}$ indicates pairs of simplices that should be identified. More precisely, if $(x, y) \in R_{f,g}$, then in the resulting simplicial set X the simplices $X_f(u(x))$ and $X_g(u(y)$ should be equal, where $u\colon G_{\mathbf{m}} \to X_{\mathbf{m}}$ sends a generating $\mathbf{m}$-simplex to its image in X. We formalize this as follows.

Definition 10.2. A pair (X, u), where $X \in \mathsf{sSet}$ and u is a family of maps of sets $u_{\mathbf{m}}\colon G_{\mathbf{m}} \to X_{\mathbf{m}}$ for every simplex $\mathbf{m}$, is a *solution for a system of generators and relations* (G, R) if $X_f(u(s)) = X_g(u(t))$ for any

$(s,t) \in R_{f,g}$. A *morphism of solutions* $(X,u) \to (X',u')$ is a simplicial map $w\colon X \to X'$ such that $\bar{w} \circ u = u'$, where $\bar{w}\colon \coprod_{\mathbf{m}} X_{\mathbf{m}} \to \coprod_{\mathbf{m}} X'_{\mathbf{m}}$ is induced from w.

Definition 10.3. The *simplicial set generated by a system of generators and relations* (g,r) is a solution (X,u) for (g,r) such that for any other solution (X',u') there is exactly one morphism of solutions $(X,u) \to (X',u')$.

Examples: spheres, real projective spaces, tori, wedges of spheres.

Example 10.4. The *simplicial sphere* S^n of dimension $n \geq -1$ is defined as follows. We set $\mathrm{S}^{-1} = \emptyset$. For $n \geq 0$ the simplicial set S^n has a generating 0-simplex v and a generating n-simplex s. The relations are $\mathrm{d}_i(s) = \mathrm{s}_0^{n-1}(v)$ for all $i \in \mathsf{U}(\mathbf{n})$. Used in 9.3*, 9.3*.

11 Simplicial chains

Informally, a simplicial chain of dimension n (or simply an n-chain) on a simplicial set X can be thought of as a map $K \to X$, where K is an n-dimensional "shape". This map need not be injective or surjective, and the dimension of X need not have any relation to n. Furthermore, this map can "cover" some parts of X several times. In particular, n-chains can be added and form an abelian group with respect to this operation. The additive inverse of an n-chain can be thought of as the same n-chain, but "traversed" in the opposite direction.

An n-chain has a boundary, which is an $(n-1)$-chain. For instance, the boundary of a 1-chain $e\colon \Delta^{\mathbf{1}} \to X$ is a formal difference of two vertices: $\partial e = \mathrm{d}_0 e - \mathrm{d}_1 e$. The boundary of an embedded circle $e\colon \mathrm{S}^1 \to X$ is empty: $\partial e = 0$.

Additionally, we expect that the boundary of a boundary is empty: $\partial(\partial(c)) = 0$ for all chains c.

To summarize, we expect the following structure: for each $n \geq 0$ we have an abelian group C_n and a homomorphism of abelian groups $\partial_n\colon C_n \to C_{n-1}$ such that $\partial_{n-1}\partial_n = 0$.

Definition 11.1. A *chain complex* C (of abelian groups) is a sequence of abelian groups C_n for all $n \in \mathbf{Z}$ together with *differentials*

$$\partial_n\colon C_n \to C_{n-1}$$

such that the map $\partial_{n-1} \circ \partial_n\colon C_n \to C_{n-2}$ is the zero homomorphism for all $n \in \mathbf{Z}$. A chain complex is *nonnegatively graded* if C_n is the zero abelian group for all $n < 0$. In this case we often suppress the mention of C_n for $n < 0$ and ∂_n for $n \leq 0$ altogether. Used in 11.2, 11.2*, 11.3, 11.3, 11.6, 11.6*, 11.7, 11.7*, 12.7, 12.7, 15.8.

Warning 11.2. The word "complex" here has a different meaning than in "simplicial complex". As we will see later, the two notions are closely related: the Dold–Kan correspondence establishes an equivalence between nonnegatively graded chain complexes and simplicial abelian groups, which are defined in the same way as simplicial sets, but using abelian groups instead of sets.

The data of a nonnegatively graded chain complex is often written from right to left as follows:

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots$$

Definition 11.3. Given a simplicial set X, its chain complex $\mathrm{C}(X)$ of (normalized) *simplicial chains* (or simply *chains*) is defined as follows. The abelian group $\mathrm{C}_n(X)$ is the quotient of the free abelian group on $X_{\mathbf{n}}$ by the subgroup generated by degenerate n-simplices. (Equivalently, one could take the free abelian group on the set of nondegenerate n-simplices of X.) The differentials $\partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ are induced by the universal property of quotients of abelian groups from the map of sets $X_{\mathbf{n}} \to \mathsf{U}(\mathrm{C}_{n-1}(X))$ that sends an n-simplex $\sigma\colon \Delta^{\mathbf{n}} \to X$ to the alternating sum $\sum_{0 \leq i \leq n} (-1)^i \mathrm{d}_i(\sigma)$, where $\mathrm{d}_i(\sigma) \in \mathrm{C}_{n-1}(X)$ via the map $X_{\mathrm{d}_i \mathbf{n}} \to \mathsf{F}(X_{n-1}) \to \mathrm{C}_{n-1}(X)$. Used in 11.0*, 11.0*, 11.0*, 11.0*, 11.0*, 11.0*, 11.0*, 11.0*, 11.5, 11.5, 11.6*, 11.6*, 13.3*, 15.0*, 15.8.

Remark 11.4. The above definition should be adjusted in a subtle way: instead of taking $X_{\mathbf{n}}$ for a fixed $\mathbf{n}$, we should take $\coprod_{\mathbf{n}} X_{\mathbf{n}}$ for *all* $\mathbf{n}$ of some fixed dimension $n \geq 0$ and quotient by the equivalence relation that identifies two simplices $\sigma \in X_{\mathbf{n}}$ and $\sigma' \in X_{\mathbf{n}'}$ if $X_f(\sigma) = \sigma'$, where $f\colon \mathbf{n}' \to \mathbf{n}$ is the unique isomorphism of simplices. This is used implicitly when we say that $\mathrm{d}_i(\sigma) \in \mathrm{C}_{n-1}(X)$ because $\mathrm{d}_i(\sigma) \in X_{\mathbf{m}}$, where $\mathbf{m}$ is obtained from $\mathbf{n}$ by removing the ith vertex.

Remark 11.5. The adjective "normalized" refers to the fact that degenerate simplices are modded out. One could also look at the nonnormalized simplicial chains, which happen to be equivalent (in the sense defined later) to the normalized chains. However, the normalized chains are far more convenient because many simplicial sets have finitely many nondegenerate simplices, but infinitely many degenerate ones.

Lemma 11.6. For any $n \geq 2$ we have $\partial_{n-1} \circ \partial_n = 0$, so the above definition indeed defines a (nonnegatively graded) chain complex.

Proof. By the universal property of free abelian groups, it suffices to verify this identity on a simplex $\sigma\colon \Delta^{\mathbf{n}} \to X$. We have

$$\begin{aligned}
\partial_{n-1}(\partial_n(\sigma)) &\stackrel{1}{=} \partial_{n-1}\left(\sum_{0\leq i\leq n}(-1)^i \mathrm{d}_i(\sigma)\right)\\
&\stackrel{2}{=} \sum_{0\leq i\leq n}(-1)^i \partial_{n-1}(\mathrm{d}_i(\sigma))\\
&\stackrel{3}{=} \sum_{0\leq i\leq n}(-1)^i \sum_{0\leq j\leq n-1}(-1)^j \mathrm{d}_j(\mathrm{d}_i(\sigma))\\
&\stackrel{4}{=} \sum_{0\leq i\leq j\leq n-1}(-1)^{i+j}\mathrm{d}_j(\mathrm{d}_i(\sigma)) + \sum_{0\leq j<i\leq n}(-1)^{i+j}\mathrm{d}_j(\mathrm{d}_i(\sigma))\\
&\stackrel{5}{=} \sum_{0\leq i\leq j\leq n-1}(-1)^{i+j}\mathrm{d}_{i,j+1}(\sigma) + \sum_{0\leq j<i\leq n}(-1)^{i+j}\mathrm{d}_{j,i}(\sigma)\\
&\stackrel{6}{=} \sum_{0\leq i<j\leq n}(-1)^{i+j-1}\mathrm{d}_{i,j}(\sigma) + \sum_{0\leq i<j\leq n}(-1)^{i+j}\mathrm{d}_{i,j}(\sigma)\\
&\stackrel{7}{=} \sum_{0\leq i<j\leq n}((-1)^{i+j-1}\mathrm{d}_{i,j}(\sigma) + (-1)^{i+j}\mathrm{d}_{i,j}(\sigma)) = 0.
\end{aligned}$$

The first equality expanded $\partial_n(\sigma)$ using the definition of ∂_n. The second equality used the fact that ∂_{n-1} is a homomorphism of abelian groups. The third equality expanded $\partial_{n-1}(\mathrm{d}_i(\sigma))$ using the definition of ∂_{n-1}. The fourth equality split the resulting double sum into two sums with $i \leq j$ and $i > j$ respectively. The fifth equality used the simplicial identities of Exercise 5.9. The sixth equality replaced j by $j-1$ in the first sum and exchanged i and j in the second sum. The seventh equality observed that both sums are now indexed in the same way and the summation terms differ only by a sign, so their sum is zero. ∎

How does the construction of simplicial chains interact with simplicial maps? As it turns out, a simplicial map induces a map between simplicial chains that commutes with the differentials.

Definition 11.7. Suppose C and D are chain complexes. A *chain map* $f\colon C \to D$ is a sequence of homomorphisms of abelian groups $f_n\colon C_n \to D_n$ such that the following square commutes for all n: Used in 11.7*, 12.0*, 12.7, 14.1, 14.2, 15.6*.

$$\begin{array}{ccc}
C_{n-1} & \xleftarrow{\partial_n^C} & C_n\\
{\scriptstyle f_{n-1}}\downarrow & & \downarrow{\scriptstyle f_n}\\
D_{n-1} & \xleftarrow{\partial_n^D} & D_n.
\end{array}$$

The data of a chain map of nonnegatively graded chain complexes is often written as follows:

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{\partial_1^C} & C_1 & \xleftarrow{\partial_2^C} & C_2 & \xleftarrow{\partial_3^C} & \cdots\\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \vdots\\
D_0 & \xleftarrow{\partial_1^D} & D_1 & \xleftarrow{\partial_2^D} & D_2 & \xleftarrow{\partial_3^D} & \cdots
\end{array}$$

Exercise 11.8. Define the *composition of chain maps* and prove that it is again a chain map. Same for identity maps. Used in 12.7.

12 Categories

Chain maps satisfy the usual properties of associativity and unitality that we are already familiar with from the definitions of simplices and simplicial sets. Rather than repeat these properties ad nauseam, we bring out the underlying abstract notion.

Definition 12.1. A *category* C is specified by the following data and properties.

- A collection $\mathsf{Ob}(\mathsf{C})$ of *objects*. We write $X \in \mathsf{C}$ instead of $X \in \mathsf{Ob}(\mathsf{C})$.
- For any objects $X, Y \in \mathsf{Ob}(\mathsf{C})$ a set of *morphisms* (alias *hom-set*) $\mathsf{Mor}_{\mathsf{C}}(X, Y)$, which can also be denoted by $\mathsf{C}(X, Y)$. We write $f: X \to Y$ instead of $f \in \mathsf{Mor}_{\mathsf{C}}(X, Y)$. We also write $\operatorname{dom} f = X$ (the *domain* of f) and $\operatorname{codom} f = Y$ (the *codomain* of f).
- For any objects $X, Y, Z \in \mathsf{Ob}(\mathsf{C})$ an operation of *composition*

$$\circ: \mathsf{Mor}_{\mathsf{C}}(Y, Z) \times \mathsf{Mor}_{\mathsf{C}}(X, Y) \to \mathsf{Mor}_{\mathsf{C}}(X, Z).$$

  We write $g \circ f$ instead of $\circ(g, f)$.
- For any object $X \in \mathsf{Ob}(\mathsf{C})$ an *identity morphism* $\mathrm{id}_X: X \to X$.
- Composition is *associative*: for any $f: W \to X$, $g: X \to Y$, $h: Y \to Z$ we have $(h \circ g) \circ f = h \circ (g \circ f)$, for which we may write $h \circ g \circ f$ instead.
- Composition is *unital*: for any $f: W \to X$ we have $\mathrm{id}_X \circ f = f \circ \mathrm{id}_W = f$.

Used in 2.0*, 3.0*, 3.1, 3.3, 3.3, 6.6, 6.6, 7.2, 7.2, 7.4, 7.4*, 7.4*, 7.4*, 7.4*, 12.0*, 12.0*, 12.3, 12.3, 12.3, 12.4, 12.4, 12.5, 12.5, 12.5, 12.5, 12.6, 12.6, 12.6, 12.6, 12.7, 12.7, 12.7*, 12.7*, 12.7*, 12.7*, 12.7*, 12.7*, 12.7*, 12.8, 12.8, 12.8, 12.9, 12.9, 12.9, 12.10, 12.10, 12.10, 12.10, 13.0*, 13.0*, 13.0*, 13.0*, 13.1, 13.1, 13.1, 13.1, 13.1, 13.1, 13.2, 13.2, 13.3, 13.3, 13.5, 13.11, 13.11, 13.11, 13.11, 13.11, 13.11, 13.11, 13.11, 13.11, 13.11, 13.11, 15.6*, 15.6*, 35.0*, 35.0*.

Remark 12.2. The word "*collection*" was used above to refer a set-like entity that can be too large to be a set. For instance, there is no set of all sets by Russell's paradox, but there is a collection of all sets. In Zermelo–Fraenkel set theory such "large" sets are known as *classes*. Used in 12.1, 12.3, 12.4, 12.5, 12.6, 12.7, 13.11, 13.11, 13.11.

Example 12.3. The primordial example of a category is the *category of sets* Set.

- $\mathsf{Ob}(\mathsf{Set})$ is the class of all sets.
- $\mathsf{Mor}_{\mathsf{Set}}(X, Y)$ is the set of all functions from X to Y.
- The operation of composition is the standard composition of functions.
- The identity morphism of a set X is the identity function on X.
- As established in elementary set theory, the composition of functions is associative and unital.

Example 12.4. The *category of abelian groups* Ab is defined as follows.

- $\mathsf{Ob}(\mathsf{Ab})$ is the class of all abelian groups.
- $\mathsf{Mor}_{\mathsf{Ab}}(X, Y)$ is the set of all homomorphisms from X to Y.
- The operation of composition is given by the composition of underlying maps of sets.
- The identity morphism of a set X is the identity homomorphism on X.
- As established in elementary algebra, the composition of homomorphisms is again a homomorphism and the resulting operation is associative and unital.

The *category of groups* Group and the *category of monoids* Monoid are defined analogously.

Example 12.5. The *category of simplices* Δ is defined as follows.

- $\mathsf{Ob}(\Delta)$ is the class of all simplices.
- $\mathsf{Mor}_{\Delta}(\mathbf{m}, \mathbf{n})$ is the set of all maps of simplices $\mathbf{m} \to \mathbf{n}$.
- The operation of composition is given by the composition of maps of simplices.
- The identity morphism of a simplex $\mathbf{m}$ is the identity map of $\mathbf{m}$.
- Associativity and unitality were shown in Exercise 3.3.

Example 12.6. The *category of simplicial sets* sSet is defined as follows.

- $\mathsf{Ob}(\mathsf{sSet})$ is the class of all simplicial sets.
- $\mathsf{Mor}_{\mathsf{sSet}}(X, Y)$ is the set of simplicial maps $X \to Y$.
- The operation of composition is given by the composition of simplicial maps.
- The identity morphism of a simplicial set X is the identity simplicial map of X.
- Associativity and unitality were verified after Definition 7.4.

Example 12.7. The *category of chain complexes* Ch is defined as follows.

- $\mathsf{Ob}(\mathsf{Ch})$ is the class of all chain complexes.
- $\mathsf{Mor}_{\mathsf{Ch}}(X, Y)$ is the set of chain maps $X \to Y$.
- Composition is indexwise: $(g \circ f)_n = g_n \circ f_n$.
- The identity morphism of a chain complex X is defined indexwise: $(\mathrm{id}_X)_n = \mathrm{id}_{X_n}$.
- Composition of chain maps is associative and unital by Exercise 11.8.

The *category of nonnegatively graded chain complexes* $\mathsf{Ch}_{\geq 0}$ is defined analogously.

In many typical examples the definition of composition and identity morphisms, as well as the verification of associativity and unitality properties is a fairly routine task (as can be seen from the above examples), and is often omitted. Accordingly, one often specifies categories by saying what their objects and morphisms are. For instance, one could say that Ch is the category of chain complexes and chain maps. Sometimes the definition of morphisms is also clear from the context, and in this case one simply specifies the objects. For instance, one could say that sSet is the category of simplicial sets. However, one must keep in mind that one can encounter in practice categories with the same collection of objects, but different morphisms. For instance, one has three very different notions of a morphism between metric spaces:

- contractive maps: $f\colon X \to Y$ is contractive if $\mathrm{d}(f(x), f(x')) \leq \mathrm{d}(x, x')$ for any points $x, x' \in X$.
- uniformly continuous maps: $f\colon X \to Y$ is uniformly continuous if for any $\epsilon > 0$ there is $\delta > 0$ such that $\mathrm{d}(x, x') < \delta$ implies $\mathrm{d}(f(x), f(x')) < \epsilon$.
- continuous maps: $f\colon X \to Y$ is continuous if for any $x \in X$ and $\epsilon > 0$ there is $\delta > 0$ such that $\mathrm{d}(x, x') < \delta$ implies $\mathrm{d}(f(x), f(x')) < \epsilon$.

These three types of maps give rise to three different categories of metric spaces:

- the category of metric spaces and contractive maps;
- the category of metric spaces and uniformly continuous maps;
- the category of metric spaces and continuous maps.

We conclude this section by giving some examples that should dispel the idea that objects in a category are "sets with structures" and morphisms are "functions that preserve structures" (as one could guess from the above examples).

Example 12.8. Suppose $(P, \leq)$ is a poset. We construct a category C as follows: $\mathsf{Ob}(\mathsf{C}) = P$ and if $x, y \in P$ then $\mathsf{Mor}_{\mathsf{C}}(x, y)$ is a singleton respectively empty set if $x \leq y$ respectively $x \not\leq y$. There is exactly one way to define compositions and identity morphisms.

Example 12.9. Suppose G is a group. We define a category $\mathrm{B}G$ as follows: $\mathsf{Ob}(\mathsf{C}) = \{*\}$ is a singleton set and $\mathsf{Mor}_{\mathsf{C}}(*, *) = \mathsf{U}(G)$. The operation of composition is given by multiplication: $\mathsf{Mor}(*, *) \times \mathsf{Mor}(*, *) = G \times G \to G = \mathsf{Mor}(*, *)$. The identity morphism of $*$ is given by the identity element of G.

Example 12.10. Suppose C is a category. The *opposite category* of C is denoted by C^{op} and is defined as follows: the set $\mathsf{Ob}(\mathsf{C}^{\mathsf{op}})$ is a disjoint copy of $\mathsf{Ob}(\mathsf{C})$ (via a map denoted $X \mapsto X^{\mathsf{op}}$) and $\mathsf{Mor}_{\mathsf{C}^{\mathsf{op}}}(X^{\mathsf{op}}, Y^{\mathsf{op}})$ is a disjoint copy of $\mathsf{Mor}_{\mathsf{C}}(Y, X)$ (via a map denoted $f \mapsto f^{\mathsf{op}}$). The composition is defined by $g^{\mathsf{op}} \circ f^{\mathsf{op}} = (f \circ g)^{\mathsf{op}}$ and $\mathrm{id}_{X^{\mathsf{op}}} = \mathrm{id}_X^{\mathsf{op}}$. (We use op to denote both the opposite category C^{op} as well as objects and morphisms in it.) Used in 13.7, 13.7, 13.7.

13 Functors

We have already encountered many constructions that preserve composition of morphisms and identity morphisms. Rather than to continue repeating these properties indefinitely, we elect to formalize them using the previously defined notion of categories.

Definition 13.1. Suppose C and D are categories. A *functor* F from C to D, denoted $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$, is specified by the following data and properties.

- A map of collections $\mathsf{Ob}(\mathsf{F})\colon \mathsf{Ob}(\mathsf{C}) \to \mathsf{Ob}(\mathsf{D})$. (We write $\mathsf{F}(X)$ or $\mathsf{F}X$ instead of $\mathsf{Ob}(\mathsf{F})(X)$.)
- A collection of maps of sets $\mathsf{Mor}_\mathsf{F}(X,Y)\colon \mathsf{Mor}_\mathsf{C}(X,Y) \to \mathsf{Mor}_\mathsf{D}(\mathsf{F}(X),\mathsf{F}(Y))$, one for each pair of objects $X, Y \in \mathsf{C}$. (We write $\mathsf{F}(f)$ or $\mathsf{F}f$ instead of $\mathsf{Mor}_\mathsf{F}(X,Y)(f)$, where $f\colon X \to Y$ is a morphism in C.)
- Composition is preserved: if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms in C, then $\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$.
- Identity morphisms are preserved: if $X \in \mathsf{C}$, then $\mathsf{F}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{F}(X)}$.

The last two properties taken together are known as *functoriality properties*. Used in 2.0*, 3.6, 4.1, 6.1, 6.1, 6.6, 7.6, 13.3*, 13.4, 13.5, 13.5, 13.6, 13.7, 13.7, 13.7, 13.7, 13.8, 13.9, 13.10, 13.11, 13.11, 13.11, 15.6*, 15.6*.

Our first two examples of categories were Set and Ab, so we start by exhibiting some functors between them.

Example 13.2. The *forgetful functor* $\mathsf{U}\colon \mathsf{Ab} \to \mathsf{Set}$ is defined as follows.

- For an abelian group $A = (S, +, -, 0)$ we set $\mathsf{U}(A) = S$, the underlying set of A.
- A homomorphism of abelian groups $f\colon A = (S, +_A, -_A, 0_A) \to B = (T, +_B, -_B, 0_B)$ is by definition a map of sets $g\colon S \to T$ that satisfies some additional properties. We set $\mathsf{U}(f) = g$, the underlying map of sets of f.
- Composition is preserved because the composition of two homomorphisms of abelian groups is by definition the composition of the underlying maps of sets.
- Identity morphisms are preserved for the same reason.

Example 13.3. The *free abelian group functor* $\mathsf{Free}\colon \mathsf{Set} \to \mathsf{Ab}$ is defined as follows.

- For a set S we set $\mathsf{Free}(S) = \{c\colon S \to \mathbf{Z} \mid \#\{s \in S \mid c(s) \neq 0\} < \infty\}$, i.e., the abelian group of finitely supported functions $S \to \mathbf{Z}$ equipped with the pointwise operations induced from the abelian group $\mathbf{Z}$.
- For a map of sets $f\colon S \to T$ we set $\mathsf{Free}(f)\colon \mathsf{Free}(S) \to \mathsf{Free}(T)$ to the homomorphism of abelian groups that sends any $c\colon S \to \mathbf{Z}$ to the map $T \to \mathbf{Z}$ that sends $t \mapsto \sum_{s \in f^{-1}(t)} c(s)$.
- As shown in elementary algebra, composition and identity morphisms are preserved by Free.

We already used the functor Free when we defined $\mathrm{C}(X)_\mathbf{n}$ as a certain quotient group of $\mathsf{Free}(X_\mathbf{n})$ in Definition 11.3.

Example 13.4. Definition 3.6 is nothing else than a definition of a *forgetful functor* $\mathsf{U}\colon \Delta \to \mathsf{Set}$. (Any functor that "forgets" structure like abelian group operations or a total ordering can be referred to as a forgetful functor and denoted by U.)

Example 13.5. Definition 4.1 combined with Remark 4.3 defines a functor $|-|\colon \Delta \to \mathsf{Space}$, where Space denotes any of the categories of "geometric spaces" mentioned in Remark 9.2. The functoriality properties are verified in Remark 4.2. Used in 13.9.

Example 13.6. The Yoneda embedding of Definition 6.5 is a functor $\Delta^\bullet\colon \Delta \to \mathsf{sSet}$.

Example 13.7. A simplicial set is nothing else than a functor $\Delta^{\mathsf{op}} \to \mathsf{Set}$. Here the superscript op denotes the opposite category of Δ defined in Example 12.10. Indeed, expanding the definitions, we see that such a functor X assigns a set $X_\mathbf{m}$ to any simplex $\mathbf{m}$, a map of sets $X_\mathbf{n} \to X_\mathbf{m}$ to any map of simplices $\mathbf{m} \to \mathbf{n}$ (the direction of the map is reversed because of the opposite category), and the functoriality conditions demand that $X_{\mathrm{id}_\mathbf{m}} = \mathrm{id}_{X_\mathbf{m}}$ for any simplex $\mathbf{m}$ and $X_{g \circ f} = X_f \circ X_g$ for any maps of simplices $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{n} \to \mathbf{p}$. This is precisely Definition 6.1. (One may wonder how simplicial maps fit into this picture. These turn out to be precisely natural transformations of functors, to be defined later.)

Example 13.8. The discrete simplicial set construction is a functor $\mathrm{dis}\colon \mathsf{Set} \to \mathsf{sSet}$, as verified in Example 7.6.

Exercise 13.9. Verify that the singular simplicial set construction is a functor $\mathrm{Sing}\colon \mathsf{Space} \to \mathsf{sSet}$, where Space is one of the categories of geometric spaces of Remark 9.2. (The nature of geometric spaces is irrelevant here, only the fact that Space is a category equipped with the functor $|-|\colon \Delta \to \mathsf{Space}$ from Example 13.5 matters.)

Exercise 13.10. Verify that the classifying simplicial set construction is a functor $\mathrm{B}\colon \mathsf{Monoid} \to \mathsf{sSet}$.

Example 13.11. Categories and functors themselves can be organized into a category, the *category of categories*, commonly denoted Cat or CAT. (These two choices correspond to requiring the collection of all objects in a category to form a set respectively a class in the Zermelo–Fraenkel set theory, a technical issue that can be ignored for the time being.) More precisely,

- $\mathsf{Ob}(\mathsf{Cat})$ is the collection of all categories whose collection of objects is a set.
- $\mathsf{Mor}_{\mathsf{Cat}}(\mathsf{X}, \mathsf{Y})$ is the set of functors $\mathsf{F}\colon \mathsf{X} \to \mathsf{Y}$.
- Composition is defined as follows: $(\mathsf{G} \circ \mathsf{F})(A) = \mathsf{G}(\mathsf{F}(A))$ for any object $A \in \operatorname{dom} \mathsf{X}$; $(\mathsf{G} \circ \mathsf{F})(f) = \mathsf{G}(\mathsf{F}(f))$ for any morphism f in $\operatorname{dom} \mathsf{X}$.
- The identity morphism of a category X is the *identity functor*: $\mathrm{id}_{\mathsf{X}}(A) = A$ for any object $A \in \mathsf{X}$; $\mathrm{id}_{\mathsf{X}}(f) = f$ for any morphism f in X.
- Composition of functors is associative and unital because composition of maps of sets is associative and unital.

14 Simplicial maps induce chain maps

Proposition 14.1. Any simplicial map $f\colon X \to Y$ induces a chain map $\mathrm{C}(f)\colon \mathrm{C}(X) \to \mathrm{C}(Y)$.

Proof. We start by constructing homomorphisms of abelian groups $\mathrm{C}(f)_{\mathbf{m}}\colon \mathrm{C}(X)_{\mathbf{m}} \to \mathrm{C}(Y)_{\mathbf{m}}$. The functor Free sends the map of sets $f_{\mathbf{m}}\colon X_{\mathbf{m}} \to Y_{\mathbf{m}}$ to the homomorphism $\mathsf{Free}(f_{\mathbf{m}})\colon \mathsf{Free}(X_{\mathbf{m}}) \to \mathsf{Free}(Y_{\mathbf{m}})$, which then descends to the quotients by the abelian subgroup generated by degenerate simplices because the map of sets $X_{\mathbf{m}} \to Y_{\mathbf{m}}$ preserves degenerate simplices. ▮

Exercise 14.2. Prove that this construction defines a chain map. Prove that C is a functor $\mathrm{C}\colon \mathsf{sSet} \to \mathsf{Ch}_{\geq 0}$.

15 Homology

Simplicial sets that we can consider to be the "same" (to be formalized later using the notion of simplicial weak equivalence) can have different (i.e., nonisomorphic) complexes of simplicial chains.

However, we can extract a graded abelian group that is invariant under simplicial weak equivalences.

In the following definition one should think of $C = \mathrm{C}(X)$ for some $X \in \mathsf{sSet}$.

Definition 15.1. Suppose $C \in \mathsf{Ch}$ and $c \in C_n$ for some $n \in \mathbf{Z}$.

- The n-chain c is a *cycle* if $\partial c = 0$. Cycles form an abelian group $\ker \partial_n$, denoted $\mathrm{Z}_n(C)$, where Z stands for the German word Zykel.
- The n-chain c is a *boundary* if there is $b \in \mathrm{C}_{n+1}(C)$ such that $\partial b = c$. Boundaries form an abelian group $\operatorname{im} \partial_{n+1}$, denoted $\mathrm{B}_n(C)$.

Lemma 15.2. For any $C \in \mathsf{Ch}$ and $n \in \mathbf{Z}$ the group $\mathrm{B}_n(C)$ is a subgroup of $\mathrm{Z}_n(C)$.

Proof. If $c \in \mathrm{B}_n(C)$, then there is $b \in \mathrm{C}_{n+1}(C)$ such that $\partial b = c$, so $\partial c = \partial(\partial(b)) = 0$, i.e., $c \in \mathrm{Z}_n(C)$. ▮

Definition 15.3. The nth *homology group* of $C \in \mathsf{Ch}$ is the quotient group $\mathrm{H}_n(C) = \mathrm{Z}_n(C)/\mathrm{B}_n(C)$. Elements of $\mathrm{H}_n(C)$ are known as *homology classes* in degree n.

Definition 15.4. Suppose $X \in \mathsf{sSet}$. We define $\mathrm{Z}_n(X) = \mathrm{Z}_n(\mathrm{C}(X))$, $\mathrm{B}_n(X) = \mathrm{B}_n(\mathrm{C}(X))$, $\mathrm{H}_n(X) = \mathrm{H}_n(\mathrm{C}(X))$. Elements of these groups are referred to as *simplicial cycles*, *simplicial boundaries*, and *simplicial homology classes*.

Sometimes it makes sense to manipulate the entire collection of the above groups for all n as a single whole. This can be formalized as follows.

Definition 15.5. Suppose I is a set. An *I-graded abelian group* is a family of abelian groups indexed by I. If A and B are I-graded abelian groups, then a homomorphism from A to B is a family of homomorphisms of abelian groups $A_i \to B_i$ for all $i \in I$. Thus, I-graded abelian groups form a category Ab^I. Used in 15.8.

Proposition 15.6. We have functors $\mathrm{Z}\colon \mathsf{Ch} \to \mathsf{Ab}^{\mathbf{Z}}$, $\mathrm{B}\colon \mathsf{Ch} \to \mathsf{Ab}^{\mathbf{Z}}$, $\mathrm{H}\colon \mathsf{Ch} \to \mathsf{Ab}^{\mathbf{Z}}$.

Proof. We defined these functors on objects of Ch above. It remains to define them on morphisms and verify the functoriality properties. Given a chain map $f\colon C \to D$, observe that $f(\mathrm{Z}_n(C)) \subset \mathrm{Z}_n(D)$ because $\partial_D(f(c)) = f(\partial_C(c)) = f(0) = 0$ for any $c \in \mathrm{Z}_n(C)$. Likewise, $f(\mathrm{B}_n(C)) \subset \mathrm{B}_n(D)$ because $f(\partial_C(c)) = \partial_D(f(c))$ for any $c \in C_{n+1}$. Thus $\mathrm{Z}(f)$ and $\mathrm{B}(f)$ can be defined as (co)restrictions of $\mathrm{C}(f)$ to the appropriate (co)domains.

To define $\mathrm{H}_n(f)\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$, we use the universal property of quotient groups for $\mathrm{H}_n(C) = \mathrm{Z}_n(C)/\mathrm{B}_n(C)$: homomorphisms of the form $\mathrm{H}_n(C) \to \mathrm{H}_n(D)$ are in bijective correspondence with homomorphisms $\mathrm{Z}_n(C) \to \mathrm{H}_n(D)$ whose restriction to $\mathrm{B}_n(C)$ vanishes. Take the homomorphism $\mathrm{Z}_n(C) \to \mathrm{H}_n(D)$ given by the composition of $\mathrm{Z}_n(f)\colon \mathrm{Z}_n(C) \to \mathrm{Z}_n(D)$ and the quotient map $q\colon \mathrm{Z}_n(D) \to \mathrm{H}_n(D)$. This composition vanishes on $\mathrm{B}_n(C)$ because $\mathrm{Z}_n(f)(\mathrm{B}_n(C)) \subset \mathrm{B}_n(D)$ and q vanishes on $\mathrm{B}_n(D)$. ▮

Exercise 15.7. Define $\mathrm{H}(f)$ and complete

Remark 15.8. The worm "*homology*" typically refers to the graded abelian group. Sometimes it is also used informally to refer to simplicial chain (as a chain complex). Used in 1.0*, 1.0*, 9.9*.

singular homology
homology of a group

16 Cohomology

cohomology cohomology of a group

17 Connected components

Definition 17.1. The *set of connected components* of a simplicial set X is a set $\pi_0(X)$ equipped with a map $q\colon X \to \operatorname{dis}\pi_0(X)$ such that for set S equipped with a map $r\colon X \to \operatorname{dis} S$ there is a unique map of sets $s\colon \pi_0(X) \to S$ such that $(\operatorname{dis} s) \circ q = r$.

The idea behind the map $X \to \operatorname{dis}\pi_0(X)$ is that it collapses every connected component of X to a single point in $\operatorname{dis}\pi_0(X)$. The universal property guarantees that different components are collapsed to different points.

Definition 17.2. A *connected simplicial set* is a simplicial set X such that $\pi_0(X)$ is a singleton set.

Examples 17.3.
- $\pi_0(\operatorname{dis} S) = S$, so $\operatorname{dis} S$ is connected if and only if the set S is a singleton.
- $\pi_0(\Delta^{\mathbf{m}}) = \{*\}$, so $\Delta^{\mathbf{m}}$ is always connected.

Exercise 17.4. Define $\pi_0(f)$ for a simplicial map $f\colon X \to Y$. Prove that this yields a functor $\pi_0\colon \mathsf{sSet} \to \mathsf{Set}$.

Exercise 17.5. Show that $\pi_0(X)$ can be computed as the quotient of $X_{\mathbf{0}}$ by the equivalence relation generated by $(\mathrm{d}_0, \mathrm{d}_1)(X_{\mathbf{1}}) \subset X_{\mathbf{0}} \times X_{\mathbf{0}}$. (Geometrically, we identify those vertices of X that are connected by a chain of 1-simplices going in any direction.)

Exercise 17.6. Prove that any simplicial set decomposes as a coproduct of connected simplicial sets. What is the indexing set of this coproduct? Formulate an appropriate notion of uniqueness for such a decomposition and prove it.

18 Fundamental groupoid and covering spaces

Supplementary sources: §2.4 and §2.5 in Joyal and Tierney [NSHT].
fundamental group covering space
natural transformation naturality isomorphism

19 Operations on simplicial sets

Hom and cartesian product.
skeleton coskeleton function complex mapping simplicial set mapping space commutative diagram cartesian square cocartesian square cartesian cocartesian pullback of simplicial sets pushout of simplicial sets base change cobase change universal property of pullbacks universal property of pushouts

20 Simplicial homotopy equivalences

simplicial homotopy equivalence

21 Kan complexes

$\mathsf{sSet}_{\mathsf{Kan}}$
Kan complex

22 The relative category of spaces

space simplicial weak equivalence weak equivalence weakly equivalent weakly contractible

23 Homotopy groups

Postnikov tower

Abstract homotopy theory

24 The Dold–Kan correspondence

Dold–Kan correspondence simplicial abelian group

25 Homotopy limits and colimits

homotopy limit homotopy colimit mapping telescope

26 Weak factorization systems

Joyal's [WFS] is a good reference.
functorial factorization

27 Model structure on simplicial sets

Barycentric subdivisions, the functors Sd and Ex. Kan's Ex^∞ functor. It preserves all 5 classes of a model structure, finite limits, filtered colimits and coproducts.

Weak equivalence: $\mathrm{Ex}^\infty f$ is a simplicial homotopy equivalence. *Cofibrations*: monomorphisms. *Acyclic cofibrations*: retracts of horn attachments. *Fibrations* and *acyclic fibrations*: lifting property. *Fibrant* objects are Kan complexes.

Closure properties of simplicial homotopy equivalences.

Right properness: weak equivalences are stable under base changes along fibrations. Apply Ex^∞.

Closure properties of left and right classes, 2-out-of-3.

Cofibrations that are weak equivalences are retracts of relative horn attachments.

Pushout product axiom for (acyclic) cofibrations of simplicial sets.

Cofibrations are closed under cobase changes, 2-out-of-3. The disjoint union of a cofibration and a simplicial set is again a cofibration. Cofibrations are closed under composition.

28 Functorial factorizations of simplicial maps

Notation 28.1. The maps $\iota_0\colon \Delta^0 \to \Delta^1$ and $\iota_1\colon \Delta^0 \to \Delta^1$ pick the two vertices of Δ^1 in increasing order. The map $\iota\colon \partial\Delta^1 \to \Delta^1$ is the boundary inclusion of Δ^1. The map $p\colon \Delta^1 \to \Delta^0$ is the unique simplicial map from Δ^1 to Δ^0.

Lemma 28.2. (The *mapping cylinder* construction.) If $X, Y \in \mathsf{sSet}$ and $f\colon X \to Y$ is a simplicial map, then the maps $X \to \operatorname{cyl}(f) \to Y$ constructed in the proof form a functorial factorization of f into a cofibration followed by a weak equivalence. Used in 28.6*, 28.7*.

Proof. Denote $\operatorname{cyl}(f) = \Delta^1 \times X \sqcup_X Y$ and $\operatorname{cyl}(X) = \operatorname{cyl}(\mathrm{id}_X) = \Delta^1 \times X$. Consider the following commutative diagram, where the square is cocartesian:

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & Y \\ & & {\scriptstyle \iota_1 \times X}\downarrow & & \downarrow \\ X & \xrightarrow{\iota_0 \times X} & \operatorname{cyl}(X) & \longrightarrow & \operatorname{cyl}(f). \end{array}$$

This diagram yields the map $X \to \operatorname{cyl}(f)$, whereas the map $\operatorname{cyl}(f) \to Y$ is induced by the universal property of pushouts from the maps $f \circ (p \times X)\colon \Delta^1 \times X \to X \to Y$ and $\mathrm{id}_Y\colon Y \to Y$. These definitions imply that the composition $X \to \operatorname{cyl}(f) \to Y$ equals f, so we indeed have a functorial factorization.

Consider the cocartesian square

$$\begin{array}{ccc} Y \sqcup Y & \xrightarrow{f \sqcup \mathrm{id}_Y} & X \sqcup Y \\ {\scriptstyle \iota \times Y}\downarrow & & \downarrow \\ \Delta^1 \times Y & \longrightarrow & X \sqcup_Y \Delta^1 \times Y \end{array}$$

The left map is a cofibration because $\iota\colon \partial\Delta^1 \to \Delta^1$ is a cofibration. Thus, the right map is a cofibration. The map $X \cong X \sqcup \emptyset \to X \sqcup Y$ is a cofibration because $\emptyset \to X$ is a cofibration. Thus, the composition of the map $X \to X \sqcup Y$ and the right map is a cofibration. The resulting map $X \to X \sqcup_Y \Delta^1 \times Y$ is the first map in the factorization.

The map $\operatorname{cyl}(f) \to Y$ is a simplicial homotopy equivalence (hence, a simplicial weak equivalence) because the composition $Y \to \operatorname{cyl}(f) \to Y$ equals id_Y and the map $Y \to \operatorname{cyl}(f)$ is a cobase change of the map $\iota_1 \times X$, which is a simplicial homotopy equivalence because ι_1 is one. ∎

Lemma 28.3. (The *mapping path space* (alias *mapping cocylinder*) construction.) If $X, Y \in \mathsf{sSet}_{\mathsf{Kan}}$ and $f\colon X \to Y$ is a simplicial map, then the maps $X \to \operatorname{cocyl}(f) \to Y$ constructed in the proof form a functorial factorization of the map f into an acyclic cofibration followed by a fibration. Used in 1.0*, 28.5*, 28.5*, 28.5*.

Remark 28.4. This lemma is formally dual to the previous lemma except for two differences. In this lemma we must assume the source and target to be Kan complexes, whereas the previous lemma imposes no such restrictions. Additionally, the previous lemma can produce weak equivalences that are not fibrations, whereas this lemma produces weak equivalences that are also cofibrations.

Proof. Denote $\operatorname{cocyl}(f) = X \times_Y Y^{\Delta^1}$ and $\operatorname{cocyl}(Y) = \operatorname{cocyl}(\mathrm{id}_Y) = Y^{\Delta^1}$. Consider the following commutative diagram, where the square is cartesian:

$$\begin{array}{ccccc} \operatorname{cocyl}(f) & \longrightarrow & \operatorname{cocyl}(Y) & \xrightarrow{\operatorname{Hom}(\iota_0, Y)} & Y \\ \downarrow & & \downarrow{\scriptstyle \operatorname{Hom}(\iota_1, Y)} & & \\ X & \xrightarrow{f} & Y. & & \end{array}$$

This diagram yields the map $\operatorname{cocyl}(f) \to Y$, whereas the map $X \to \operatorname{cocyl}(f)$ is induced by the universal property of pullbacks from the maps $\mathrm{id}_X\colon X \to X$ and $\operatorname{Hom}(p, Y) \circ f\colon X \to Y \to Y^{\Delta^1}$. These definitions imply that the composition $X \to \operatorname{cocyl}(f) \to Y$ equals f, so we indeed have a functorial factorization.

Consider the cartesian square

$$\begin{array}{ccc} X\times_Y Y^{\Delta^1} & \longrightarrow & Y^{\Delta^1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(\iota,Y)} \\ X\times Y & \xrightarrow{f\times \mathrm{id}_Y} & Y\times Y \end{array}$$

The right map is a fibration because $\iota\colon\partial\Delta^1\to\Delta^1$ is a cofibration and Y is fibrant. Thus, the left map is a fibration. The map $X\times Y\to 1\times Y\cong Y$ is a fibration because $X\to 1$ is a fibration. Thus, the composition $\mathrm{cocyl}(f)=X\times_Y Y^{\Delta^1}\to X\times Y\to Y$ is also a fibration and it is the second map in the factorization.

The map $X\to\mathrm{cocyl}(f)$ is a simplicial homotopy equivalence (hence, a simplicial weak equivalence) because the composition $X\to\mathrm{cocyl}(f)\to X$ equals id_X and the map $X\to\mathrm{cocyl}(f)$ is a base change of the map $\mathrm{Hom}(\iota_1,X)$, which is a simplicial homotopy equivalence because ι_1 is one.

Finally, the map $X\to\mathrm{cocyl}(f)=X\times_Y Y^{\Delta^1}$ is a cofibration because its composition with the projection $X\times_Y Y^{\Delta^1}\to X$ equals id_X. ▮

We now remove the requirement for the source and target to be Kan complexes.

Proposition 28.5. (The *derived mapping path space* (alias *derived mapping cocylinder*) construction.) If $f\colon X\to Y$ is a simplicial map, then the maps $X\to\mathrm{Rcocyl}(f)\to Y$ constructed in the proof form a functorial factorization of the map f into an acyclic cofibration followed by a fibration. Used in 28.6.

Proof. Consider the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow & & \big\downarrow \\ \mathrm{Ex}^\infty X & \xrightarrow{\mathrm{Ex}^\infty f} & \mathrm{Ex}^\infty Y. \end{array}$$

We apply the mapping path space construction of Lemma 28.3 to the bottom map (its source and target are fibrant) and complete the resulting diagram as depicted below, with the right square being cartesian and the top map in the left square induced by the universal property of pullbacks.

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{Rcocyl}(f) & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{Ex}^\infty X & \longrightarrow & \mathrm{cocyl}(\mathrm{Ex}^\infty f) & \longrightarrow & \mathrm{Ex}^\infty Y. \end{array}$$

By Lemma 28.3 the bottom left map is an acyclic cofibration and the bottom right map is a fibration. Accordingly, the top right map is a fibration because it is a base change of a fibration.

The middle map is a weak equivalence because the right map is one and weak equivalences are stable under base changes along fibrations. Thus, in the left square all maps except for the top one are weak equivalences, hence the top map is also a weak equivalence.

Both maps $X\to\mathrm{Ex}^\infty X\to\mathrm{cocyl}(\mathrm{Ex}^\infty f)$ are cofibrations. Hence, their composition is also a cofibration. Therefore, the top left map $X\to\mathrm{Rcocyl}(f)$ is also a cofibration. ▮

The 2-out-of-3 property of simplicial weak equivalences immediately implies the following claim.

Corollary 28.6. If $f\colon X\to Y$ is a simplicial weak equivalence, then the maps $X\to\mathrm{Rcocyl}(f)\to Y$ constructed in Proposition 28.5 form a functorial factorization of the map f into an acyclic cofibration followed by an acyclic fibration. Used in 28.7*.

We now improve Lemma 28.2, allowing the second map to be an acyclic fibration and not just a simplicial weak equivalence.

Proposition 28.7. If $X,Y\in\mathsf{sSet}$ and $f\colon X\to Y$ is a simplicial map, then the maps $X\to\mathrm{Rcyl}(f)\to Y$ constructed in the proof form a functorial factorization of f into a cofibration followed by an acyclic fibration.

Proof. Use Lemma 28.2 to factor the map f as $X\to\mathrm{cyl}(f)\to Y$, where the first map is a cofibration and the second map is a weak equivalence. Use Corollary 28.6 to factor the map $g\colon\mathrm{cyl}(f)\to Y$ as $\mathrm{cyl}(f)\to$

$\mathrm{Rcocyl}(g) \to Y$, where the first map is an acyclic cofibration and the second map is an acyclic fibration. Composing $X \to \mathrm{cyl}(f)$ and $\mathrm{cyl}(f) \to \mathrm{Rcocyl}(g)$, we get a functorial factorization of the map f as $X \to \mathrm{Rcocyl}(g) \to Y$, where the first map is a cofibration and the second map is an acyclic fibration. ▮

Summary 28.8. Suppose $f\colon X \to Y$ is a simplicial map. Expanding the above constructions, we obtain the following formulas for the functorial factorizations of f. Acyclic cofibration followed by a fibration:

$$X \to \mathrm{Ex}^\infty X \times_{\mathrm{Ex}^\infty Y} (\mathrm{Ex}^\infty Y)^{\Delta^1} \times_{\mathrm{Ex}^\infty Y} Y \to Y.$$

Cofibration followed by an acyclic fibration:

$$X \to \mathrm{Ex}^\infty((\Delta^1 \times X) \sqcup_X Y) \times_{\mathrm{Ex}^\infty Y} (\mathrm{Ex}^\infty Y)^{\Delta^1} \times_{\mathrm{Ex}^\infty Y} Y \to Y.$$

Stable homotopy theory

29 Generalized homology theories

generalized homology theory generalized cohomology theory

30 Model categories

model category model structure derived mapping space derived right proper

31 K-theory

K-theory

32 Spectra

spectrum

Local homotopy theory

33 Sheaf cohomology

Sheaf cohomology

34 Further topics

Cohomology. Homology and cohomology with coefficients in an abelian group. The fundamental groupoid. The van Kampen theorem. Covering spaces. Products and coproducts of simplicial sets. Quotients and subspaces of simplicial sets. Cup product in cohomology. Combinatorial manifolds. Cap product. Poincaré duality. Intersection product in homology. Local systems. Homology and cohomology with coefficients in local systems. Poincaré duality for nonorientable manifolds. Pushforward and pullback of local systems. Verdier duality. Homotopy groups. The Hurewicz isomorphism. Long exact sequence of a fibration. Blakers–Massey theorem. Freudenthal suspension theorem. Fiber and cofiber sequences of homotopy groups. Simplicial homotopies. Kan complexes. Kan fibrant replacement functor. Weak equivalences of simplicial sets. Smith–Dugger–Isaksen criterion. Derived mapping spaces. Homotopy limits and colimits of simplicial sets. Chain complexes and their homotopy (co)limits. The Dold–Kan correspondence. Eilenberg–MacLane spaces. Interaction of homotopy (co)limits with simplicial (co)homology. Homology and cohomology theories as (co)continuous functors. Eilenberg–Steenrod axioms. Topological spaces. Geometric realization and singular simplicial set functor. Quillen–Kan–Serre–Milnor equivalence. Topological K-theory. Model categories. The Smith recognition theorem. Example: simplicial sets, chain complexes, spectra. Simplicial symmetric spectra. Representability of homology and cohomology theories by spectra. Smash product and internal hom of spectra. Multiplicative cohomology theories. Spanier–Whitehead duality. Thom spectra. Atiyah duality. Simplicial presheaves. Sheaf cohomology. De Rham cohomology, de Rham theorem.

35 Appendix: sets and functions

The *ordered pair* (a,b) is defined as $\{\{\{a\}\},\{\{b\},\{\emptyset\}\}\}$. The reasoning behind this definition is that $(a,b)=(a',b')$ if and only if $a=a'$ and $b=b'$. The *product of sets* A and B is the set $A\times B=\{z\mid \exists a\in A, b\in B\colon z=(a,b)\}$. An *ordered triple* (a,b,c) can now be defined as $((a,b),c)$ and likewise for n-tuples.

A *relation* is a triple (A,B,R), where A and B are sets and $R\subset A\times B$. We emphasize that A and B form a part of the data of a relation. We also say that R is a relation from A to B. We often write aRb instead of $(a,b)\in R$. Relations can be composed: if R is a relation from A to B and S is a relation from B to C, then $S\circ R$ is a relation from A to C for which $(a,c)\in S\circ R$ if and only if there is $b\in B$ such that aRb and bSc. We have $(T\circ S)\circ R=T\circ(S\circ R)$, for which we simply write $T\circ S\circ R$. The *identity relation* id_A from A to A satisfies $(a,a')\in \mathrm{id}_A \iff a=a'$. We have $\mathrm{id}_B\circ R=R=R\circ\mathrm{id}_A$. An *equivalence relation* on a set A is a relation R from A to A such that aRa for all $a\in A$, aRb implies bRa for all $a,b\in A$, and aRb and bRc implies aRc for all $a,b,c\in A$. The *equivalence class* of an element $a\in A$ with respect to an equivalence relation R on A is the set $[a]:=\{x\in A\mid aRx\}$.

A *map of sets* from A to B is defined as a *functional relation* from A to B, namely, a relation f from A to B with an additional property that for any $a\in A$ there is exactly one $b\in B$ (denoted $f(a)$) such that $(a,b)\in f$. We refer to A as the domain of f (denoted $\operatorname{dom} f$) and B as the codomain of f (denoted $\operatorname{codom} f$). The composition of two functional relations is again functional, which allows us to define compositions of maps via compositions of relations.

In the modern mathematical parlance, the word "*function*" is exactly synonymous with "map" (of sets). Historically, though, a very different meaning was used: "E is a function of x" meant that x is a variable, and substituting some value for x in the expression E would give us different values, denoted $E(x)$, which therefore are "functions" of x. Of course, the historical meaning is closely related to the modern meaning: if $f\colon A\to B$ is a map of sets, then $f(x)$ is a function of $x\in A$. Vice versa, if E is a function of $x\in A$ and we are given a set B such that $E(x)\in B$ for all $x\in A$, then the set of pairs $\{(x,E(x))\mid x\in A\}$ defines a functional relation from A to B, i.e., a map of sets $A\to B$. The passage from functions in the old sense to maps is ambiguous: B has to be given separately. Sometimes, A is also omitted and must be guessed from the context. Occasionally, even x is suppressed, which may be quite confusing: is x^2+y a function of x, of y, or both x and y? Even more confusing is the situation when the old and new meanings are freely mixed together and both of them referred to as "function". This is the case for many high school and lower-division undegraduate mathematics textbooks, which is an endless source of frustration for students.

We denote $2^A=\{S\subset A\}$, the set of all subsets of A. Given a map of sets $f\colon A\to B$, we have two induced maps: the *pushforward* $f_*\colon 2^A\to 2^B$ and *pullback* $f^{-1}=f^*\colon 2^B\to 2^A$. Given $A'\in 2^A$ (i.e., $A'\subset A$), we set $f_*(A')=\{b\in B\mid \exists a\in A'\colon f(a)=b\}$. Given $B'\in 2^B$ (i.e., $B'\subset B$), we set $f^{-1}(B')=\{a\in A\mid f(a)\in B'\}$.

A map of sets $f\colon A\to B$ is *injective* if $f(a)=f(a')$ implies $a=a'$, *surjective* if for any $b\in B$ there is $a\in A$ such that $f(a)=b$, and *bijective* if it is injective and surjective.

An *inclusion of sets* is an (automatically injective) map $f\colon A\to B$ such that $f(a)=a$ for all $a\in A$.

A *quotient map of sets* is an (automatically surjective) map $f\colon A\to B$ such that $\emptyset\notin B$ and for any $b\in B$ we have $b=f^{-1}(b)$. *quotient set* Such maps can be identified with equivalence relations on A: a quotient map f yields an equivalence relation R on A such that $aRa' \iff f(a)=f(a')$. Vice versa, an equivalence relation R on A gives rise to a quotient map $f\colon A\to B$, where $B=\{P\subset A\mid \exists a\in A\colon P=\{a'\in A\mid aRa'\}\}$ and $f(a)=\{a'\in A\mid aRa'\}$. Equivalently, such maps can be identified with partitions into disjoint nonempty subsets of A: a quotient map f yields a partition whose elements are $f^{-1}(b)$ for all $b\in B$.

Many maps of sets that seem to be inclusions of sets are in fact merely injective. For instance, one could say that any integer number is also a rational number. Naively, such a claim could be formalized as $\mathbf{Z}\subset\mathbf{Q}$. However, this is false for the most common construction of $\mathbf{Q}$ as a quotient set of $\mathbf{Z}\times(\mathbf{Z}\setminus\{0\})$ with respect to the equivalence relation $(p,q)\sim(p',q') \iff pq'=p'q$. (Instead of (p,q) one could write $\frac{p}{q}$, in which case the above relation reads

$$\frac{p}{q}\sim\frac{p'}{q'} \iff pq'=p'q,$$

a fundamental property of fractions that is taken as a definition here.) However, we do have a canonical injective map $\mathbf{Z}\to\mathbf{Q}$, which sends $n\in\mathbf{Z}$ to the set $\{(nk,k)=\frac{nk}{k}\mid k\in\mathbf{Z}\setminus\{0\}\}$. Thus, although $\mathbf{Z}\not\subset\mathbf{Q}$, we can still pretend that $\mathbf{Z}\subset\mathbf{Q}$ by implicitly applying the injective map $\mathbf{Z}\to\mathbf{Q}$ whenever necessary.

Similar reasoning applies to quotient maps of sets. For instance, the map $\exp\colon i\mathbf{R} \to \mathrm{U}(1) = \{z \in \mathbf{C} \mid |z| = 1\}$ is not a quotient map of sets (or groups) because elements of U(1) are not subsets of $i\mathbf{R}$, but the difference is superficial: the group quotient map $i\mathbf{R} \to i\mathbf{R}/2\pi i\mathbf{Z}$ (i.e., $i\mathbf{R}/{\sim}$, where $x \sim y \iff x-y = 2\pi ik$ for some $k \in \mathbf{Z}$) is a quotient map of sets, and there is a canonical isomorphism $i\mathbf{R}/2\pi i\mathbf{Z} \to \mathrm{U}(1)$, so we can pretend that U(1) is a quotient of $i\mathbf{R}$.

The *disjoint union* of sets A and B is defined as $A \sqcup B = A \times \{\emptyset\} \cup B \times \{\{\emptyset\}\}$. We have canonical injection maps $\iota_A\colon A \to A \sqcup B$ $(a \mapsto (a, \emptyset))$ and $\iota_B\colon B \to A \sqcup B$ $(b \mapsto (b, \{\emptyset\}))$. The sets $\{\emptyset\}$ and $\{\{\emptyset\}\}$ could be replaced by any pair of distinct singleton sets. The point of this construction is that A and B are replaced by isomorphic copies of themselves that happen to be disjoint (hence the name "disjoint union"). In particular, we have a canonical map of sets $A \sqcup B \to A \cup B$ $((a, \emptyset) \mapsto a,\ (b, \{\emptyset\}) \mapsto b)$, which is an isomorphism if and only if $A \cap B = \emptyset$.

If I is a set, then an I-indexed *family of sets* is a map of sets $f\colon T \to I$. The underlying idea of this definition is that we assign to $i \in I$ the set $f^{-1}(\{i\})$. An equivalent definition: an I-indexed family of sets is a surjective map of sets $g\colon I \to W$. The underlying idea of this definition is that we assign to $i \in I$ the set $g(i)$. In order to construct g from f, we define $W = \{S \subset T \mid \exists i \in I\colon S = f^{-1}(\{i\})$ and set $g(i) = f^{-1}(\{i\})$. In order to construct f from g, we use the family version of the disjoint union construction discussed above. Set $T = \bigcup_{i\in I} g(i) \times \{i\}$ and for any $t \in T$ set $f(t) = i$, where $t = (x, i)$ for some $i \in I$ and $x \in g(i)$.

36 Appendix: abelian groups

Informally, abelian groups are "vector spaces over integers".

Definition 36.1. An *abelian group* is a tuple $(S, +, -, 0)$, where S is a set, $+\colon S \times S \to S$, $-\colon S \to S$, and $0 \in S$ are such that the following properties are satisfied for all $a, b, c \in S$: $(a + b) + c = a + (b + c)$ (associativity), $a+0 = 0+a = a$ (unitality), $-a+a = 0$ (existence of inverses), $a+b = b+a$ (commutativity). Used in 36.5.

Example 36.2. The (additive) abelian group of *integers* is $(\mathbf{Z}, +, -, 0)$, where $\mathbf{Z}$ denotes the set of integer numbers and the three operations are the familiar operations on integers. We denote this group simply by $\mathbf{Z}$.

Example 36.3. Analogously to the previous example, we have the (additive) abelian groups of *rationals* $\mathbf{Q}$, *reals* $\mathbf{R}$, and *complex numbers* $\mathbf{C}$.

Example 36.4. We can also define the *multiplicative groups* for the above sets of numbers. Their elements are *invertible numbers*, i.e., x is *invertible* if there is y such that $x \cdot y = 1$. For $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$, the sets of invertible elements are $\{-1, 1\}$, $\mathbf{Q} \setminus \{0\}$, $\mathbf{R} \setminus \{0\}$, and $\mathbf{C} \setminus \{0\}$. In all four cases, the three operations are given respectively by multiplication, reciprocal, and the element 1. The multiplicative groups are typically denoted using a superscript $\times$: $\mathbf{Z}^\times$, $\mathbf{Q}^\times$, $\mathbf{R}^\times$, $\mathbf{C}^\times$. Used in 36.4.

Definition 36.5. Suppose $A = (S, +, -, 0)$ and $A' = (S', +', -', 0')$ are abelian groups. A *homomorphism of abelian groups* from A to A' is a map of sets $f\colon S \to S'$ such that the following properties are satisfied for all $a, b \in S$: $f(a + b) = f(a) +' f(b)$ (additivity), $f(-a) = -'f(a)$ (preservation of inverses), $f(0) = 0'$ (preservation of zeros). Used in 15.5, 36.6, 36.7, 36.8, 36.9, 36.9, 36.9, 36.9, 36.9, 36.9, 36.9, 36.9, 36.10, 36.10, 36.10.

Example 36.6. The following maps are homomorphisms of abelian groups.

- $\mathbf{Z} \to \mathbf{C}$, $n \mapsto an$ for some fixed $a \in \mathbf{C}$.
- $\mathbf{Z} \to \mathbf{C}^\times$, $n \mapsto a^n$ for some fixed $a \in \mathbf{C}^\times$.
- $\mathbf{C} \to \mathbf{C}^\times$, $z \mapsto \exp(az)$ for some fixed $a \in \mathbf{C}$.
- $\mathbf{C}^\times \to \mathbf{C}^\times$, $z \mapsto z^n$ for some fixed $n \in \mathbf{Z}$.
- $\mathbf{C}^\times \to \mathbf{C}$, $z \mapsto a \log |z|$ for some fixed $a \in \mathbf{C}$.

Definition 36.7. Suppose $f\colon A \to A'$ is a homomorphism of abelian groups. If f is an inclusion of sets, we say that A is a *subgroup* of A'. If f is a quotient map of sets, we say that A' is a *quotient group* of A. If f is an bijection, we say that A is isomorphic to A'. Used in 13.3*, 15.3, 36.8, 36.8, 36.8.

Definition 36.8. Suppose $f\colon A \to A'$ is a homomorphism of abelian groups.

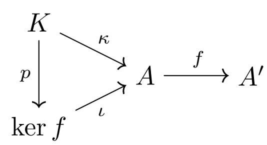
- The *kernel* of f is the subgroup $\ker f$ of A with the underlying set $\{a \in A \mid f(a) = 0\}$ and all operations induced from A.

- The *image* of f is the subgroup $\operatorname{im} f$ of A' with the underlying set $\{a' \in A' \mid \exists a \in A\colon f(a) = a'\}$.
- The *cokernel* of f is the quotient group $\operatorname{coker} f$ of A' whose underlying set is the quotient of A' by the equivalence relation $x \sim y \iff x - y \in \operatorname{im} f$ and all operations induced from A'.

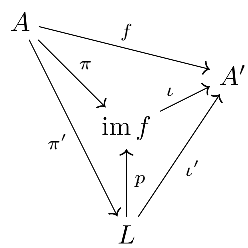
Used in 36.10, 36.10, 36.10, 36.10.

Proposition 36.9. The three groups defined above for $f\colon A \to A'$ can be equivalently characterized by the following universal properties.

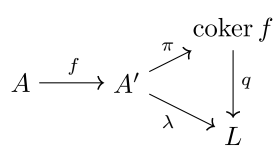
- The homomorphism $\iota\colon \ker f \to A$ satisfies $f\iota = 0$. Furthermore, if $\kappa\colon K \to A$ is another homomorphism such that $f\kappa = 0$, then there is a unique homomorphism $p\colon K \to \ker f$ such that $\iota p = \kappa$.



- The homomorphisms $\pi\colon A \to \operatorname{im} f$ and $\iota\colon \operatorname{im} f \to A'$ satisfy $\iota\pi = f$. Furthermore, if $\pi'\colon A \to L$ and $\iota'\colon L \to A'$ is another pair of homomorphisms such that $\iota'\pi' = f$, then there is a unique $p\colon L \to \operatorname{im} f$ such that $p\pi' = \pi$, hence also $\iota' = \iota p$.



- The homomorphism $\pi\colon A' \to \operatorname{coker} f$ satisfies $\pi f = 0$. Furthermore, if $\lambda\colon A' \to L$ is another homomorphism such that $\lambda f = 0$, then there is a unique homomorphism $q\colon \operatorname{coker} f \to L$ such that $q\pi = \lambda$.



Used in 15.6*.

Remark 36.10. Consider a homomorphism of abelian groups $f\colon A \to B$ with the associated homomorphisms

$$\ker f \to A \to \operatorname{im} f \to A' \to \operatorname{coker} f.$$

- The cokernel of the homomorphism $\ker f \to A$ is isomorphic to $\operatorname{im} f$.
- The kernel of the homomorphism $A' \to \operatorname{coker} f$ is isomorphic to $\operatorname{im} f$.
- The kernel of $A \to \operatorname{im} f$ is isomorphic to $\ker f$.
- The cokernel of $\operatorname{im} f \to A'$ is isomorphic to $\operatorname{coker} f$.

Definition 36.11. Suppose $\{A_i\}_{i\in I}$ is a family of abelian groups indexed by a (possibly infinite) set I. The *direct sum* of A is the abelian group $\bigoplus_i A_i = (S, +, -, 0)$, where $S \subset \prod_{i\in I} A_i$ is the set of all elements $f \in \prod_{i\in I} A_i$ such that $\{i \in I \mid f(i) \neq 0\}$ is a finite set. The operations are defined indexwise. The *direct product* $\prod_i A_i$ is defined in the same way, but with the finiteness condition dropped.

37 References

[SSCSH] Samuel Eilenberg, Joseph A. Zilber. Semi-simplicial complexes and singular homology. Annals of Mathematics 51:3 (1950), 499–513. 6.4, 8.4.

[KanCSS] Daniel M. Kan. On c.s.s. complexes. American Journal of Mathematics 79:3 (1957), 449–476. 6.4.

[Approx] Paul Alexandroff. Simpliziale Approximationen in der allgemeinen Topologie. Mathematische Annalen 96:1 (1927), 489–511. http://dx.doi.org/10.1007/bf01209183 9.4*.

[ZH] Leopold Vietoris. Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen. Mathematische Annalen 97:1 (1927), 454–472. 9.4*.

[SHTe] Samuel Eilenberg. Singular homology theory. Annals of Mathematics (Second Series) 45:3 (1944), 407–447. MR0010970, Zbl:0061.40603, doi:10.2307/1969185. 3.8.

[HGR] Clifford H. Dowker. Homology groups of relations. Annals of Mathematics 56:1 (1952), 84–95. 9.7*.

[WFS] André Joyal. Weak factorisation systems. Joyal's CatLab. https://ncatlab.org/joyalscatlab/published/Weak+factorisation+systems. 26.0*.

[NSHT] André Joyal, Myles Tierney. Notes on simplicial homotopy theory. March 7, 2008. http://mat.uab.cat/∼kock/crm/hocat/advanced-course/Quadern47.pdf. 2.0*, 18.0*.

[SHTgj] Paul G. Goerss, John F. Jardine. Simplicial homotopy theory. Progress in Mathematics 174 (1999). 2.0*.

[CFHT] Peter Gabriel, Michel Zisman. Calculus of fractions and homotopy theory. Ergebnisse der Mathematik und ihrer Grenzgebiete 35 (1967). 2.0*.

[SOAT] J. Peter May. Simplicial objects in algebraic topology. Van Nostrand Mathematical Studies 11 (1967). 2.0*.

[SHTc] Edward B. Curtis. Simplicial homotopy theory. Advances in Mathematics 6 (1971), 107–209. 2.0*.

[SAT] Klaus Lamotke. Semisimpliziale algebraische Topologie. Grundlehren der mathematischen Wissenschaften 147 (1968). 2.0*.