Mathematics 5324 (Topology)

Final exam

If you are unable to do the case of an arbitrary abelian group A of coefficients, you may assume that $A = \mathbf{Z}$, with a reduction of points awarded.

1. Consider the 2-simplex σ with all three vertices identified, i.e., $\mathrm{d}_1\mathrm{d}_2(\sigma) = \mathrm{d}_0\mathrm{d}_2(\sigma) = \mathrm{d}_0\mathrm{d}_1(\sigma)$.

- Draw a picture of the resulting simplicial set X.
- Compute the homology and cohomology of X with coefficients in an arbitrary abelian group A.

2. Consider the 2-sphere with its north and south pole identified.

- Write down a simplicial set X that models such a construction.
- Compute the homology and cohomology of X with coefficients in an arbitrary abelian group A.

3. The *orange slice* with n carpel slices ($n \geq 1$) is defined as follows. There are n generating 2-simplices, denoted t_i ($i \in \mathbf{Z}/n\mathbf{Z}$). The relations are as follows: $\mathrm{d}_1(t_i) = \mathrm{d}_2(t_{i+1})$ ($i \in \mathbf{Z}/n\mathbf{Z}$).

- Draw a picture of the orange slice for $n = 3$.

We now add the following relations: $\mathrm{d}_0(t_i) = \mathrm{d}_0(t_{i+1})$ ($i \in \mathbf{Z}/n\mathbf{Z}$).

- Draw a picture of the resulting simplicial set X_n for $n = 3$. (Identifications that are difficult to visualize may be denoted using letters, like for the real projective plane.)
- Compute the homology and cohomology with coefficients in an arbitrary abelian group A of the resulting simplicial set X_n for all $n \geq 1$. (If you are unable to do the general case, you may assume $n = 3$, with a reduction of points.)

4. The (solid) *orange* with n carpels ($n \geq 1$) is defined as follows. There are n generating 3-simplices, denoted c_i ($i \in \mathbf{Z}/n\mathbf{Z}$). The relations are as follows: $\mathrm{d}_2(c_i) = \mathrm{d}_3(c_{i+1})$.

- Draw a picture of an orange with 3 carpels.

We now add the following relations: $\mathrm{d}_1(c_i) = \mathrm{d}_0(c_{i+s})$, where $s \in \mathbf{Z}/n\mathbf{Z}$ is a generator (i.e., the elements s, $s+s$, $s+s+s$, etc., exhaust the entire group $\mathbf{Z}/n\mathbf{Z}$).

- Draw a picture of the resulting simplicial set X_n for $n = 3$. (Identifications that are difficult to visualize may be denoted using letters, like for the real projective plane.)
- Compute the homology and cohomology with coefficients in an arbitrary abelian group A of the resulting simplicial set X_n for all $n \geq 1$ and an arbitrary generator $s \in \mathbf{Z}/n\mathbf{Z}$. (If you are unable to do the general case, you may assume $n = 3$ and $s = 1$, with a reduction of points.)

5. Consider $X = \mathrm{S}^1\mathrm{S}^2$ and the simplicial subset $Y \subset X$ given by $Y = (\mathrm{S}^1*) \cup (*\mathrm{S}^2)$, where $*$ denotes the 0-simplices of S^1 and S^2.

- Draw a picture of Y, X, and the inclusion $Y \subset X$.

Consider the coequalizer Q of $f, g\colon Y \rightrightarrows X$, where f is the inclusion $Y \subset X$ and g is the composition $Y \to \Delta^{\mathbf{0}} \to \mathrm{S}^1\mathrm{S}^2$, where the second map is unique because its target has only one 0-simplex.

- Compute the homology of the map $X \to Q$ with coefficients in A.

6. Suppose that $f\colon X \to Y$ is a simplicial map and $n \geq 0$ is such that $\mathsf{H}_n(X) \cong \mathbf{Z}$ and $\mathsf{H}_n(Y) \cong 0$. We are interested in the existence of a simplicial map $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$. Determine which of the following possibilities is true and prove your claim.

- The map g always exists.
- The map g never exists.
- For some choices of f the map g exists, and for some it does not.

7. Prove or disprove: if A and B are simplicial sets with finitely many nondegenerate simplices (so that $\chi(A)$ and $\chi(B)$ are defined), then $\chi(AB) = \chi(A)\chi(B)$.

8. Compute the group homology $\mathsf{H}_1(\mathbf{Z}/3\mathbf{Z})$.