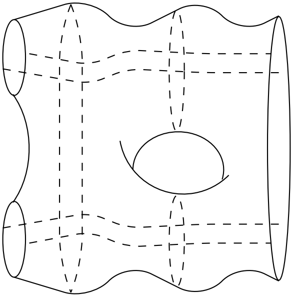
The geometric cobordism hypothesis
Lecture 5a: The leftovers

Daniel Grady, Dmitri Pavlov (Texas Tech University, Lubbock, TX)

These slides: `https://dmitripavlov.org/lecture-5a.pdf`



Overview

- Tuesday: definitions
- Wednesday: locality and how to use it to prove one half of the GCH
- Thursday: the framed GCH (the other half)
- Today: contractibility of moduli spaces of cuts and its applications to GCH

Homotopy cocontinuity of $\mathfrak{Bord}_d$

Proposition (G.–P. (formal))

Given $d \geq 0$, we have a left Quillen functor

$$\mathrm{sPSh}(\mathsf{FEmb}_d)_{\mathsf{inj}} \to \mathrm{sPSh}(\mathsf{Cart} \times \Gamma \times \Delta^{\times d})_{\mathsf{loc}}, \qquad \mathcal{S} \mapsto \mathfrak{Bord}_d^{\mathcal{S}}.$$

Theorem (G.–P.)

Given $d \geq 0$, the left derived functor of the left Quillen functor

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sends Čech nerves of open covers in FEmb_d to weak equivalences.

The codescent property

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sends the Čech nerve of an open cover $\{W_a \to U_a\}_{a \in A}$ of $(W \to U) \in \mathsf{FEmb}_d$ to a weak equivalence:

$$\operatorname*{hocolim}_{n \in \Delta^{\mathrm{op}}} \coprod_{\alpha:[n] \to A} \mathfrak{Bord}_d^{W_\alpha \to U_\alpha} \xrightarrow{\sim} \mathfrak{Bord}_d^{W \to U},$$

where $W_\alpha = W_{\alpha_0} \cap \cdots \cap W_{\alpha_n}$.

The codescent property: main steps

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Step 4 Prove all maps in the filtration are weak equivalences.

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Definition

Given $d \geq 0$ and $(W = \mathbf{R}^d \times U \to U) \in \mathrm{FEmb}_d^{\mathrm{op}}$, the set $\mathfrak{Bord}_d^{\mathbf{R}^d\times U\to U}(V, \langle \ell \rangle, \mathbf{m})_n$ has elements:

- a smooth manifold M;
- a V-family of embeddings $M \to \mathbf{R}^d$;
- a $V \times \mathbf{\Delta}^n$-family of cut tuples with $m_1 \times \cdots \times m_d$ cells;
- $P\colon M \to \langle \ell \rangle$;
- smooth map $V \to U$;

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Proposition (Formal)

The map $\mathrm{colim} \to B_0$ *is a weak equivalence in* $\mathrm{sPSh}(\Gamma \times \Delta^{\times d})_{\mathrm{loc}}$.

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Proof.

- Evaluate $B_{i-1} \to B_i$ on an arbitrary object X of $\Gamma \times \Delta^{\times(d-1)}$, obtaining a map $B_{i-1}(X) \to B_i(X)$ in $\mathrm{sPSh}(\Delta)$;

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- Extract the kth simplicial degree (for some $k \geq 0$), obtaining a map in $\mathrm{PSh}(\Delta) = \mathrm{sSet}$;

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- Extract the kth simplicial degree (for some $k \geq 0$), obtaining a map in $\mathrm{PSh}(\Delta) = \mathrm{sSet}$;
- The resulting simplicial set has
  - vertices: germs of cuts (embedded in W);
  - edges: bordisms between cuts (embedded in W);
  - 2-simplices: composition of bordisms;
  - everything is in smooth families indexed by Δ^k;
  - bordisms must belong to B_{i-1} respectively B_i.

Want to show: $B_{i-1} \to B_i$ is a categorical weak equivalence in the Joyal model structure on simplicial sets. □

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- Answer: Dugger–Spivak necklace categories.

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- Necklaces from x to y: composable chains of bordisms in B_{i-1} (or B_i) joined together by joint cuts.

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- Observation: the ambient composed bordism never changes $\implies$ can fix it in advance.

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- This implies $B_{i-1} \to B_i$ is a weak equivalence.

The big picture

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- All neighborhoods can be chosen to be subordinate to the open cover of W.
- How does this help us to show contractibility of necklace categories?

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Theorem (Simplicial Whitehead theorem)

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- Apply this theorem to the fibrant replacement X of the nerve of the necklace category of B_{i-1} (or B_i) from x to y.

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- Chop up M as explained on the previous slide.

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- By induction on the Morse decomposition, push the cuts past each small region in the Morse decomposition, with some cutting and gluing of cuts.

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- Chop up M as explained on the previous slide.
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- At the final step, all cuts have been collapsed to the source cut of M.

How is this used in the framed GCH?

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- everything except for the index k handle is in B_{k-1};

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Cutting out handles

Theorem (G.–P.)

For any $d \geq 0$ and $0 \leq k \leq d$, the following squares are homotopy cocartesian in $\mathrm{C}^\infty\mathrm{Cat}^{\otimes,\vee}_{\infty,d}$:

$$\begin{array}{ccccccc} O_{k-1} & \longrightarrow & \overline{O}_{k-1} & \longrightarrow & \widetilde{O}_{k-1} & \longrightarrow & B_{k-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_k & \longrightarrow & \overline{H}_k & \longrightarrow & \widetilde{H}_k & \longrightarrow & B_k. \end{array}$$

- *H_k: index $k-1$ (counit) and index k (unit) handles;*
- *O_{k-1}: index $k-1$ (counit) handle;*
- *$\overline{H}$, $\overline{O}$: same, with tails attached in the $(d-2)$nd direction;*
- *$\widetilde{H}$, $\widetilde{O}$: same, with tails attached in the $(d-1)$st direction.*
- *left two squares: insert cuts close to the handle;*
- *right square: invoke the same proof as for locality, using a new open cover.*

Nebulous visions of the future...

- Prequantum/classical: the book "Natural operations in differential geometry" (Kolář–Michor–Slovák) constructs a lot of functorial field theories...
- Quantum: quantization and path integrals (for fully extended FFTs) via the GCH;
- Further compute the right hand side of GCH via ∞-Lie theory;

Explore possible value categories:

- geometric factorization algebras (Peña, based on Carmona–Flores–Muro);
- closed symmetric monoidal category with duals of complete vector spaces (?).