Noncommutative L^p-spaces of W*-categories and their applications to L^p-modules

Dmitri Pavlov

Mathematical Institute, University of Münster

https://dmitripavlov.org/

1 Introduction.

The main result of this paper can be formulated as follows.

Theorem 1.1. For any von Neumann algebra M and for any d and e in $\mathbf{R}_{\geq 0}$ the category of right $\mathrm{L}_d(M)$-modules is equivalent to the category of right $\mathrm{L}_{d+e}(M)$-modules. The equivalences are implemented by the algebraic tensor product and the algebraic internal hom with $\mathrm{L}_e(M)$. In particular, all categories of right $\mathrm{L}_d(M)$-modules are equivalent to each other and to the category of representations of M on Hilbert spaces.

Here an $\mathrm{L}_d(M)$-module is an algebraic M-module equipped with an inner product valued in $\mathrm{L}_{2d}(M)$ satisfying a natural set of properties, as defined by Junge and Sherman [JuSh]. For any fixed $d \in \mathbf{R}_{\geq 0}$ the category of $\mathrm{L}_d(M)$-modules is a W*-category, and as we explain below, $\mathrm{L}_d(M)$-modules are tightly related to the noncommutative L_d-spaces of certain W*-categories (the linking categories of $\mathrm{L}_d(M)$-modules).

We pause briefly to explain the smooth commutative analog of this result, which is a good source of intuition for the noncommutative case.

Consider a smooth bundle V of complex Hilbert spaces over a smooth manifold X. The inner product on V can be expressed as a morphism of bundles $\bar{V} \otimes V \to \mathrm{Dens}_0(X)$, where $\bar{V}$ denotes the conjugate bundle of V. We consider vector bundles equipped with a more general type of inner product with values in $\mathrm{Dens}_{2d}(X)$ for some $d \in \mathbf{R}$. Here it is essential that d is real because we need $\mathrm{Dens}_{2d}^+(X)$ for the positivity property. Such an inner product equips every fiber V_x of V with an inner product with values in the one-dimensional vector space $\mathrm{Dens}_{2d}(\mathrm{T}_x X)$ and we require that all fibers are complete with respect to this inner product. In particular, all fibers are Hilbertable, i.e., their topology is induced by some $\mathbf{C}$-valued inner product, but unless $d = 0$ there is no canonical way to choose such an inner product. We refer to such bundles as *d-bundles*. In particular, for $d = 0$ the space $\mathrm{Dens}_{2d}(U)$ is canonically isomorphic to $\mathbf{C}$ and we get the usual smooth bundles of complex Hilbert spaces.

Examples of d-bundles abound in differential geometry. For example, the Dirac operator on a conformal spin n-manifold is a differential operator from a $(1 - 1/n)/2$-bundle to a $(1 + 1/n)/2$-bundle as explained by Stephan Stolz and Peter Teichner in [StTe, Definition 2.3.9 in the article version, Definition 2.22 in the book version].

The easiest example of a d-bundle is supplied by the bundle $\mathrm{Dens}_a(X)$, where $\Re a = d$. The inner product is given by the composition $\overline{\mathrm{Dens}_a(X)} \otimes \mathrm{Dens}_a(X) \to \mathrm{Dens}_{\bar{a}}(X) \otimes \mathrm{Dens}_a(X) \to \mathrm{Dens}_{a+\bar{a}}(X) = \mathrm{Dens}_{2d}(X)$. Other examples can be obtained by tensoring a bundle of Hilbert spaces with $\mathrm{Dens}_d(X)$. In fact, these examples exhaust all possible d-bundles, for if V is a d-bundle, then $V \otimes \mathrm{Dens}_{-d}(X) = \mathrm{Hom}(\mathrm{Dens}_d(X), V)$ is a 0-bundle.

More generally, if V is a d-bundle, then $V \otimes \mathrm{Dens}_e(X)$ is a $(d + e)$-bundle for any $e \in \mathbf{R}$. This correspondence extends to a functor, which establishes an equivalence between the categories of d-bundles and $(d+e)$-bundles. In particular, all categories of d-bundles for various values of d are canonically equivalent to each other.

Just as compactly supported smooth sections of the bundle of d-densities of X can be completed to the space $\mathrm{L}_d(X)$, compactly supported smooth sections of an arbitrary d-bundle on X can be completed to an $\mathrm{L}_d(X)$-*module*, provided that $d \geq 0$.

More precisely, if M is a von Neumann algebra and $d \in \mathbf{R}_{\geq 0}$, then a *right $\mathrm{L}_d(M)$-module* is an algebraic right M-module V equipped with an inner product with values in $\mathrm{L}_{2d}(M)$ satisfying the usual algebraic properties (bilinearity, positivity, nondegeneracy) that is complete in the *measurable topology*, which is the weakest topology on V such that the maps $y \in V \mapsto (x, y) \in \mathrm{L}_{2d}(M)$ are continuous for all $x \in V$, where $\mathrm{L}_{2d}(M)$ is equipped with the measurable topology. A *morphism* of $\mathrm{L}_d(M)$-modules is a continuous morphism of algebraic M-modules. P. Ghez, Ricardo Lima, and John Roberts in [GhLiRo, Example 1.4] equip the category of representations of M on Hilbert spaces with a structure of a W*-category. As we will see later,

this category is equivalent to the category of $\mathrm{L}_d(M)$-modules for any d as a *-category, thus $\mathrm{L}_d(M)$-modules also form a W*-category.

Suppose X is a smooth manifold. Then $\mathrm{L}_0(X)$ (the algebra of bounded measurable functions on the underlying measurable space of X) is a von Neumann algebra and by an $\mathrm{L}_d(X)$-module we mean an $\mathrm{L}_d(\mathrm{L}_0(X))$-module. The measurable topology on the space of compactly supported smooth sections of a d-bundle V is the weakest topology such that the maps $y \in \mathrm{C}^\infty_{\mathrm{cs}}(V) \mapsto (x, y) \in \mathrm{C}^\infty_{\mathrm{cs}}(\mathrm{Dens}_{2d}(X))$ are continuous for all $x \in \mathrm{C}^\infty_{\mathrm{cs}}(\bar{V})$, where $\mathrm{C}^\infty_{\mathrm{cs}}(\mathrm{Dens}_{2d}(X))$ is equipped with the measurable topology. Completing this space gives us an $\mathrm{L}_d(X)$-module with the measurable topology. Every L_d-module over any commutative von Neumann algebra can be obtained in this way.

Combining together equivalences of categories of d-bundles and facts about algebraic tensor products and internal homs of spaces $\mathrm{L}_a(M)$ we arrive at the following statement, which is the second main result of this paper: (1) If V is a right $\mathrm{L}_d(M)$-module, then $V \otimes_M \mathrm{L}_e(M)$ can be equipped in a natural way with a structure of a right $\mathrm{L}_{d+e}(M)$-module, in particular it is automatically complete; (2) If V is a right $\mathrm{L}_{d+e}(M)$-module, then $\mathrm{Hom}_M(\mathrm{L}_e(M), V)$ is naturally a right $\mathrm{L}_d(M)$-module, in particular it is automatically complete; (3) The above constructions can be extended to an adjoint unitary W*-equivalence of W*-categories. Here d and e are arbitrary elements of $\mathbf{R}_{\geq 0}$. The above results can be summarized as follows.

Theorem 1.2. For any von Neumann algebra M and for any d and e in $\mathbf{R}_{\geq 0}$ the category of right $\mathrm{L}_d(M)$-modules is equivalent to the category of right $\mathrm{L}_{d+e}(M)$-modules. The equivalences are implemented by the algebraic tensor product and the algebraic internal hom with $\mathrm{L}_e(M)$. In particular, all categories of right $\mathrm{L}_d(M)$-modules are equivalent to each other and to the category of representations of M on Hilbert spaces.

This theorem can also be extended to bimodules. An M-$\mathrm{L}_d(N)$-bimodule is a right $\mathrm{L}_d(N)$-module V equipped with a morphism of von Neumann algebras $M \to \mathrm{End}(V)$. Here $\mathrm{End}(V)$ denotes the von Neumann algebra of all continuous N-linear endomorphisms of V. An $\mathrm{L}_d(M)$-N-bimodule is defined similarly.

Theorem 1.3. For any von Neumann algebras M and N the category of $\mathrm{L}_d(M)$-N-bimodules, the category of M-$\mathrm{L}_d(N)$-bimodules, and the category of birepresentations of M and N (i.e., commuting representations of M^{op} and N) on Hilbert spaces are all equivalent to each other. The equivalences for different values of d are implemented by the algebraic tensor product and the algebraic internal hom with the relevant space $\mathrm{L}_e(M)$. The equivalence between $\mathrm{L}_d(M)$-N-bimodules and M-$\mathrm{L}_d(N)$-bimodules is implemented by passing from an $\mathrm{L}_d(M)$-N-bimodule to an $\mathrm{L}_{1/2}(M)$-N-bimodule, then reinterpreting the latter module as an M-N-birepresentation,

2 W*-categories and their linking algebras.

Our reference for W*-categories is the paper by Ghez, Lima, and Roberts [GhLiRo].

Intuitively, *-categories, Banach *-categories, C*-categories, and W*-categories should be thought of as many-objects versions (horizontal categorifications) of *-algebras, Banach *-algebras, C*-algebras, and von Neumann algebras. In particular, in the case when the set of objects has one element these notions coincide with the corresponding notions of algebras, provided that we either ignore 2-morphisms on the categorical side or add intertwining elements as 2-morphisms on the algebraic side. A *-category* is a category enriched over the symmetric monoidal category of complex vector spaces with the algebraic tensor product, equipped with a contravariant complex-antilinear involution on morphisms: $*\colon \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X)$ for any pair of objects X and Y. The involution has to satisfy the usual identities: $\mathrm{id}_X^* = \mathrm{id}_X$ for any object X and $(fg)^* = g^*f^*$ for any composable morphisms f and g. A *Banach *-category* is a *-category whose spaces of morphisms $\mathrm{Hom}(X,Y)$ are equipped with Banach norms such that $\|\mathrm{id}_X\| = 1$ for any object X, $\|fg\| \le \|f\| \cdot \|g\|$ for any composable morphisms f and g, and $\|f^*\| = \|f\|$ for any morphism f. Alternatively, one can define a Banach *-category as a category enriched over the symmetric monoidal category of complex Banach spaces and *contractive* linear maps (maps of norm at most 1) with the projective tensor product and equipped with a compatible involution. A *C*-category* is a Banach *-category that for any morphism f satisfies the C*-identity $\|f^*f\| = \|f\|^2$ and the positivity condition $f^*f \ge 0$ (as an element of the C*-algebra $\mathrm{End}(X)$, which is a C*-algebra because of the C*-identity). The positivity condition does not follow from the other properties. Just as for C*-algebras, norms of morphisms in a C*-category can be recovered uniquely from the categorical structure. Finally, a *W*-category* is a C*-category such that the Banach space of morphisms between any pair of objects has a predual, which is necessarily unique.

Functors between *-categories are enriched functors that commute with the involution. For a functor between Banach *-categories we require that all maps of spaces of morphisms are contractive. In the framework of enrichment over Banach spaces these are simply enriched functors that commute with the involution. For C*-categories the contractivity condition is satisfied automatically. Finally, for W*-categories we require functors to be continuous in the ultraweak topology (the weak topology induced by the unique predual) on morphisms. Equivalently, we can require maps of spaces of morphisms to possess a predual.

No additional conditions have to be satisfied for natural transformations between *-categories. For Banach *-categories, C*-categories, and W*-categories we require that the norms of component morphisms of a natural transformation are uniformly bounded. This condition follows naturally once we consider the enriched functor category for the enrichment defined above.

The above definitions can also be motivated by the fact that functors between small categories of one of the above types again form a small category of the same type. For example, functors between small W*-categories form a small W*-category. More generally, we can substitute the appropriate version of (local) presentability or accessibility for smallness. For example, we will see below that the W*-category of modules over a von Neumann algebra is W*-presentable, hence functors between such categories form a W*-category, which in fact is equivalent to the W*-category of bimodules (correspondences) between these algebras.

Thus we obtained four bicategories, whose objects are *-categories, Banach *-categories, C*-categories, and W*-categories respectively. The words 'functor' and 'natural transformation' are always used in the sense defined above when applied to a category of the type discussed above.

We now explain how the concept of linking algebras allows us to reduce the theory of small W*-categories to the theory of von Neumann algebras. This reduction can be seen as a complement of the obvious reduction from von Neumann algebras to small W*-categories (every von Neumann algebra can be thought of as a W*-category with one object). From now on we concentrate on W*-categories, although the theory of linking algebras can take place in any of the four settings mentioned above.

A *linking algebra* is a pair (A, p), where A is a von Neumann algebra and $p\colon I \to A$ is a family of projections in A indexed by a set (not a proper class) I such that $\sum_{i \in I} p_i = 1 \in A$ and $p_i p_j = 0$ for all $i \in I$ and $j \in I$ such that $i \ne j$.

Consider a small W*-category C. The *linking algebra of* C is the pair (A, p) defined as follows. The indexing set I of p is the set of objects of C. The von Neumann algebra A consists of square matrices with indexing set I. The entry with index (X, Y) has to belong to $\mathrm{Hom}(Y, X)$ (recall that X and Y are objects

of C). For example, in the case of two objects we have matrices of the form

$$\begin{pmatrix} \operatorname{End}(X) & \operatorname{Hom}(Y,X) \\ \operatorname{Hom}(X,Y) & \operatorname{End}(Y) \end{pmatrix}.$$

If I is infinite, we require the matrix to be bounded in the sense that norms of finite square submatrices have to be uniformly bounded. Alternatively, one can require that the matrix acts as a bounded operator on the Hilbert space constructed from Haagerup's standard forms of $\operatorname{Hom}(Y,X)$. The identity element is given by the diagonal matrix with identity morphisms on the diagonal. Multiplication obeys the matrix rules, with products of individual elements coming from the composition in C. The involution transposes the matrix and applies the involution in C to individual elements. For an object X in C we define the projection $p_X \in A$ to be the matrix whose entries are all zero except for the diagonal entry indexed by X, which is id_X. We have $\sum_X p_X = 1$ and $p_X p_Y = 0$ for all $X \neq Y$, hence (A, p) is indeed a linking algebra.

The crucial property of W*-categories is that the algebra A constructed above is indeed a von Neumann algebra. In fact, the opposite is also true, the structure of a small W*-category can be recovered from its linking algebra. Given a linking algebra (A, p) we construct from it a small W*-category C as follows. The set of objects of C is the indexing set of p. Individual spaces of morphisms are the corresponding corners of A: $\operatorname{Hom}(X, Y) := p_Y A p_X$. We have $\operatorname{id}_X = p_X$, composition is induced by the multiplication on A, involution is induced by the involution on A, norms on spaces of morphisms are induced from the norm on A, the predual of $p_Y A p_X$ is $p_X A_* p_Y$. The properties of a W*-category are satisfied because A is a von Neumann algebra.

The above correspondence can be extended to functors between W*-categories whose induced map on objects is injective. In a certain sense we do not lose anything by restricting ourselves to injective functors, because we can always "pump" the set of objects in such a way that we have enough freedom to replace every functor by an isomorphic injective functor.

More precisely, consider a small full subbicategory U of the bicategory of small W*-categories. For example, in the framework of Grothendieck universes one can take all bicategories whose sets of objects are elements of the given Grothendieck universe. We will construct another small full subbicategory V of the bicategory of small W*-categories with the following properties. Denote by W the subbicategory of V consisting of all objects, injective functors, and all natural transformations. We will construct explicit biequivalences $I \colon U \to W$, $W \to V$, $V \to U$ that induce isomorphisms on the sets of objects. The biequivalence $W \to V$ is simply the inclusion of W into V. In other words, we can use the functor I to "pump" the sets of objects of W*-categories in U to get a biequivalent bicategory V whose constituent W*-categories have big enough sets of objects for injective functors to represent all functors.

The full subbicategory V is constructed as follows. For every W*-category C in U we explain how to construct the corresponding W*-category $D = I(C)$ in V. Denote by Q the cartesian product

$$\prod_{C \in \operatorname{Ob}(U)} \operatorname{FM}(\operatorname{Ob}(C)).$$

Here $\operatorname{FM}(X)$ denotes the free monoid on a set X. In other words, an element of Q contains a finite tuple of objects of C for each W*-category C in U. The set of objects of D is defined to be $\operatorname{Ob}(C) \times Q$. The structure of a W*-category of D is obtained by pulling back the structure of a W*-category of C along the projection $\operatorname{Ob}(C) \times Q \to \operatorname{Ob}(C)$. The second component Q is only needed to ensure injectivity of functors later. We have a canonical equivalence $D \to C$ induced by the above projection on objects. Furthermore, we have a canonical equivalence $C \to D$ induced by the map on objects that simply adds an element of Q consisting of empty tuples.

The functor $I \colon U \to W$ has already been defined on objects. Given a functor $F \colon C \to D$ in U, the object map $\operatorname{Ob}(I(F)) \colon \operatorname{Ob}(C) \times Q \to \operatorname{Ob}(D) \times Q$ of the functor $I(F)$ is defined as follows. The first component $\operatorname{Ob}(C) \times Q \to \operatorname{Ob}(D)$ is the composition $\operatorname{Ob}(C) \times Q \to \operatorname{Ob}(C) \to \operatorname{Ob}(D)$, where the first map is a projection. The second component $\operatorname{Ob}(C) \times Q \to Q$ is the canonical injection given by inserting an element of $\operatorname{Ob}(C)$ at the beginning of the tuple of elements of $\operatorname{Ob}(C)$. The second component guarantees the injectivity of the resulting map $\operatorname{Ob}(C) \times Q \to \operatorname{Ob}(D) \times Q$. Finally, a natural transformation between two functors F and G from C to D immediately induces a natural transformation between $I(F)$ and $I(G)$.

As we already mentioned above, the functor $W \to V$ is simply the inclusion of W into V. Finally, the functor $V \to U$ is defined as follows. On objects it is given by the inverse of the map $\operatorname{Ob}(I)$. Given a

functor G in V, we can represent its domain and target in the form $I(C)$ and $I(D)$ respectively (the functor itself need not be in the image of I). Composing G with the equvialences $C \to I(C)$ and $I(D) \to D$ defined above gives us the corresponding functor from C to D in U. This construction also defines the map on natural transformations.

The cyclic composition $U \to W \to V \to U$ is simply the identity map. The other two cyclic compositions induce the identity maps on objects and are fully faithful on 2-morphisms, but they are only essentially surjective on 1-morphisms. Thus all three cyclic compositions are biequivalences, hence the original three functors are also biequivalences.

Without the smallness condition we run into cardinality problems, and in fact the corresponding bicategories are not biequivalent.

A *morphism* $(A, p) \to (B, q)$ of linking algebras with indexing sets I and J respectively is a pair (f, u), where $f : A \to rBr$ is a morphism of von Neumann algebras, $u : I \to J$ is an injection such that $f(ap_i) = f(a)q_{u(i)}$ and $f(p_i a) = q_{u(i)} f(a)$ for all $a \in A$ and $i \in I$, and $r := \mathrm{cosupp}(f, u) := \sum_{i \in I} q_{u(i)} \in B$ is the "cosupport" of (f, u).

The *linking morphism* of an injective functor between small W*-categories is the morphism between their linking algebras given by the pair (f, u), where u is the underlying injective map on objects and f is induced by applying the functor to individual matrix entries. Vice versa, any morphism of linking algebras of some small W*-categories is the linking morphism of an injective functor between them.

If $(f, u) : (A, p) \to (B, q)$ and $(g, v) : (A, p) \to (B, q)$ are two morphisms of linking algebras, then a *2-morphism* $h : (f, u) \to (g, v)$ is an element $h \in sBr$ such that $hf(a) = g(a)h$ for all $a \in A$, where $r := \mathrm{cosupp}(f, u)$ and $s := \mathrm{cosupp}(g, v)$.

The *linking intertwining element* of a natural transformation between injective functors is given by the matrix whose nonzero elements are the components of the natural transformation. More precisely, given a natural transformation $H : F \to G$ between functors $F, G : C \to D$ with linking morphisms

$$(f, u), (g, v) : (A, p) \to (B, q),$$

the linking intertwining element of H has H_X in the column $u(X)$ and row $v(X)$ for any object X of C and zeros elsewhere. Again, the injectivity of F and G is crucial for this construction. Vice versa, all 2-morphisms between the linking morphisms of some injective functors come from natural transformations.

Given a subbicategory W of the bicategory of small W*-categories such that all 1-morphisms of W are injective functors, the above linking constructions assemble into a functor from W to the bicategory of linking von Neumann algebras.

Altogether, the above constructions assemble into a fully faithful functor from any small subbicategory U of the bicategory of small W*-categories to the bicategory of linking von Neumann algebras, their morphisms, and 2-morphisms. This functor is defined to be the composition of the pumping functor $I : U \to W$ constructed above and the linking functor for W.

3 L_a-spaces of W*-categories.

In this section we extend the theory of modular algebras and L_a-spaces to W*-categories. Applications of this theory include the theory of L_a-modules, which can be reduced to the theory of L_a-spaces of W*-categories using the notion of the linking W*-category of an L_0-module.

The extension is done using the theory of linking algebras developed in the previous section by passing from a W*-category to its linking algebra, applying the relevant construction to the resulting von Neumann algebra, and enforcing compatibility with the given family of projections as appropriate. In particular, we can immediately lift the entire theory of modular algebras to the setting of W*-categories. The resulting theory is functorial with respect to the W*-categorical analog of operator valued weights, which we define below. As usual, operator valued weights come in two flavors: positive unbounded and (not necessarily positive or even self-adjoint) bounded. The former are usually simply referred to as operator valued weights. We start with bounded operator valued weights, because they are easier to deal with.

Definition 3.1. A *bounded operator valued weight* associated to a morphism of linking algebras

$$(f, u) \colon (A, p) \to (B, q)$$

is a bounded operator valued weight T associated to $f \colon A \to rBr$, where $r = \operatorname{cosupp}(f, u)$.

Remark 3.2. The above definition immediately allows us to derive that $T(q_{u(i)} b q_{u(j)}) = p_i T(b) p_j$ for all $b \in rBr$, $i \in I$, $j \in I$. Thus a bounded operator valued weight respects the additional structure of linking algebras.

Definition 3.3. Given a functor $F \colon C \to D$ of small W*-categories we define a *bounded operator valued weight* associated to F as a bounded operator valued weight T associated to the linking morphism

$$(g, h) \colon (A, p) \to (B, q)$$

of F.

Remark 3.4. Here we assume that either F is injective or the pumping machinery of the previous section is applied first.

Definition 3.5. An *operator valued weight* associated to a morphism of linking algebras $(f, u) \colon (A, p) \to (B, q)$ is an operator valued weight T associated to $f \colon A \to rBr$, where $r = \operatorname{cosupp}(f, u)$.

Remark 3.6. Similarly to the bounded case, the above definition implies that $T(q_{u(i)} b q_{u(i)}) = p_i T(b) p_i$ for all $b \in rBr$, $i \in I$ and $T((y q_{u(i)} + z q_{u(j)})^* b (y q_{u(i)} + z q_{u(j)})) = (y p_i + z p_j)^* T(b)(y p_i + z p_j)$ for all $b \in rBr$, $i \in I$, $j \in I$, $y \in \mathbf{C}$, $z \in \mathbf{C}$, which means that operator valued weights also respect the additional structure of linking algebras.

Definition 3.7. Given a functor $F \colon C \to D$ of small W*-categories we define an *operator valued weight* associated to F as an operator valued weight T associated to the linking morphism $(f, u) \colon (A, p) \to (B, q)$ of F.

Remark 3.8. Although an operator valued weight associated to a functor between W*-categories does induce an operator valued weight between the corresponding von Neumann algebras of endomorphisms of objects, in particular, it induces a map between their extended positive cones, the same cannot be said about spaces of morphisms between different objects. This is due to the fact that arbitrary operator valued weights are defined only on extended positive cones, and a nonzero morphism between different objects is never positive (or even self-adjoint) in the linking algebra. Even if we rectify the problem with self-adjointness by considering a pair of morphisms $f \colon X \to Y$ and $g \colon Y \to X$ such that the matrix $\begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix}$ is a self-adjoint element (i.e., $f = g^*$) in the von Neumann linking algebra of the full W*-subcategory spanned by X and Y, this matrix is positive if and only if $f = g = 0$. However, *bounded* operator valued weights do give maps between all spaces of morphisms, as explained above.

Remark 3.9. Although the above definitions are easy to obtain using the linking machinery of the previous section, they become less elegant when the functor F is not injective, in which case we have to run the

pumping machinery to convert F to an injective functor first. This can be done canonically by applying the pumping machinery to the full subbicategory of the bicategory of small W*-categories spanned by the source and target of F. However, one gets additional issues when one starts composing operator valued weights. Furthermore, the above definitions only work for small categories. To resolve these problems we use the insight provided by the above definitions to construct new definitions that work for all functors.

Definition 3.10. A *bounded operator valued weight* associated to a functor $F\colon C \to D$ of W*-categories is given by specifying for every pair of objects X and Y of C an ultraweakly continuous linear map (or a map with a predual) $T_{X,Y}\colon \mathrm{Hom}_D(F(X), F(Y)) \to \mathrm{Hom}_C(X,Y)$ such that for any $f\colon X \to Y$ we have $T_{X,Y}(fF(g)) = T_{X,Y}(f)g$ for all $g\colon W \to X$ and $T_{X,Y}(F(g)f) = gT_{X,Y}(f)$ for all $g\colon Y \to Z$. Furthermore, the individual components $T_{X,Y}$ must assemble into a bounded operator-valued weight on any linking algebra composed of objects in C.

Definition 3.11. An *operator valued weight* associated to a functor $F\colon C \to D$ of W*-categories is given by specifying for every finite family $R\colon I \to \mathrm{Ob}(C)$ of objects in C an operator valued weight T_R associated to the morphism of von Neumann algebras $\mathrm{L}(R^*F)\colon \mathrm{L}(R^*C) \to \mathrm{L}(R^*D)$, where R^*C and R^*D denote the W*-categories with I as the set of objects obtained by pulling back the structures of W*-categories on C and D via the map R, R^*F denotes the functor induced from F by pulling back along R (here R^*F induces the identity map on objects), and L denotes the linking functor. Given an arbitrary injection $\rho\colon J \to I$ we require that T_R restricts to $T_{R\rho}$.

Remark 3.12. The above definition is more complicated than the definition for bounded operator valued weights because arbitrary operator valued weights can only be evaluated on elements in extended positive cones, which necessitates the usage of 2×2 matrices. To ensure compatibility with conjugation we need 3×3 matrices. Bigger matrices are necessary for positivity.

Remark 3.13. The above definition makes sense for nonsmall categories, because for any (proper) class C finite families of elements of C again form a (proper) class. We could have also taken arbitrary set-indexed families.

Remark 3.14. Operator valued weights whose components are bounded are precisely positive bounded operator valued weights. Here a bounded operator valued weight T is *positive* if the map T_R is positive for any finite family $R\colon I \to \mathrm{Ob}(C)$, where T_R is given by the matrix whose entry with index (X,Y) is $T_{X,Y}$.

Remark 3.15. A unitary isomorphism between two objects in a W*-category C identifies the corresponding components of a (bounded or positive unbounded) operator valued weight associated to a functor $F\colon C \to D$. This follows from the multiplicative respectively conjugation property of bounded respectively unbounded operator valued weights.

Definition 3.16. Given two W*-categories C and D, two functors F and G from C to D, and two bounded operator valued weights T and U associated to F and G respectively, a *natural transformation* from (F,T) to (G,U) is a pair (H,V), where H is a natural transformation from F to G and $V\colon \mathrm{Ob}(C)\times\mathrm{Ob}(C) \to \mathrm{Mor}(C)$ is a function that sends a pair of objects (X,Y) in C to the map $V_{X,Y}\colon \mathrm{Hom}_D(FX, GY) \to \mathrm{Hom}_C(X,Y)$ such that

Remark 3.17. W*-categories, functors with associated bounded operator valued weights, and natural transformations form a bicategory. Identity and composition are specified componentwise.

Definition 3.18. A (bounded or positive unbounded) operator valued weight is *faithful* respectively *semifinite* if its individual components are faithful respectively semifinite. Bounded weights are automatically semifinite.

4 Linking categories.

Recall that a (Banach/C/W) *-category with one object is precisely a (Banach/C/W) *-algebra, once we deal with 2-morphisms properly.

5 L_a-modules.

In this section we prove the main theorem of this paper. The case of densities 0 and 1/2 (without automatic completeness or continuity) is covered by Theorem 2.2 in Baillet, Denizeau, and Havet [BaDeHa]; many ideas were already present in Rieffel [Ri].

Remark 5.1. If M is a complex *-algebra and X is an algebraic right M-module, then denote by $X^\sharp$ the algebraic left M-module whose underlying abelian group is that of X and the left multiplication map is the composition $M \otimes X \to M \otimes X \to X \otimes M \to X$, where the first map is the tensor product of the involution on M and the identity map on X, the second map is the braiding map, and the third map is the right multiplication map on M. There is a canonical complex antilinear isomorphism of vector spaces $X \to X^\sharp$, which equals the identity on the underlying abelian groups. Henceforth we denote this isomorphism by $x \mapsto x^\sharp$. The functor $X \mapsto X^\sharp$ is an equivalence of the categories of right and left algebraic M-modules. Whenever we equip M-modules with additional structures or properties we extend the functor $X \mapsto X^\sharp$ to the new category of modules without explicitly mentioning it.

Definition 5.2. (Junge and Sherman, [JuSh].) Suppose $d \in \mathbf{R}_{\geq 0}$ and M is an arbitrary von Neumann algebra. A *right pre-$L_d(M)$-module* is an algebraic right M-module X together with an M-M-bilinear inner product $\mu\colon X^\sharp \otimes X \to L_{2d}(M)$ such that for all u and v in X we have $(u,v)^* = (v,u)$ and for all $w \in X$ we have $(w,w) \geq 0$ and $(w,w) = 0$ implies $w = 0$. Here $(x,y) := \mu(x^\sharp \otimes y)$. *Left* pre-$L_d(M)$-modules are defined similarly.

Remark 5.3. A canonical example of a right $L_d(M)$-module is given by the space $L_a(M)$, where $\Re a = d$ and the inner product is given by $(x,y) := x^*y$. Observe that d has to be real because the notion of positive element in $L_a(M)$ only makes sense for $a \in \mathbf{R}_{\geq 0}$.

Definition 5.4. Suppose $d \in \mathbf{R}_{\geq 0}$, M is a von Neumann algebra, and X and Y are right pre-$L_d(M)$-modules. A morphism f from X to Y is a morphism of the underlying algebraic right M-modules of X and Y that is continuous in the topologies given by the quasinorm $x \in X \mapsto \|(x,x)^{1/2}\| \in \mathbf{R}_{\geq 0}$ on X and likewise for Y. See Proposition 3.2 in Junge and Sherman [JuSh] for a proof that this is a quasinorm.

Definition 5.5. Suppose M is a von Neumann algebra and $d \in \mathbf{R}_{\geq 0}$. A *right $L_d(M)$-module* is a right pre-$L_d(M)$-module X such that every bounded (in the corresponding quasinorms) morphism from X to $L_{2d}(M)$ has the form $y \in X \mapsto (x,y) \in L_{2d}(M)$ for some $x \in X$. The category of right $L_d(M)$-modules is the full subcategory of the category of right pre-$L_d(M)$-modules consisting of $L_d(M)$-modules. We also refer to right $L_0(M)$-modules as *W*-modules over M*.

Proposition 5.6. Suppose M is a von Neumann algebra and $d \in \mathbf{R}_{\geq 0}$. The full subcategory of right $L_d(M)$-modules in the category of right pre-$L_d(M)$-modules is reflective. The reflector (i.e., the left adjoint to the inclusion functor) sends a right pre-$L_d(M)$-module to its completion in the measurable topology (which can also be described algebraically as $\mathrm{CHom}(X, L_{2d}(M))^\sharp$) and a morphism of right pre-$L_d(M)$-module to its unique extension to completed modules, which admits a similar algebraic description. The unit of the adjunction embeds a right pre-$L_d(M)$-module into its completion. This embedding of categories is exact (the reflector preserves finite products) and bireflective (the unit of the adjunction is both a monomorphism and an epimorphism).

Proposition 5.7. Suppose M is a von Neumann algebra and $d \in \mathbf{R}_{\geq 0}$. Consider the contravariant endofunctor $*$ on the category of right $L_d(M)$-modules that sends every object to itself and every morphism $f\colon X \to Y$ of right $L_d(M)$-modules X and Y to the morphism f^* given by the composition $Y \to \mathrm{CHom}_M(Y, L_{2d}(M))^\sharp \to \mathrm{CHom}_M(X, L_{2d}(M))^\sharp \to X$, where the middle morphism is given by the precomposition with f and the other two isomorphisms come from the definition of right $L_d(M)$-modules. Here CHom denotes morphisms that are continuous in the quasinorm topology (i.e., bounded). This functor is an involutive contravariant endoequivalence of the category of right $L_d(M)$-modules, i.e., the category of right $L_d(M)$-modules is a *-category. Moreover, the morphism f^* is uniquely characterized by the equation $(f(x), y) = (x, f^*(y))$ for all $x \in X$ and $y \in Y$.

Proof. If there is another such map g, then we have $(x, (f^* - g)(y)) = 0$ for all $x \in X$ and $y \in Y$, in particular, for $x = (f^* - g)(y)$ we have $((f^* - g)(y), (f^* - g)(y)) = 0$, hence $(f^* - g)(y) = 0$ for all $y \in Y$, therefore $f^* = g$.

Denote the three morphisms in the above composition by v, h, and u. Observe that $(x, f^*(y)) = (x, uhv(y)) = (uhv(y), x)^* = h(v(y))(x^\sharp)^* = v(y)(f^\sharp(x^\sharp))^* = v(y)(f(x)^\sharp)^* = (y, f(x))^* = (f(x), y)$, as desired.

The map $f \mapsto f^*$ preserves identities and composition, hence it defines a contravariant endofunctor. Now $(x, f^{**}(y)) = (f^*(x), y) = (y, f^*(x))^* = (f(y), x)^* = (x, f(y))$ for all $x \in X$, hence $f^{**}(y) = f(y)$ for all $y \in X$, thus $f^{**} = f$ and the functor $*$ is an involutive equivalence. Since it is also **C**-antilinear on morphisms, it turns the category of right $L_d(M)$-modules into a *-category. ∎

Proposition 5.8. Suppose M is a von Neumann algebra, $d \in \mathbf{R}_{\geq 0}$, and $e \in \mathbf{R}_{\geq 0}$. If X is a right $L_d(M)$-module with the inner product $\mu: X^\sharp \otimes X \to L_{2d}(M)$, then $X \otimes_M L_e(M)$ has an inner product ν given by the composition $(X \otimes_M L_e(M))^\sharp \otimes (X \otimes_M L_e(M)) \to L_e(M)^\sharp \otimes_M X^\sharp \otimes X \otimes_M L_e(M) \to L_e(M)^\sharp \otimes_M L_{2d}(M) \otimes_M L_e(M) \to L_{2(d+e)}(M)$, where the first map comes from the functor $\sharp$, the second map is the inner product on X tensored with identity maps, and the last map is the multiplication map combined with the canonical isomorphism $L_e(M)^\sharp \to L_e(M)$ given by the involution. This inner product turns $X \otimes_M L_e(M)$ into a right $L_{d+e}(M)$-module. Furthermore, a morphism $f: X \to Y$ of right $L_d(M)$-modules induces a morphism $f \otimes_M \mathrm{id}_{L_e(M)}: X \otimes_M L_e(M) \to Y \otimes_M L_e(M)$ of right $L_{d+e}(M)$-modules. The above constructions combine into a *-functor from the category of right $L_d(M)$-modules to the category of right $L_{d+e}(M)$-modules.

Proof. First we prove that the morphism ν defined above is an inner product on $X \otimes_M L_e(M)$. By the rank 1 theorem every element of $X \otimes_M L_e(M)$ can be represented in the form $x \otimes_M u$ for some $x \in X$ and $u \in L_e(M)$. We have $(x \otimes_M u, y \otimes_M v)^* = (u^*(x, y)v)^* = v^*(x, y)^*u = v^*(y, x)u = (y \otimes_M v, x \otimes_M u)$ for all x and y in X and u and v in $L_e(M)$. Moreover, $(x \otimes_M u, x \otimes_M u) = u^*(x, x)u \geq 0$ because $(x, x) \geq 0$ and conjugation preserves positivity. Finally, if $(x \otimes_M u, x \otimes_M u) = u^*(x, x)u = 0$, then $p^*(x, x)p = 0$, where p is the left support of u, hence $p^*(x, x)p = (x \otimes_M p, x \otimes_M p) = (xp, xp) = 0$, therefore $xp = 0$, which implies that $x \otimes_M u = x \otimes_M pu = xp \otimes_M u = 0 \otimes_M u = 0$.

The module $X \otimes_M L_e(M)$ is complete because every bounded map from it to $L_{2d+2e}(M)$ is given by an inner product with some element.

Recall that a morphism $f: X \to Y$ of algebraic right M-modules is continuous if and only if it has an adjoint. For $f \otimes_M \mathrm{id}_{L_e(M)}$ we have $((f \otimes_M \mathrm{id}_{L_e(M)})(x \otimes_M u), y \otimes_M v) = u^*(f(x), y)v = u^*(x, f^*(y))v(x \otimes_M u, f^*(y) \otimes_M v) = (x \otimes_M u, (f^* \otimes_M \mathrm{id}_{L_e(M)})(y \otimes_M v))$ for all $x \in X$, $y \in Y$, u and v in $L_e(M)$, therefore the adjoint of $f \otimes_M \mathrm{id}_{L_e(M)}$ is $f^* \otimes_M \mathrm{id}_{L_e(M)}$, hence $f \otimes_M \mathrm{id}_{L_e(M)}$ is continuous.

Finally, the map $f \mapsto f \otimes_M \mathrm{id}_{L_e(M)}$ preserves the involution and therefore is a *-functor: $((f \otimes_M \mathrm{id}_{L_e(M)})(x \otimes_M u), y \otimes_M v) = (f(x) \otimes_M u, y \otimes_M v) = u^*(f(x), y)v = u^*(x, f^*(y))v = (x \otimes_M u, (f^* \otimes_M \mathrm{id}_{L_e(M)})(y \otimes_M v))$ for all $x \in X$, $y \in Y$, and u and v in $L_e(M)$, hence $(f \otimes_M \mathrm{id}_{L_e(M)})^* = (f^* \otimes_M \mathrm{id}_{L_e(M)})$. ∎

Proposition 5.9. Suppose M is a von Neumann algebra, $d \in \mathbf{R}_{\geq 0}$, and $e \in \mathbf{R}_{\geq 0}$. Denote by ${}_M\mathrm{Hom}$ the algebraic internal hom of left M-modules. If X is a right $L_{d+e}(M)$-module with the inner product $\mu: X^\sharp \otimes X \to L_{2d+2e}(M)$, then $\mathrm{Hom}_M(L_e(M), X)$ has an inner product ν given by the composition

$$\mathrm{Hom}_M(L_e(M), X)^\sharp \otimes \mathrm{Hom}_M(L_e(M), X) \to {}_M\mathrm{Hom}(L_e(M)^\sharp, X^\sharp) \otimes \mathrm{Hom}_M(L_e(M), X)$$
$$\to {}_M\mathrm{Hom}_M(L_e(M)^\sharp \otimes L_e(M), X^\sharp \otimes X)$$
$$\to {}_M\mathrm{Hom}_M(L_e(M)^\sharp \otimes L_e(M), L_{2d+2e}(M))$$
$$\to \mathrm{Hom}_M(L_e(M), {}_M\mathrm{Hom}(L_e(M)^\sharp, L_{2d+2e}(M)))$$
$$\to \mathrm{Hom}_M(L_e(M), L_{2d+e}(M))$$
$$\to L_{2d}(M),$$

where the first map comes from the functor $\sharp$, the second map is the tensor product of morphisms, the third map is the composition with the inner product on X, the fourth map is the usual tensor-hom adjunction map, the fifth map is given by the algebraic hom isomorphism theorem combined with the canonical isomorphism $L_e(M)^\sharp \to L_e(M)$ given by the involution, and the last map is again given by the algebraic hom isomorphism theorem. Alternatively, if x and y are in $\mathrm{Hom}_M(L_e(M), X)$, then their inner product is the unique element $w \in L_{2d}(M)$ such that $(x(u), y(v)) = u^*wv$ for all u and v in $L_e(M)$. This inner product turns $\mathrm{Hom}_M(L_e(M), X)$ into a right $L_d(M)$-module. Furthermore, a morphism $f: X \to Y$

9

of right $L_{d+e}(M)$-modules induces via composition a morphism $\mathrm{Hom}_M(L_e(M), f)\colon \mathrm{Hom}_M(L_e(M), X) \to \mathrm{Hom}_M(L_e(M), Y)$ of right $L_{d+e}(M)$-modules. The above constructions combine into a *-functor from the category of right $L_{d+e}(M)$-modules to the category of right $L_d(M)$-modules.

Proof. Suppose that x and y are in $\mathrm{Hom}_M(L_e(M), X)$. We have $u^*(x, y)v = (x(u), y(v))$ for all u and v in $L_e(M)$, therefore $v^*(x, y)^*u = (y(v), x(u))$, hence by the alternative definition of the inner product we have $(x, y)^* = (y, x)$. If $x \in \mathrm{Hom}_M(L_e(M), X)$, then $u^*(x, x)u = (x(u), x(u)) \geq 0$ for all $u \in L_e(M)$. An element of $L_{2d}(M)$ whose conjugation by any element of $L_e(M)$ is positive must itself be positive, hence $(x, x) \geq 0$. Finally, if $(x, x) = 0$, then $(x(u), x(u)) = u^*(x, x)u = 0$, hence $x(u) = 0$ for all $u \in L_e(M)$, therefore $x = 0$.

The module $\mathrm{Hom}_M(L_e(M), X)$ is complete because every bounded map from it to the module $L_d(M)$ comes from an inner product with some element.

The morphism $\mathrm{Hom}_M(L_e(M), f)$ is continuous for every morphism f, because

$$u^*(\mathrm{Hom}_M(L_e(M), f)(x), y)v = (f(x(u)), y(v)) = (x(u), f^*(y(v))) = u^*(x, \mathrm{Hom}_M(L_e(M), f^*)(y))v$$

for all $x \in X$, $y \in Y$, u and v in $L_e(M)$, therefore the adjoint of $\mathrm{Hom}_M(L_e(M), f)$ is $\mathrm{Hom}_M(L_e(M), f^*)$, and maps that admit an adjoint are continuous.

Finally, the map $f \mapsto \mathrm{Hom}_M(L_e(M), f)$ preserves the involution: $u^*(\mathrm{Hom}_M(L_e(M), f)(x), y)v = (f(x(u)), y(v)) = (x(u), f^*(y(v))) = u^*(x, \mathrm{Hom}_M(L_e(M), f^*)(y))v$ for all u and v in $L_e(M)$, hence we have $(\mathrm{Hom}_M(L_e(M), f)(x), y) = (x, \mathrm{Hom}_M(L_e(M), f^*)(y))$ and $\mathrm{Hom}_M(L_e(M), f)^* = \mathrm{Hom}_M(L_e(M), f^*)$ for all $f\colon X \to Y$, $x \in X$, and $y \in Y$. ∎

Proposition 5.10. Suppose M is a von Neumann algebra, $d \in \mathbf{R}_{\geq 0}$, and $e \in \mathbf{R}_{\geq 0}$. If X is a right $L_{d+e}(M)$-module, then the evaluation map $\mathrm{ev}\colon \mathrm{Hom}_M(L_e(M), X) \otimes_M L_e(M) \to X$ is a unitary isomorphism. Moreover, these maps combine into a unitary natural isomorphism of *-functors.

Proof. The alternative definition of the inner product on $\mathrm{Hom}_M(L_e(M), X)$ immediately proves that the evaluation map preserves the inner product, in particular it is injective: $(\mathrm{ev}(x \otimes_M u), \mathrm{ev}(y \otimes_M v)) = (x(u), y(v)) = u^*(x, y)v = (x \otimes_M u, y \otimes_M v)$ for all x and y in X and u and v in $L_e(M)$.

Consider an arbitrary element $x \in X$ and denote by Z the closed submodule of X generated by x. The map q from the algebraic submodule of Z generated by x to the algebraic right submodule of $L_{d+e}(M)$ generated by $(x, x)^{1/2}$ given by sending an element of the form $xp \in X$ for some $p \in M$ to the element $(x, x)^{1/2}p \in L_{d+e}(M)$ is well-defined with respect to the choice of p and preserves the inner product, hence it extends to an isomorphism from Z to $sL_{d+e}(M)$, where $s \in M$ is the right support of x, i.e., the support of $(x, x)^{1/2}$. The restriction of ev to the map of the form $\mathrm{Hom}_M(L_e(M), Z) \otimes_M L_e(M) \to Z$ is an isomorphism. Indeed, the map $sL_d(M) \otimes_M L_e(M) \to \mathrm{Hom}_M(L_e(M), sL_{d+e}(M)) \otimes_M L_e(M) \to \mathrm{Hom}_M(L_e(M), Z) \otimes_M L_e(M) \to Z \to sL_{d+e}(M)$ is given by the multiplication map, which is an isomorphism, hence the above restriction of ev is an isomorphism and therefore ev is a surjection. ∎

Proposition 5.11. Suppose M is a von Neumann algebra, $d \in \mathbf{R}_{\geq 0}$, and $e \in \mathbf{R}_{\geq 0}$. If X is a right $L_d(M)$-module, then the left multiplication map $\mathrm{lm}\colon X \to \mathrm{Hom}_M(L_e(M), X \otimes_M L_e(M))$ is a unitary isomorphism. Moreover, these maps combine into a unitary natural isomorphism of *-functors.

Proof. The left multiplication map preserves the inner product, hence it is injective: $u^*(\mathrm{lm}(x), \mathrm{lm}(y))v = (\mathrm{lm}(x)(u), \mathrm{lm}(y)(v)) = (x \otimes_M u, y \otimes_M v) = u^*(x, y)v$ for all x and y in X and u and v in $L_e(M)$. Since u and v are arbitrary, it follows that $(\mathrm{lm}(x), \mathrm{lm}(y)) = (x, y)$. Finally, the left multiplication map is surjective and hence it is an isomorphism. ∎

Theorem 5.12. Suppose M is a von Neumann algebra, $d \in \mathbf{R}_{\geq 0}$, and $e \in \mathbf{R}_{\geq 0}$. The *-category of right $L_d(M)$-modules is a W*-category. Moreover, the functors of tensor product and internal hom with $L_e(M)$ between the categories of right $L_d(M)$-modules and $L_{d+e}(M)$-modules together with the unitary natural isomorphisms of evaluation and left multiplication form an adjoint unitary W*-equivalence of the W*-categories of right $L_d(M)$-modules and right $L_{d+e}(M)$-modules.

Proof. The above propositions establish that the functors and natural isomorphisms under consideration constitute a unitary *-equivalence of the corresponding *-categories. The *-category of right $L_0(M)$-modules

10

is a W*-category, hence the *-category of right $L_d(M)$-modules is also a W*-category. A *-equivalence of W*-categories is automatically normal, i.e., it is a W*-equivalence, and a unitary natural transformation is automatically bounded, i.e., it is a unitary W*-natural transformation. Thus we only have to prove the adjunction property. This amounts to checking the unit-counit equations. For equivalences, one of the equations implies the other one, but here we check them both.

The first property states that the composition of the morphisms

$$X \otimes_M L_e(M) \to \operatorname{Hom}_M(L_e(M), X \otimes_M L_e(M)) \otimes_M L_e(M)$$

(the tensor product of the left multiplication map of X and the identity morphism of $L_e(M)$) and

$$\operatorname{Hom}_M(L_e(M), X \otimes_M L_e(M)) \otimes_M L_e(M) \to X \otimes_M L_e(M)$$

(the evaluation map of $X \otimes_M L_e(M)$) is the identity morphism of $X \otimes_M L_e(M)$. The first map sends an element $x \otimes_M u \in X \otimes_M L_e(M)$ to the element $(v \in L_e(M) \mapsto x \otimes_M v \in X \otimes_M L_e(M)) \otimes_M u$, which is then evaluated to $x \otimes_M u$.

The second property states that the composition of the morphism

$$\operatorname{Hom}_M(L_e(M), X) \to \operatorname{Hom}_M(L_e(M), \operatorname{Hom}_M(L_e(M), X) \otimes_M L_e(M))$$

(the left multiplication map of the module $\operatorname{Hom}_M(L_e(M), X)$) and the morphism

$$\operatorname{Hom}_M(L_e(M), \operatorname{Hom}_M(L_e(M), X) \otimes_M L_e(M)) \to \operatorname{Hom}_M(L_e(M), X)$$

(the composition with the evaluation map of X) is the identity morphism of $\operatorname{Hom}_M(L_e(M), X)$. The first map sends an element $f \in \operatorname{Hom}_M(L_e(M), X)$ to $u \in L_e(M) \mapsto f \otimes_M u \in \operatorname{Hom}_M(L_e(M), X) \otimes_M L_e(M)$, which is then mapped to $(u \in L_e(M) \mapsto f(u) \in X) = f$. ∎

Definition 5.13. Suppose M and N are von Neumann algebras and $d \in \mathbf{R}_{\geq 0}$. An M-$L_d(N)$-bimodule is a right $L_d(N)$-module X equipped with a morphism of von Neumann algebras $M \to \operatorname{End}(X)$. A morphism of M-$L_d(N)$-bimodules is a morphism of the underlying right $L_d(N)$-modules that commutes with the left action of M. Similarly, an $L_d(M)$-N-bimodule is a left $L_d(M)$-module X equipped with a morphism of von Neumann algebras $N \to \operatorname{End}(X)$. We also refer to right M-$L_0(N)$-bimodules as *right M-N-W*-bimodules* and similarly for left bimodules.

Theorem 5.14. For any von Neumann algebras M and N and any $d \in \mathbf{R}_{\geq 0}$ and $e \in \mathbf{R}_{\geq 0}$ there are canonical adjoint unitary W*-equivalences of the W*-categories of M-$L_d(N)$-bimodules and M-$L_e(N)$-bimodules.

Proof. All functors and natural isomorphisms under consideration are W*-functors and unitary W*-natural isomorphisms and therefore they preserve the left action of M and give an adjoint unitary equivalence of the corresponding W*-categories. ∎

We briefly review the equivalence of $L_{1/2}(M)$-modules and representations of M on a Hilbert space. See Example 3.4.(ii) in Junge and Sherman [JuSh].

Definition 5.15. A *right representation* of a von Neumann algebra M on a Hilbert space H is a morphism of von Neumann algebras $M \to B(H)^{\operatorname{op}}$, where multiplication of elements of $B(H)$ corresponds to the composition of operators in the usual reverse order (i.e., xy means apply y first, then apply x). We denote the W*-category of right representations of M and their bounded intertwiners by Rep_M. Likewise, a *left representation* is a morphism $M \to B(H)$ and all left representations form a category $_M\operatorname{Rep}$. Finally, a *birepresentation* of von Neumann algebras M and N on a Hilbert space H is a pair of morphisms $M \to B(H)$ and $N \to B(H)^{\operatorname{op}}$ with commuting images.

Definition 5.16. For every von Neumann algebra M we define a W*-functor $F \colon \operatorname{Mod}_{L_{1/2}(M)} \to \operatorname{Rep}_M$: if X is a right $L_{1/2}(M)$-module, then we turn it into a Hilbert space with the inner product given by composing the $L_1(M)$-valued inner product on X with the Haagerup trace $L_1(M) \to \mathbf{C}$. We also define a W*-functor going in the opposite direction $\operatorname{Rep}_M \to \operatorname{Mod}_{L_{1/2}(M)}$: if X is a right representation of M, then for a pair of

elements u and v in X the value of $L_1(M)$-valued inner product (u, v) evaluated at $p \in M$ is $\langle u, vp \rangle$. Observe that we have unitary W*-natural isomorphisms id $\to GF$ and $FG \to$ id.

Theorem 5.17. Two W*-functors defined above together with the corresponding W*-natural isomorphisms form an adjoint unitary W*-equivalence of W*-categories of right $L_{1/2}(M)$-modules and right representations of M. Likewise for left modules and representations. Moreover, the above adjoint unitary W*-equivalences of W*-categories yield adjoint unitary W*-equivalences of the W*-categories of M-$L_{1/2}(N)$-bimodules, birepresentations of M and N, and $L_{1/2}(M)$-N-bimodules.

Corollary 5.18. There are adjoint unitary W*-equivalences of the W*-categories of M-$L_d(N)$-bimodules, $L_d(M)$-N-bimodules, and birepresentations of M and N for all von Neumann algebras M and N and for all $d \in \mathbf{R}_{\geq 0}$.

If we pass from an M-$L_d(N)$-bimodule X to the corresponding $L_d(M)$-N-bimodule Y, then X is generally not isomorphic to Y as an algebraic M-N-bimodule unless $d = 1/2$. For example, take $d = 0$, $M = \mathbf{C}$, $N = \mathrm{B}(H)$, $X = \mathrm{B}(H)$ for some infinite dimensional Hilbert space H. Then $Y = L_{1/2}(\mathrm{B}(H))$ as a $L_{1/2}(\mathbf{C})$-$\mathrm{B}(H)$-bimodule and Y is not isomorphic to X as an algebraic right $\mathrm{B}(H)$-module.

6   References.

[GhLiRo] P. Ghez, Ricardo Lima, John E. Roberts. W*-categories. Pacific Journal of Mathematics 120:1 (1985), 79–109. doi:10.2140/pjm.1985.120.79. 1.1*, 2.0*.

[StTe] Stephan Stolz, Peter Teichner. What is an elliptic object? Chapter 10 of Topology, geometry and quantum field theory. Proceedings of the 2002 Oxford symposium in the honour of the 60th birthday of Graeme Segal. London Mathematical Society Lecture Notes Series 308 (2004), 247–343. https://math.berkeley.edu/~teichner/Papers/Oxford.pdf, doi:10.1017/CBO9780511526398.013. 1.1*.

[BaDeHa] Michel Baillet, Yves Denizeau, Jean-François Havet. Indice d'une espérance conditionelle. Compositio Mathematica 66:2 (1988), 199–236. http://www.numdam.org/item/?id=CM_1988__66_2_199_0. 5.0*.

[JuSh] Marius Junge, David Sherman. Noncommutative L^p modules. Journal of Operator Theory 53:1 (2005), 3–34. MR2132686, Zbl:1078.46046, arXiv:math/0301044v2, https://www.jstor.org/stable/24715642. 1.1*, 5.2, 5.4, 5.14*.

[Ri] Marc Rieffel. Morita equivalence for C*-algebras and W*-algebras. Journal of Pure and Applied Algebra 5:1 (1974), 51–96. doi:10.1016/0022–4049(74)90003–6, https://dmitripavlov.org/scans/rieffel.pdf. 5.0*.