The geometric cobordism hypothesis
Lecture 4: The geometrically framed case

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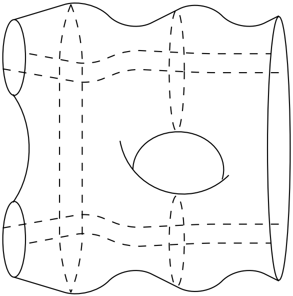
These slides: `https://dmitripavlov.org/lecture-4.pdf`



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- For fixed $U \in \mathsf{Cart}$, $\langle \ell \rangle \in \Gamma$ and $\mathbf{m} \in \Delta^{\times d}$, a vertex in

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- An l-simplex is a smooth deformation of the $\mathbf{m}$-cut tuples, parametrized by Δ^l. The face maps restrict the germ of the core, as needed.

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Definition

Let $(\mathsf{C}, \otimes, 1)$ be a symmetric monoidal cat. Let $c \in \mathsf{C}$. Then a dual for c is an object $c^\vee$ along with maps

1. $\epsilon\colon c^\vee \otimes c \to 1$
2. $\eta\colon 1 \to c \otimes c^\vee$

s.t.

$$c^\vee \cong c^\vee \otimes 1 \to c^\vee \otimes (c \otimes c^\vee) \cong (c^\vee \otimes c) \otimes c^\vee \to 1 \otimes c^\vee \cong c^\vee$$

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is the identity. Same with c and $c^\vee$ reversed.

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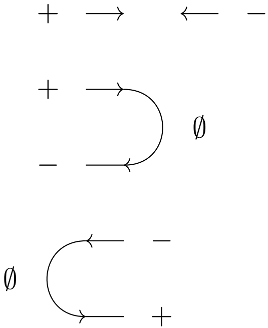
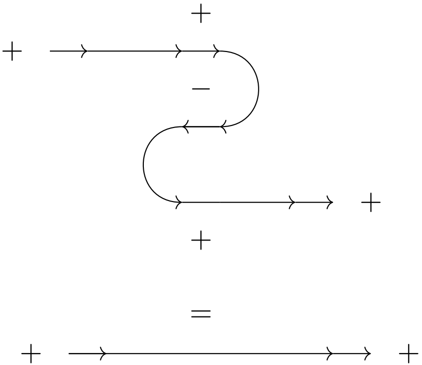
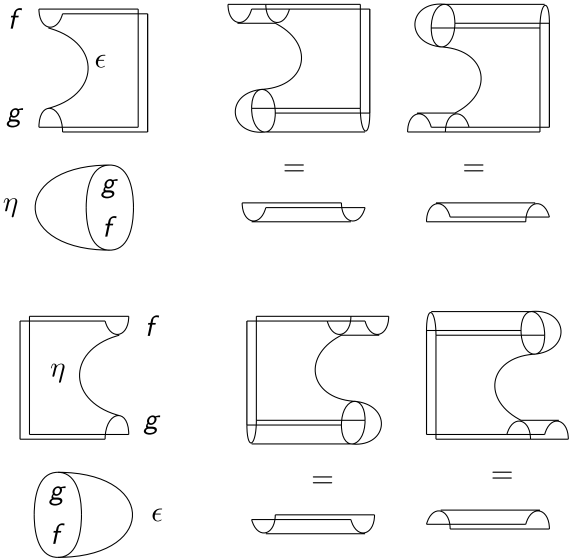
- For a 2-category, one can also define duals for morphisms: they are adjunctions. Can extend to higher cats by induction.

Duals in $\mathrm{Bord}_1^{\mathbb{R}\times U\to U}$





Duals in $\mathrm{Bord}_2^{\mathbb{R}^2\times U\to U}$



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subject to the relations given by the two triangle identities.

- We write $f \to \mathrm{Adj}$, $\eta \to \mathrm{Adj}$, $\epsilon \to \mathrm{Adj}$ for the sub 2-categories generated by f, (f, g, η) and (f, g, ϵ), respectively.

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Definition

We say that C has all duals if it has duals for all k-morphisms with $1 \leq k \leq d-1$.

- Adding the maps $f \to \mathrm{Adj}$ (after applying the left adjoint to the evaluation at $(U, \langle \ell \rangle, (\mathbf{m}, -, \mathbf{0})))$ to the list of maps at which we localize, we get a new model category:

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$$\begin{array}{ccc} f & \longrightarrow & \mathsf{C}_{\mathbf{m},*,*,\mathbf{0}}(U, \langle \ell \rangle) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Adj} & \longrightarrow & * \end{array}$$

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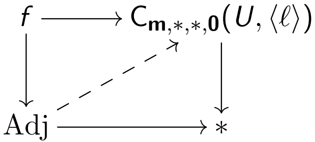

- So every k-morphism is the left adjoint for an adjunction.

The geometric cobordism hypothesis: framed case

Theorem (G.-P.)

Let $d \geq 0$, and let C be a fibrant object in $\mathrm{C}^\infty\mathsf{Cat}_{\infty,d}^{\otimes,\vee}$. Then evaluation at the (positive) point yields a weak equivalence

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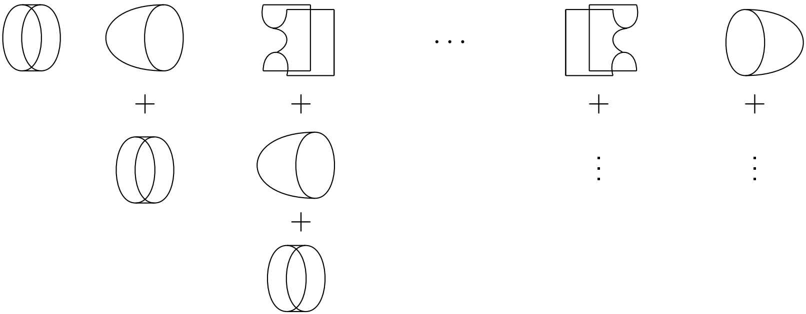

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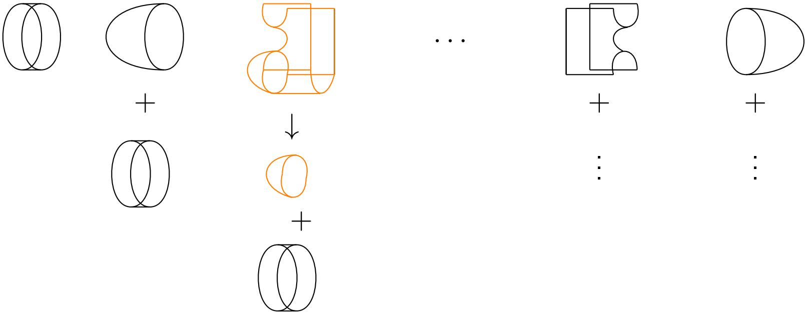

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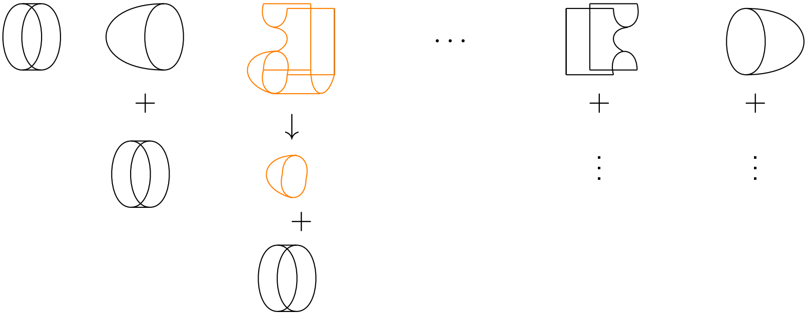

- Problem: What if the disc of index 0 is not sent to the unit of an adjunction?

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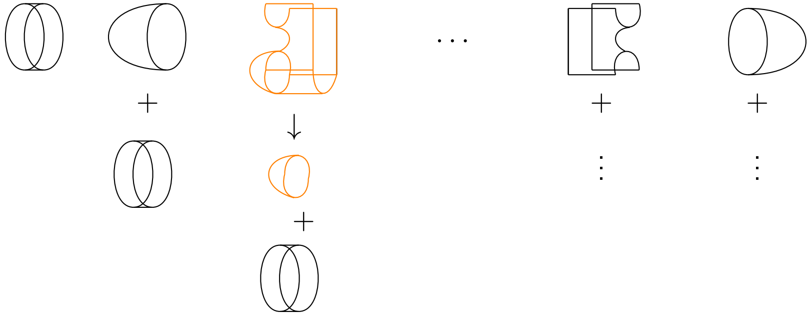

- Problem: What if the disc of index 0 is not sent to the unit of an adjunction?
- For higher index, this is not a problem (use exchange principle).

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- We again work in families over Δ^l.

Handles of index $k-1$

- We define $O_{k-1} \subset H_k$ as the further subobject just containing the bisimplices in the left column.

Theorem (G.-P., Propositions 4.2.33, 4.3.2)

For $k \geq 1$, we have a homotopy pushout diagram

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- This statement can be regarded as a generalization of Lurie's claims 3.4.12 and 3.4.17: B_k is freely generated from B_{k-1} by the addition of $\mathrm{O}(d-k)$ worth of handles of index k and a handle cancellation for each index k-handle.

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Theorem (G.-P., Proposition 4.2.24)

For $k \geq 1$, the map $O_{k-1} \to H_k$ induces weak equivalence

$$\mathrm{Fun}^{\otimes}(H_k, \mathsf{C}) \to \mathrm{Fun}^{\otimes}(O_{k-1}, \mathsf{C})_{\mathrm{unit}}$$

Another crucial ingredient:

Theorem (G.-P., Proposition 4.2.24)

For $k \geq 1$, the map $O_{k-1} \to H_k$ induces weak equivalence

$$\mathrm{Fun}^{\otimes}(H_k, \mathsf{C}) \to \mathrm{Fun}^{\otimes}(O_{k-1}, \mathsf{C})_{\mathrm{unit}}$$

Corollary

For $d \geq 1$ and C a fibrant object in $\mathrm{C}^{\infty}\mathsf{Cat}^{\otimes,\vee}_{\infty,d}$, There are weak equivalences

$$\mathrm{Fun}^{\otimes}(B_k, \mathsf{C}) \to \mathrm{Fun}^{\otimes}(B_{k-1}, \mathsf{C})$$

for $k \geq 2$ and a weak equivalence

$$\mathrm{Fun}^{\otimes}(B_1, \mathsf{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\otimes}(B_{-1}, \mathsf{C}) \times_{\mathsf{Fun}^{\otimes}(O_{-1},\mathsf{C})} \mathrm{Fun}^{\otimes}(H_0, \mathsf{C})_{\mathrm{unit}}$$

Proof of corollary

- The homotopy pushout square

$$\begin{array}{ccc} O_{k-1} & \longrightarrow & B_{k-1} \\ \downarrow & & \downarrow \\ H_k & \longrightarrow & B_k \end{array}$$

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- By definition, the left map factors through the coproduct summand of units.

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- Hence, for the first claim, it suffices to prove that the top map factors through the coproduct summand of units.

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Let X be a $d \geq 3$-fold complete Segal space and suppose we have the following multisimplices

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Suppose both u and u' are units of an adjunction.

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Suppose both u and u' are units of an adjunction. Then there are morphisms $\gamma\colon \mathrm{id}_{f^\dagger \circ f} \to u \circ_2 v'$ *and* $\beta\colon u' \circ_2 v \to \mathrm{id}_{f \circ f^\dagger}$ *associated to α via equivalences.*

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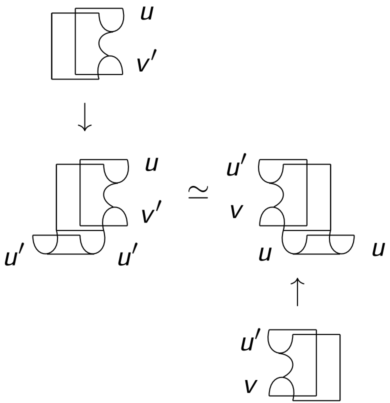
Suppose both u and u' are units of an adjunction. Then there are morphisms $\gamma\colon \mathrm{id}_{f^\dagger \circ f} \to u \circ_2 v'$ and $\beta\colon u' \circ_2 v \to \mathrm{id}_{f \circ f^\dagger}$ associated to α via equivalences. Moreover, β is the counit of an adjunction if and only if γ is the unit of an adjunction.

Exchanging a counit for a unit



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- This, combined with the pullback diagram for $k = 0$ proves claim 2.

Proof of GCH

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- By the corollary we have an equivalences

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{\mathbb{R}^d \times U \to U}, \mathsf{C}) \simeq \mathrm{Fun}^{\otimes}(B_1, \mathsf{C})$$

  and

$$\mathrm{Fun}^{\otimes}(B_1, \mathsf{C}) \simeq \mathrm{Fun}^{\otimes}(B_{-1}, \mathsf{C}) \times_{\mathrm{Fun}^{\otimes}(O_{-1}, \mathsf{C})} \mathrm{Fun}^{\otimes}(H_0, \mathsf{C})_{\mathrm{unit}}$$

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Definition

We define a functor $\iota_{d-1}\colon \mathsf{FEmb}_{d-1} \to \mathsf{FEmb}_d$ by sending a submersion $M \to U$ to $M \times \mathbb{R} \to U$.

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- We have $B_{-1} \simeq \mathfrak{Bord}_{d-1}^{\iota_{d-1}^*(\mathbb{R}^d\times U\to U)}$.
- By GCH in dimension $d-1$, we have an equivalence

$$\mathrm{Fun}^{\otimes}(B_1, \mathsf{C}) \simeq \mathrm{Map}(\iota_{d-1}^*(\mathbb{R}^d \times U \to U), \mathsf{C}^{\times}) \times_{\mathrm{Fun}^{\otimes}(O_{-1},\mathsf{C})} \mathrm{Fun}^{\otimes}(H_0, \mathsf{C})_{\mathrm{unit}} \quad (\spadesuit)$$

A homotopy pushout for geometric structures

Lemma

We have a homotopy pushout diagram in $\mathsf{sPSh}_\Delta(\mathsf{FEmb}_{d-1})_{\check{\mathrm{C}},\mathrm{flc}}$*:*

$$\begin{array}{ccc} (\iota_{d-2})_! \iota_{d-2}^*(\mathbb{R}^{d-1} \times U \to U) & \longrightarrow & \mathbb{R}^{d-1} \times U \to U \\ \downarrow & & \downarrow \\ \mathbb{R}^{d-1} \times U \to U & \longrightarrow & \iota_{d-1}^*(\mathbb{R}^{d} \times U \to U) \end{array}$$

Proof: Idea is to move to $\mathrm{O}(d-1)$-equivariant presheaves by a zig-zag of Quillen equivalences.

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Proof: Idea is to move to $\mathrm{O}(d-1)$-equivariant presheaves by a zig-zag of Quillen equivalences. This turns the above diagram into a homotopy pushout diagram in $\mathsf{sPSh}_\Delta(\mathsf{Cart};\mathsf{sSet}^{\mathrm{O}(d-1)})$:

$$\begin{array}{ccc} \mathrm{O}(d-1) \sqcup_{\mathrm{O}(d-2)} \mathrm{O}(d-1) \times U & \longrightarrow & \mathrm{O}(d-1) \times U \\ \downarrow & & \downarrow \\ \mathrm{O}(d-1) \times U & \longrightarrow & \mathrm{O}(d) \times U \end{array}$$

- This can be shown to be a homotopy pushout square of equivariant spaces (Lurie's proof of Proposition 2.4.6).

Moving down to codimension 2

Since $\mathsf{C}^{\times}_{d-1}$ is fiberwise locally constant (by the induction hypothesis), we have a homotopy pullback diagram

$$\begin{array}{ccc} \mathrm{Map}((\iota_{d-1})_! \iota^*_{d-1}(\mathbb{R}^{d-1} \times U), \mathsf{C}^{\times}_{d-1}) & \longleftarrow & \mathrm{Map}(\mathbb{R}^{d-1} \times U, \mathsf{C}^{\times}_{d-1}) \\ \uparrow & & \uparrow \\ \mathrm{Map}(\mathbb{R}^{d-1} \times U, \mathsf{C}^{\times}_{d-1}) & \longleftarrow & \mathrm{Map}(\iota^*_{d-1}(\mathbb{R}^{d} \times U), \mathsf{C}^{\times}_{d-1}) \end{array}$$

- The two off-diagonal corners are equivalent to $\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{d-1}^{\mathbb{R}^{d-1} \times U \to U}, \mathsf{C})$ by the induction hypothesis.

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\mathrm{Map}(\mathbb{R}^{d-1}\times U, \mathsf{C}_{d-1}^{\times}) & \longleftarrow & \mathrm{Map}(\iota_{d-1}^*(\mathbb{R}^{d}\times U), \mathsf{C}_{d-1}^{\times})
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- Then we invoke the corollary in dimension $d-1$ to get

$$\begin{aligned}
&\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{d-1}^{\mathbb{R}^{d-1}\times U\to U}, \mathsf{C}) \\
\simeq &\mathrm{Map}(\iota_{d-2}^*(\mathbb{R}^{d-1}\times U), \mathsf{C}_{d-1}^{\times}) \times_{\mathrm{Fun}^{\otimes}(O_{-1,d-1},\mathsf{C})} \mathrm{Fun}^{\otimes}(H_{0,d-1}, \mathsf{C})_{\mathrm{unit}}
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$$\begin{array}{ccccc}
C^{O_{-1,d-1}} & \longleftarrow & \mathrm{Map}(\iota^*_{d-2}(\mathbb{R}^{d-1} \times U), \mathsf{C}^{\times}_{d-2}) & \longleftarrow & \mathrm{Map}(\iota^*_{d-2}(\mathbb{R}^{d-1} \times U), \mathsf{C}^{\times}) \times_{\mathsf{C}^{O_{-1,d-1}}} \mathsf{C}_u^{H_{0,d-1}} \\
\uparrow & & \uparrow & & \uparrow \\
\mathsf{C}_u^{H_{0,d-1}} & \longleftarrow & \mathrm{Map}(\iota^*_{d-2}(\mathbb{R}^{d-1} \times U), \mathsf{C}^{\times}_{d-2}) \times_{\mathsf{C}^{O_{-1,d-1}}} \mathsf{C}_u^{H_{0,d-1}} & \longleftarrow & \mathrm{Map}(\iota^*_{d-1}(\mathbb{R}^{d} \times U), \mathsf{C}^{\times}_{d-1})
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$$\mathrm{Map}(\iota_{d-1}^*(\mathbb{R}^d \times U \to U), \mathsf{C}_{d-1}^{\times}) \simeq$$
$$\mathrm{Map}(\iota_{d-2}^*(\mathbb{R}^{d-1} \times U \to U), \mathsf{C}_{d-2}^{\times}) \times_{\mathsf{C}^{O_{-1,d-1}}} \mathsf{C}_u^{H_{0,d-1}} \times_{\mathsf{C}^{O_{-1,d-1}}} \times \mathsf{C}_u^{H_{0,d-1}}$$

- Plugging back into (♠), we get an equivalence

$$\mathrm{Fun}^{\otimes}(B_1, \mathsf{C}^{\times}) \simeq$$
$$\mathrm{Map}(\iota_{d-2}^*(\mathbb{R}^{d-1} \times U \to U), \mathsf{C}_{d-2}^{\times}) \times_{\mathsf{C}^{O_{-1,d-1}}} \mathsf{C}_u^{H_{0,d-1}} \times_{\mathsf{C}^{O_{-1,d-1}}} \times \mathsf{C}_u^{H_{0,d-1}} \times_{\mathsf{C}^{O_{-1}}} \mathsf{C}_u^{H_0} \quad (\heartsuit)$$

- Focusing on the triple pullback

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is an equivalence (since C has duals).

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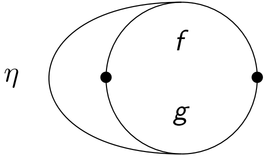

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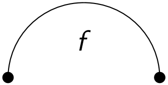

- Finally, combining with (♡), we have equivalences

$$
\begin{aligned}
&\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{d-1}^{\mathbb{R}^d \times U \to U}, \mathsf{C}) \simeq \mathrm{Fun}^{\otimes}(B_1, \mathsf{C}) \\
\simeq &\mathrm{Map}(\iota_{d-2}^{*}(\mathbb{R}^{d-1} \times U \to U), \mathsf{C}_{d-2}^{\times}) \times_{\mathsf{C}^{O_{-1,d-1}}} \mathsf{C}^{H_{0,d-1}} \times_{\mathsf{C}^{O_{-1,d-1}}} \times \mathsf{C}^{H_{0,d-1}} \times_{\mathsf{C}^{O_{-1}}} \mathsf{C}^{H_0} \\
\simeq &\mathrm{Map}(\iota_{d-2}^{*}(\mathbb{R}^{d-1} \times U \to U), \mathsf{C}_{d-2}^{\times}) \times_{\mathsf{C}^{O_{-1,d-1}}} \mathsf{C}^{H_{0,d-1}} \\
\simeq &\mathrm{Fun}^{\otimes}(B_{1,d-1}, \mathsf{C}) \\
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\end{aligned}
$$

- The induction is complete.

Proof of the key lemmas

Lemma (Proposition 4.2.24)

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Proof: We let $\mathcal{M} \subset \mathrm{Map}(\mathrm{Adj}, B_k)$ be the coproduct summand of maps that send the left adjoint to bordisms of the form f. We claim we have a homotopy pushout

$$\begin{array}{ccc} \mathcal{M} \times \eta_{d-1} & \longrightarrow & O_{k-1} \\ \downarrow & & \downarrow \\ \mathcal{M} \times \mathrm{Adj}_{d-1} & \longrightarrow & H_k \end{array}$$

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- The composition $F_\epsilon \to G_0 \to F_\epsilon$ is homotopic to identity.

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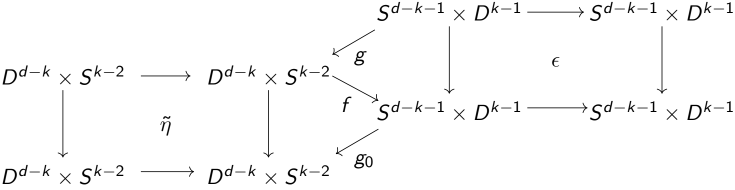

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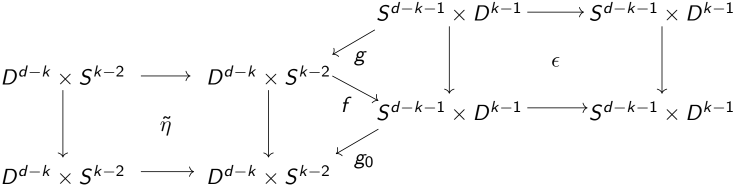

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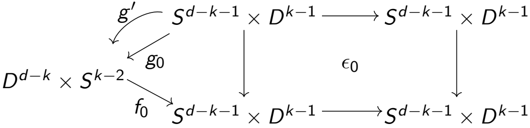

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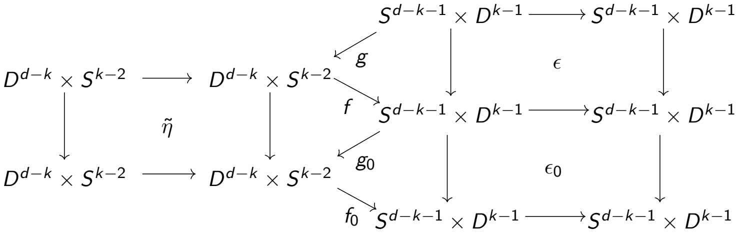

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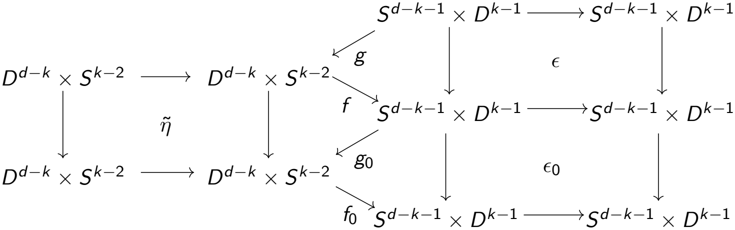

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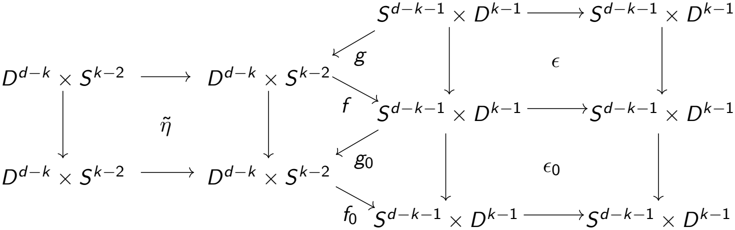

- After composing the bottom portion of the diagram, the entire diagram contracts to just (ϵ, f, g), by a homotopy (corresponding to one of the triangle identities).
- Hence, $F_\epsilon \to G_0 \to F_\epsilon$ is homotopic to identity.
- Since G_0 is contractible, this proves the claim.

Lemma (Propositions 4.2.33, 4.3.2)

For $k \geq 1$, we have a homotopy pushout diagram

$$\begin{array}{ccc} \mathrm{O}_{k-1} & \longrightarrow & B_{k-1} \\ \downarrow & & \downarrow \\ H_k & \longrightarrow & B_k \end{array}$$

Proof: We use introduce intermediate objects $H_k \subset \overline{H}_k \subset \widetilde{H}_k$ and $O_{k-1} \subset \overline{O}_{k-1} \subset \widetilde{O}_{k-1}$.

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$$\begin{array}{ccccccc} \mathrm{O}_{k-1} & \longrightarrow & \overline{O}_{k-1} & \longrightarrow & \widetilde{H}_k & \longrightarrow & B_{k-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_k & \longrightarrow & \overline{H}_k & \longrightarrow & \widetilde{H}_k & \longrightarrow & B_k \end{array}$$

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