$d = 1$, Riemannian metrics

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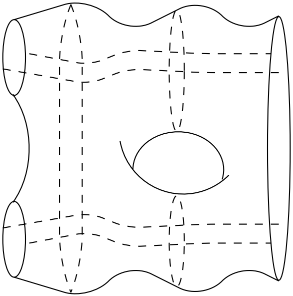
These slides: `https://dmitripavlov.org/lecture-5b.pdf`



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$$\mathrm{Map}(\mathbf{B}\mathbb{R}, \mathsf{C}^{\times}),$$

where the mapping simplicial set is taken in $\mathrm{PSh}_{\Delta}(\mathsf{Cart})$. Recall, $\mathsf{C}^{\times}$ is a simplicial presheaf on Cart.

- There is a homotopy cocontinuous functor

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- So we need to compute $\mathcal{C}_1(\mathcal{R})$.

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$$\alpha' \circ h \geq \alpha + \beta$$

Here, $\beta(u)$ is keeping track of the "offset" of an interval of length $\alpha(u)$ embedded into a larger interval.

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- We know that $\mathcal{C}_1(\mathbb{R} \times U \to U) \simeq \mathbb{Z}/2 \times U$.
- So passing $\mathcal{C}_1$ inside the homotopy colimit, we get

$$\mathrm{hocolim}_{(0,\alpha) \to \mathbb{R}_{>0}} \mathbb{Z}/2 \times U,$$

which we still need to compute.

- We compute the colimit objectwise by taking the nerve of the Grothendieck construction, applied to
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- We can split off a factor of $\mathbb{Z}/2$ to get $G_W = E_W \times \mathbb{Z}/2$.
- We define a functor

$$F\colon E_W \to \mathcal{B}(C^\infty(W, \mathbb{R}), +))$$

  that essentially throws away all the data except β.
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- We recall that β's add when morphisms are composed.
- We show that F induces an equivalence on nerves, using Quillen's theorem A. The point is that Kan fibrantly replacing E_W formally adds inverses for $\beta\colon W \to \mathbb{R}_{\geq 0}$.

- So we get we have computes the homotopy colimit. We get

$$\mathcal{C}_1(\mathcal{R}) \simeq \mathbb{Z}/2 \times \mathbf{B}\mathbb{R}$$