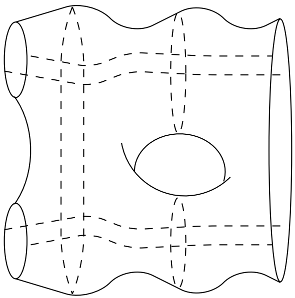
The geometric cobordism hypothesis
Lecture 1: Introduction

Daniel Grady, Dmitri Pavlov (Texas Tech University, Lubbock, TX)

These slides: `https://dmitripavlov.org/lecture-1.pdf`



Origins of functorial field theory

Origins of functorial field theory

- 1948 (Feynman): path integral formulation of quantum mechanics
- 1949 (Feynman–Kac): the Feynman–Kac formula

Origins of functorial field theory

- 1948 (Feynman): path integral formulation of quantum mechanics
- 1949 (Feynman–Kac): the Feynman–Kac formula
- Later: path integral used in QFT, no longer rigorous

Origins of functorial field theory

- 1948 (Feynman): path integral formulation of quantum mechanics
- 1949 (Feynman–Kac): the Feynman–Kac formula
- Later: path integral used in QFT, no longer rigorous
- 1980s (Witten): properties of path integrals for (conformal) field theory

Origins of functorial field theory

- 1948 (Feynman): path integral formulation of quantum mechanics
- 1949 (Feynman–Kac): the Feynman–Kac formula
- Later: path integral used in QFT, no longer rigorous
- 1980s (Witten): properties of path integrals for (conformal) field theory
- 1980s (Segal): mathematical formulation of conformal field theory

Definition

A conformal field theory is a symmetric monoidal functor

$$\mathrm{Bord} \to \mathrm{Vect}.$$

Bord: 1-manifolds and conformal 2-bordisms, with $\sqcup$.
Vect: vector spaces and linear maps, with $\otimes$.

Origins of functorial field theory

Definition

A conformal field theory is a symmetric monoidal functor

$$\mathrm{Bord} \to \mathrm{Vect}.$$

Bord: 1-manifolds and conformal 2-bordisms, with $\sqcup$.
Vect: vector spaces and linear maps, with $\otimes$.

Origins of functorial field theory

Definition

A conformal field theory is a symmetric monoidal functor

$$\mathrm{Bord} \to \mathrm{Vect}.$$

Bord: 1-manifolds and conformal 2-bordisms, with $\sqcup$.
Vect: vector spaces and linear maps, with $\otimes$.

Example (The σ-model)

- Σ: a worldvolume (later: an arbitrary bordism);
- X: a target space (later: a simplicial presheaf on manifolds; for Chern–Simons $X = \mathrm{B}_\nabla G$);
- $X \to B$: background gauge field (e.g., $B = \mathrm{B}^d_\nabla \mathrm{U}(1)$).

Origins of functorial field theory

Definition

A conformal field theory is a symmetric monoidal functor

$$\mathrm{Bord} \to \mathrm{Vect}.$$

Example (The σ-model)

- Σ: a worldvolume (later: an arbitrary bordism);
- X: a target space (later: a simplicial presheaf on manifolds; for Chern–Simons $X = \mathsf{B}_\nabla G$);
- $X \to B$: background gauge field (e.g., $B = \mathsf{B}^d_\nabla \mathrm{U}(1)$).

The σ-model studies $\mathrm{Map}(\Sigma, X)$ equipped with the action functional induced by $X \to B$ (e.g., holonomy).

Origins of functorial field theory

Example (The σ-model)

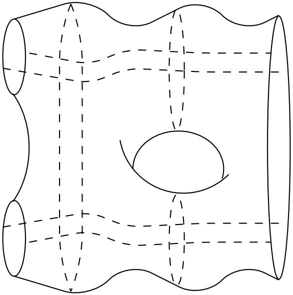
- Σ: a worldvolume (later: an arbitrary bordism);
- X: a target space (later: a simplicial presheaf on manifolds; for Chern–Simons $X = \mathrm{B}_\nabla G$);
- $X \to B$: background gauge field (e.g., $B = \mathrm{B}^d_\nabla \mathrm{U}(1)$).

The σ-model studies $\mathrm{Map}(\Sigma, X)$ equipped with the action functional induced by $X \to B$ (e.g., holonomy).

- 1-manifold $S \mapsto$ functions on $\mathrm{Map}(S, X)$
- 2-bordism $B\colon S_1 \to S_2 \mapsto$ linear map (pull-push)

$$\mathrm{Map}(S_1, X) \to \mathrm{Map}(B, X) \to \mathrm{Map}(S_2, X).$$

How to compose bordisms



Further developments

Further developments

- 1980s (Atiyah, Kontsevich, ...): topological theories: easier to construct and study, but less relevant to physics

Further developments

- 1980s (Atiyah, Kontsevich, . . .): topological theories: easier to construct and study, but less relevant to physics
- 1992 (Freed, Lawrence): extended field theories (correspond to locality in physics)

Further developments

- 1980s (Atiyah, Kontsevich, . . .): topological theories: easier to construct and study, but less relevant to physics
- 1992 (Freed, Lawrence): extended field theories (correspond to locality in physics)
- 1995 (Baez–Dolan): the topological cobordism and tangle hypotheses

Further developments

- 1980s (Atiyah, Kontsevich, ...): topological theories: easier to construct and study, but less relevant to physics
- 1992 (Freed, Lawrence): extended field theories (correspond to locality in physics)
- 1995 (Baez–Dolan): the topological cobordism and tangle hypotheses
- 2002 (Stolz–Teichner): modern formulation of nontopological field theories (including supersymmetry); the Stolz–Teichner program

Further developments

- 1980s (Atiyah, Kontsevich, ...): topological theories: easier to construct and study, but less relevant to physics
- 1992 (Freed, Lawrence): extended field theories (correspond to locality in physics)
- 1995 (Baez–Dolan): the topological cobordism and tangle hypotheses
- 2002 (Stolz–Teichner): modern formulation of nontopological field theories (including supersymmetry); the Stolz–Teichner program
- 2004 (Costello): the $(\infty, 2)$-category of topological 2-dimensional bordisms
- 2006 (Hopkins–Lurie); 2015 (Calaque–Scheimbauer): the (∞, d)-category of topological bordisms

Summary of necessary features

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Homotopy: need chain complexes to encode BV-BRST

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Homotopy: need chain complexes to encode BV-BRST
  $\Longrightarrow$ must encode (higher) diffeomorphisms between bordisms

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Homotopy: need chain complexes to encode BV-BRST
  $\Longrightarrow$ must encode (higher) diffeomorphisms between bordisms
  $\Longrightarrow$ symmetric monoidal (∞, d)-categories

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Homotopy: need chain complexes to encode BV-BRST
  $\Longrightarrow$ must encode (higher) diffeomorphisms between bordisms
  $\Longrightarrow$ symmetric monoidal (∞, d)-categories
- Geometric (nontopological) structures on bordisms: Riemannian/Lorentzian metrics, complex/conformal/symplectic structures,

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Homotopy: need chain complexes to encode BV-BRST
  $\Longrightarrow$ must encode (higher) diffeomorphisms between bordisms
  $\Longrightarrow$ symmetric monoidal (∞, d)-categories
- Geometric (nontopological) structures on bordisms:
  Riemannian/Lorentzian metrics,
  complex/conformal/symplectic structures,
  principal G-bundles with connection,

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Homotopy: need chain complexes to encode BV-BRST
  $\Longrightarrow$ must encode (higher) diffeomorphisms between bordisms
  $\Longrightarrow$ symmetric monoidal (∞, d)-categories
- Geometric (nontopological) structures on bordisms:
  Riemannian/Lorentzian metrics,
  complex/conformal/symplectic structures,
  principal G-bundles with connection,
  higher gauge fields (Kalb–Ramond, Ramond–Ramond)

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\implies$ symmetric monoidal d-category of bordisms
- Homotopy: need chain complexes to encode BV-BRST
  $\implies$ must encode (higher) diffeomorphisms between bordisms
  $\implies$ symmetric monoidal (∞, d)-categories
- Geometric (nontopological) structures on bordisms:
  Riemannian/Lorentzian metrics,
  complex/conformal/symplectic structures,
  principal G-bundles with connection,
  higher gauge fields (Kalb–Ramond, Ramond–Ramond)
  $\implies$ an $(\infty, 1)$-sheaf of geometric structures

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Homotopy: need chain complexes to encode BV-BRST
  $\Longrightarrow$ must encode (higher) diffeomorphisms between bordisms
  $\Longrightarrow$ symmetric monoidal (∞, d)-categories
- Geometric (nontopological) structures on bordisms:
  Riemannian/Lorentzian metrics,
  complex/conformal/symplectic structures,
  principal G-bundles with connection,
  higher gauge fields (Kalb–Ramond, Ramond–Ramond)
  $\Longrightarrow$ an $(\infty, 1)$-sheaf of geometric structures
- Smoothness: values of field theories depend smoothly on bordisms

Summary of necessary features

- Locality: need d-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Homotopy: need chain complexes to encode BV-BRST
  $\Longrightarrow$ must encode (higher) diffeomorphisms between bordisms
  $\Longrightarrow$ symmetric monoidal (∞, d)-categories
- Geometric (nontopological) structures on bordisms:
  Riemannian/Lorentzian metrics,
  complex/conformal/symplectic structures,
  principal G-bundles with connection,
  higher gauge fields (Kalb–Ramond, Ramond–Ramond)
  $\Longrightarrow$ an $(\infty, 1)$-sheaf of geometric structures
- Smoothness: values of field theories depend smoothly on bordisms
  $\Longrightarrow$ $(\infty, 1)$-sheaf of (∞, d)-categories of bordisms

Previous results on the topological cobordism hypothesis

- 2008 (Lurie): outline of a proof of the topological cobordism hypothesis
- 2017 (Ayala–Francis): a different proof, conditional on a conjecture
- 2004 (Costello), 2009 (Schommer-Pries):
  the 2-dimensional topological cobordism hypothesis
- 2006 (Galatius–Madsen–Tillmann–Weiss);
  2011 (Bökstedt–Madsen); 2017 (Schommer-Pries):
  the invertible case

Low-dimensional nontopological field theories

Examples of 2-dimensional nonextended nontopological field theories:

- 2007 (Pickrell): Riemannian 2-dimensional field theory
- 2018 (Runkel–Szegedy): volume-dependent 2-dimensional field theory

Classifications of holonomy maps, transport functors, and 1-dimensional nontopological field theories:

- 1990 (Barrett), 1994 (Caetano–Picken), 2007 (Schreiber–Waldorf): parallel transport for bundles
- 2000 (Mackaay–Picken), 2004 (Picken), 2008 (Schreiber–Waldorf): parallel transport for gerbes
- 2015 (Berwick-Evans–P.), 2020 (Ludewig–Stoffel): 1-dimensional field theories

Canonical example: Schrödinger quantum mechanics

Input data:

- $\mathcal{H}$: a vector space (state space);
- $H\colon \mathcal{H} \to \mathcal{H}$: a linear map (Hamiltonian).

Output data: a 1-dimensional oriented Riemannian functorial field theory:

- $(\mathbf{R}^0, +) \mapsto \mathcal{H}$, $(\mathbf{R}^0, -) \mapsto \mathcal{H}^*$;
- $I_t\colon (\mathbf{R}^0, +) \to (\mathbf{R}^0, 0) \mapsto \exp(tH/i\hbar)\colon \mathcal{H} \to \mathcal{H}$.

So what does “nontopological” mean?

So what does "nontopological" mean?

Definition

Given $d \geq 0$, the site Emb_d has

So what does “nontopological” mean?

Definition

Given $d \geq 0$, the site Emb_d has

- Objects: d-manifolds;

So what does "nontopological" mean?

Definition

Given $d \geq 0$, the site Emb_d has

- Objects: d-manifolds;
- Morphisms: open embeddings;

So what does "nontopological" mean?

Definition

Given $d \geq 0$, the site Emb_d has

- Objects: d-manifolds;
- Morphisms: open embeddings;
- Covering families: open covers.

So what does "nontopological" mean?

Definition

Given $d \geq 0$, the site Emb_d has

- Objects: d-manifolds;
- Morphisms: open embeddings;
- Covering families: open covers.

Definition

(Preliminary.) Given $d \geq 0$, a d-dimensional geometric structure is a simplicial presheaf on Emb_d:

$$\mathsf{Emb}_d^{\mathrm{op}} \to \mathrm{sSet}.$$

So what does “nontopological” mean?

Definition

(Preliminary.) Given $d \geq 0$, a d-dimensional geometric structure is a simplicial presheaf on Emb_d:

$$\mathsf{Emb}_d^{\mathrm{op}} \to \mathrm{sSet}.$$

(Topological structures send isotopic maps to homotopic maps.)

So what does "nontopological" mean?

Definition

(Preliminary.) Given $d \geq 0$, a d-dimensional geometric structure is a simplicial presheaf on Emb_d:

$$\mathsf{Emb}_d^{\mathrm{op}} \to \mathrm{sSet}.$$

Example

- Send a d-manifold M to the set of Riemannian metrics on M;
- Send an open embedding $M \to N$ of d-manifolds to the map that restricts a metric from N to M.

Examples of geometric structures

- Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;

Examples of geometric structures

- Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- conformal, complex, symplectic, contact, Kähler structures;

Examples of geometric structures

- Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- conformal, complex, symplectic, contact, Kähler structures;
- foliations, possibly with transversal metrics;

Examples of geometric structures

- Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- conformal, complex, symplectic, contact, Kähler structures;
- foliations, possibly with transversal metrics;
- smooth map to a target manifold M (traditional σ-model);

Examples of geometric structures

- Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- conformal, complex, symplectic, contact, Kähler structures;
- foliations, possibly with transversal metrics;
- smooth map to a target manifold M (traditional σ-model);
- smooth map to an orbifold or ∞-sheaf on manifolds;

Examples of geometric structures

- Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- conformal, complex, symplectic, contact, Kähler structures;
- foliations, possibly with transversal metrics;
- smooth map to a target manifold M (traditional σ-model);
- smooth map to an orbifold or ∞-sheaf on manifolds;
- etale map or an open embedding into a target manifold N;

Examples of geometric structures

- Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- conformal, complex, symplectic, contact, Kähler structures;
- foliations, possibly with transversal metrics;
- smooth map to a target manifold M (traditional σ-model);
- smooth map to an orbifold or ∞-sheaf on manifolds;
- etale map or an open embedding into a target manifold N;
- topological structures: orientation, framing, etc.

Examples of geometric structures

- Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- conformal, complex, symplectic, contact, Kähler structures;
- foliations, possibly with transversal metrics;
- smooth map to a target manifold M (traditional σ-model);
- smooth map to an orbifold or ∞-sheaf on manifolds;
- etale map or an open embedding into a target manifold N;
- topological structures: orientation, framing, etc.
- differential n-forms (possibly closed).

Examples of geometric structures: gauge transformations

Examples of geometric structures: gauge transformations

Definition

Examples of geometric structures: gauge transformations

Definition

- Send a d-manifold M to (the nerve of) the groupoid $\mathrm{B}_{\nabla} G(M)$:

Examples of geometric structures: gauge transformations

Definition

- Send a d-manifold M to (the nerve of) the groupoid $\mathrm{B}_{\nabla} G(M)$:
  - Objects: principal G-bundles with connection on M (gauge fields);

Examples of geometric structures: gauge transformations

Definition

- Send a d-manifold M to (the nerve of) the groupoid $\mathrm{B}_\nabla G(M)$:
    - Objects: principal G-bundles with connection on M (gauge fields);
    - Morphisms: connection-preserving isomorphisms (gauge transformations).

Examples of geometric structures: gauge transformations

Definition

- Send a d-manifold M to (the nerve of) the groupoid $\mathrm{B}_{\nabla}G(M)$:
  - Objects: principal G-bundles with connection on M (gauge fields);
  - Morphisms: connection-preserving isomorphisms (gauge transformations).
- Send an open embedding of d-manifolds $M \to N$ to the restriction functor.

Examples of geometric structures: (higher) gauge transformations

Examples of geometric structures: (higher) gauge transformations

- Principal G-bundles with connection on M (gauge fields, e.g., the electromagnetic field);

Examples of geometric structures: (higher) gauge transformations

- Principal G-bundles with connection on M (gauge fields, e.g., the electromagnetic field);
- Bundle gerbe with connection on M (B-field, Kalb–Ramond field).

Examples of geometric structures: (higher) gauge transformations

- Principal G-bundles with connection on M (gauge fields, e.g., the electromagnetic field);
- Bundle gerbe with connection on M (B-field, Kalb–Ramond field).
- Bundle 2-gerbe with connection on M (supergravity C-field).

Examples of geometric structures: (higher) gauge transformations

- Principal G-bundles with connection on M (gauge fields, e.g., the electromagnetic field);
- Bundle gerbe with connection on M (B-field, Kalb–Ramond field).
- Bundle 2-gerbe with connection on M (supergravity C-field).
- Bundle $(d-1)$-gerbes with connection on M (Deligne cohomology, Cheeger–Simons characters, ordinary differential cohomology, circle d-bundles).

Examples of geometric structures: (higher) gauge transformations

- Principal G-bundles with connection on M (gauge fields, e.g., the electromagnetic field);
- Bundle gerbe with connection on M (B-field, Kalb–Ramond field).
- Bundle 2-gerbe with connection on M (supergravity C-field).
- Bundle $(d-1)$-gerbes with connection on M (Deligne cohomology, Cheeger–Simons characters, ordinary differential cohomology, circle d-bundles).
- Geometric tangential structures: geometric Spin^c-structure, String (Waldorf), Fivebrane (Sati–Schreiber–Stasheff), Ninebrane (Sati). (Vanishing of anomaly.)

Examples of geometric structures: (higher) gauge transformations

- Principal G-bundles with connection on M (gauge fields, e.g., the electromagnetic field);
- Bundle gerbe with connection on M (B-field, Kalb–Ramond field).
- Bundle 2-gerbe with connection on M (supergravity C-field).
- Bundle $(d-1)$-gerbes with connection on M (Deligne cohomology, Cheeger–Simons characters, ordinary differential cohomology, circle d-bundles).
- Geometric tangential structures: geometric Spin^c-structure, String (Waldorf), Fivebrane (Sati–Schreiber–Stasheff), Ninebrane (Sati). (Vanishing of anomaly.)
- differential K-theory (Ramond–Ramond field). Requires ∞-groupoids.

Smoothness of geometric structures

Smoothness of geometric structures

Defect: a FFT can map a smooth family of geometric structures to a nonsmooth family of values.

Smoothness of geometric structures

Defect: a FFT can map a smooth family of geometric structures to a nonsmooth family of values.
Solution: make everything fibered over smooth manifolds (equivalently: cartesian spaces)

Smoothness of geometric structures

Defect: a FFT can map a smooth family of geometric structures to a nonsmooth family of values.
Solution: make everything fibered over smooth manifolds (equivalently: cartesian spaces)

Definition

Given $d \geq 0$, the site FEmb_d has

Smoothness of geometric structures

Defect: a FFT can map a smooth family of geometric structures to a nonsmooth family of values.
Solution: make everything fibered over smooth manifolds (equivalently: cartesian spaces)

Definition

Given $d \geq 0$, the site FEmb_d has

- Objects: submersions $T \to U$ with d-dimensional fibers, where $U \cong \mathbf{R}^n$ is a cartesian manifold;

Smoothness of geometric structures

Defect: a FFT can map a smooth family of geometric structures to a nonsmooth family of values.
Solution: make everything fibered over smooth manifolds (equivalently: cartesian spaces)

Definition

Given $d \geq 0$, the site FEmb_d has

- Objects: submersions $T \to U$ with d-dimensional fibers, where $U \cong \mathbf{R}^n$ is a cartesian manifold;
- Morphisms: commutative squares with $T \to T'$ a fiberwise open embedding over a smooth map $U \to U'$;

Smoothness of geometric structures

Definition

Given $d \geq 0$, the site FEmb_d has

- Objects: submersions $T \to U$ with d-dimensional fibers, where $U \cong \mathbf{R}^n$ is a cartesian manifold;
- Morphisms: commutative squares with $T \to T'$ a fiberwise open embedding over a smooth map $U \to U'$;
- Covering families: open covers on total spaces T.

Definition

Given $d \geq 0$, a d-dimensional geometric structure is a simplicial presheaf on FEmb_d:

$$\mathcal{S} \colon \mathsf{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}.$$

Smoothness of geometric structures

Definition

Given $d \geq 0$, the site FEmb_d has

- Objects: submersions with d-dimensional fibers;
- Morphisms: fiberwise open embeddings;
- Covering families: open covers on total spaces T.

Definition

Given $d \geq 0$, a d-dimensional geometric structure is a simplicial presheaf $\mathcal{S}\colon \mathsf{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$.

Example:

- $T \to U \mapsto$ the set of fiberwise Riemannian metrics on $T \to U$;
- $(T \to T', U \to U') \mapsto$ the restriction map from T' to T.

Smoothness in action

- Fix a target manifold M;

Smoothness in action

- Fix a target manifold M;
- $\mathcal{S}(T \to U) = \mathrm{C}^{\infty}(T, M)$ with a choice of fiberwise orientation on $T \to U$;

Smoothness in action

- Fix a target manifold M;
- $\mathcal{S}(T \to U) = \mathrm{C}^\infty(T, M)$ with a choice of fiberwise orientation on $T \to U$;
- T: the smooth category of vector spaces;

Smoothness in action

- Fix a target manifold M;
- $\mathcal{S}(T \to U) = \mathrm{C}^\infty(T, M)$ with a choice of fiberwise orientation on $T \to U$;
- T: the smooth category of vector spaces;
- $d = 1$.

Smoothness in action

- Fix a target manifold M;
- $\mathcal{S}(T \to U) = \mathrm{C}^\infty(T, M)$ with a choice of fiberwise orientation on $T \to U$;
- T: the smooth category of vector spaces;
- $d = 1$.

Theorem (Kobayashi, Schreiber–Waldorf, Berwick-Evans–P., Ludewig–Stoffel)

The groupoid $\mathsf{FFT}_{1,\mathcal{S},T}$ is equivalent to the groupoid of finite-dimensional vector bundles with connection over M and connection-preserving isomorphisms.

Smoothness in action

- Fix a target manifold M;
- $\mathcal{S}(T \to U) = \mathrm{C}^\infty(T, M)$ with fiberwise orientation;
- T: the smooth category of vector spaces;

Theorem (Kobayashi, Schreiber–Waldorf, Berwick-Evans–P., Ludewig–Stoffel)

The groupoid $\mathsf{FFT}_{1,\mathcal{S},T}$ *is equivalent to the groupoid of finite-dimensional vector bundles with connection over* M *and connection-preserving isomorphisms.*

Construction:

- Start with a vector bundle $V \to M$ with connection.

Smoothness in action

- Fix a target manifold M;
- $\mathcal{S}(T \to U) = \mathrm{C}^\infty(T, M)$ with fiberwise orientation;
- T: the smooth category of vector spaces;

Theorem (Kobayashi, Schreiber–Waldorf, Berwick-Evans–P., Ludewig–Stoffel)

The groupoid $\mathsf{FFT}_{1,\mathcal{S},T}$ *is equivalent to the groupoid of finite-dimensional vector bundles with connection over* M *and connection-preserving isomorphisms.*

Construction:

- Start with a vector bundle $V \to M$ with connection.
- Send an object $P\colon \mathbf{R}^0 \to M$ to the fiber $V_P \in T(\mathbf{R}^0)$.

Smoothness in action

- Fix a target manifold M;
- $\mathcal{S}(T \to U) = \mathrm{C}^\infty(T, M)$ with fiberwise orientation;
- T: the smooth category of vector spaces;

Theorem (Kobayashi, Schreiber–Waldorf, Berwick-Evans–P., Ludewig–Stoffel)

The groupoid $\mathsf{FFT}_{1,\mathcal{S},T}$ is equivalent to the groupoid of finite-dimensional vector bundles with connection over M and connection-preserving isomorphisms.

- Start with a vector bundle $V \to M$ with connection.
- Send an object $P\colon \mathbf{R}^0 \to M$ to the fiber $V_P \in T(\mathbf{R}^0)$.
- Send a smooth family of objects $P\colon \mathbf{R}^n \to M$ to the pullback vector bundle $P^*V \in T(\mathbf{R}^n)$.

Smoothness in action

- Fix a target manifold M;
- $\mathcal{S}(T \to U) = \mathrm{C}^\infty(T, M)$ with fiberwise orientation;
- T: the smooth category of vector spaces;

Theorem (Kobayashi, Schreiber–Waldorf, Berwick-Evans–P., Ludewig–Stoffel)

The groupoid $\mathrm{FFT}_{1,\mathcal{S},T}$ is equivalent to the groupoid of finite-dimensional vector bundles with connection over M and connection-preserving isomorphisms.

- Start with a vector bundle $V \to M$ with connection.
- Send an object $P\colon \mathbf{R}^0 \to M$ to the fiber $V_P \in T(\mathbf{R}^0)$.
- Send a smooth family of objects $P\colon \mathbf{R}^n \to M$ to the pullback vector bundle $P^*V \in T(\mathbf{R}^n)$.
- Send a bordism $p\colon [0,1] \to M$ to the parallel transport map $V_{p(0)} \to V_{p(1)}$.

Smoothness in action

- Fix a target manifold M;
- $\mathcal{S}(T \to U) = \mathrm{C}^\infty(T, M)$ with fiberwise orientation;
- T: the smooth category of vector spaces;

Theorem (Kobayashi, Schreiber–Waldorf, Berwick-Evans–P., Ludewig–Stoffel)

The groupoid $\mathsf{FFT}_{1,\mathcal{S},T}$ is equivalent to the groupoid of finite-dimensional vector bundles with connection over M and connection-preserving isomorphisms.

- Start with a vector bundle $V \to M$ with connection.
- smooth family of objects $(P\colon \mathbf{R}^n \to M) \mapsto P^*V \in T(\mathbf{R}^n)$.
- Send a bordism $p\colon [0,1] \to M$ to the parallel transport map $V_{p(0)} \to V_{p(1)}$.
- Send a smooth family of bordisms $p\colon \mathbf{R}^n \times [0,1] \to M$ to the smooth map of bundles $p(-,0)^*V \to p(-,1)^*V$.

The main theorem

Ingredients:

The main theorem

Ingredients:

- A dimension $d \geq 0$.

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
  - k-morphisms $(k \leq d)$: k-dimensional bordisms

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
  - k-morphisms $(k \leq d)$: k-dimensional bordisms
    - with d-dimensional germs of an $\mathcal{S}$-structure;

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
    - k-morphisms $(k \leq d)$: k-dimensional bordisms
        - with d-dimensional germs of an $\mathcal{S}$-structure;
        - corners of all codimensions, arranged in a d-dimensional grid;

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
  - k-morphisms ($k \leq d$): k-dimensional bordisms
    - with d-dimensional germs of an $\mathcal{S}$-structure;
    - corners of all codimensions, arranged in a d-dimensional grid;
  - Monoidal product: disjoint union of bordisms.

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
  - k-morphisms ($k \leq d$): k-dimensional bordisms
    - with d-dimensional germs of an $\mathcal{S}$-structure;
    - corners of all codimensions, arranged in a d-dimensional grid;
  - Monoidal product: disjoint union of bordisms.
  - Composition: gluing of bordisms (in any of the d directions).

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
  - k-morphisms ($k \leq d$): k-dimensional bordisms
    - with d-dimensional germs of an $\mathcal{S}$-structure;
    - corners of all codimensions, arranged in a d-dimensional grid;
  - Monoidal product: disjoint union of bordisms.
  - Composition: gluing of bordisms (in any of the d directions).
  - Bordisms come in smooth families over V:

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
  - k-morphisms ($k \leq d$): k-dimensional bordisms
    - with d-dimensional germs of an $\mathcal{S}$-structure;
    - corners of all codimensions, arranged in a d-dimensional grid;
  - Monoidal product: disjoint union of bordisms.
  - Composition: gluing of bordisms (in any of the d directions).
  - Bordisms come in smooth families over V:
    - can be pulled back along smooth maps $U \to V$;

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
  - k-morphisms ($k \leq d$): k-dimensional bordisms
    - with d-dimensional germs of an $\mathcal{S}$-structure;
    - corners of all codimensions, arranged in a d-dimensional grid;
  - Monoidal product: disjoint union of bordisms.
  - Composition: gluing of bordisms (in any of the d directions).
  - Bordisms come in smooth families over V:
    - can be pulled back along smooth maps $U \to V$;
    - can be glued along open covers of V;

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
  - k-morphisms ($k \leq d$): k-dimensional bordisms
    - with d-dimensional germs of an $\mathcal{S}$-structure;
    - corners of all codimensions, arranged in a d-dimensional grid;
  - Monoidal product: disjoint union of bordisms.
  - Composition: gluing of bordisms (in any of the d directions).
  - Bordisms come in smooth families over V:
    - can be pulled back along smooth maps $U \to V$;
    - can be glued along open covers of V;
  - k-morphisms ($k > d$): higher gauge transformations from $\mathcal{S}$;

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
  - k-morphisms $(k \leq d)$: k-dimensional bordisms
    - with d-dimensional germs of an $\mathcal{S}$-structure;
    - corners of all codimensions, arranged in a d-dimensional grid;
  - Monoidal product: disjoint union of bordisms.
  - Composition: gluing of bordisms (in any of the d directions).
  - Bordisms come in smooth families over V:
    - can be pulled back along smooth maps $U \to V$;
    - can be glued along open covers of V;
  - k-morphisms $(k > d)$: higher gauge transformations from $\mathcal{S}$;
  - k-morphisms $(k > d)$: homotopies of cuts.

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}$.
- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.

Definition

A d-dimensional functorial field theory valued in $\mathcal{V}$ with geometric structure $\mathcal{S}$ is a smooth symmetric monoidal (∞, d)-functor

$$\mathfrak{Bord}_d^{\mathcal{S}} \to \mathcal{V}.$$

The main theorem

Definition

A *d-dimensional functorial field theory valued in* $\mathcal{V}$ *with geometric structure* $\mathcal{S}$ is a smooth symmetric monoidal (∞, d)-functor

$$\mathfrak{Bord}_d^{\mathcal{S}} \to \mathcal{V}.$$

Definition

The simplicial set of d-dimensional functorial field theories valued in $\mathcal{V}$ with geometric structure $\mathcal{S}$ is the derived mapping simplicial set

$$\mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) = \mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}).$$

Can be refined to a derived internal hom.

The main theorem

Definition

The simplicial set of d-dimensional functorial field theories valued in $\mathcal{V}$ with geometric structure $\mathcal{S}$ is the derived mapping simplicial set

$$\mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) = \mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}).$$

Can be refined to a derived internal hom.

Theorem (G.–P., The geometric cobordism hypothesis)

Part I (Lecture 3): $\mathfrak{Bord}_d$ is a left adjoint functor:

$$\mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}) \simeq \mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}_d^{\times}),$$

where $\mathcal{V}_d^{\times} = \mathsf{FFT}_{d,\mathcal{V}}$, i.e., $\mathcal{V}_d^{\times}(T \to U) = \mathsf{FFT}_{d,\mathcal{V}}(T \to U)$.

The main theorem

Theorem (G.–P., The geometric cobordism hypothesis)

Part I (Lecture 3): $\mathfrak{Bord}_d$ is a left adjoint functor:

$$\mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}) \simeq \mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}_d^{\times}),$$

where $\mathcal{V}_d^{\times} = \mathsf{FFT}_{d,\mathcal{V}}$, i.e., $\mathcal{V}_d^{\times}(T \to U) = \mathsf{FFT}_{d,\mathcal{V}}(T \to U)$.

Part II (Lecture 4): The evaluation-at-points map

$$\mathcal{V}_d^{\times}(\mathbf{R}^d \times U \to U) = \mathsf{FFT}_{d,\mathcal{V}}(\mathbf{R}^d \times U \to U) \to \mathcal{V}^{\times}(U)$$

is a weak equivalence of simplicial sets.

Example: the prequantum Chern–Simons theory

Example: the prequantum Chern–Simons theory

Input data:

Example: the prequantum Chern–Simons theory

Input data:

- G: a Lie group with Lie algebra $\mathfrak{g}$;

Example: the prequantum Chern–Simons theory

Input data:

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;

Example: the prequantum Chern–Simons theory

Input data:

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathrm{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -, - \rangle]$.

Example: the prequantum Chern–Simons theory

Input data:

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathrm{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -, - \rangle]$.
- $\mathcal{S} = \mathrm{B}_\nabla G$ (principal G-bundles with connection; gauge fields);

Example: the prequantum Chern–Simons theory

Input data:

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathrm{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -, - \rangle]$.
- $\mathcal{S} = \mathrm{B}_\nabla G$ (principal G-bundles with connection; gauge fields);
- $\mathcal{V} = \mathrm{B}^3\mathrm{U}(1)$ (a single k-morphism for $k < 3$; 3-morphisms are $\mathrm{U}(1)$).

Example: the prequantum Chern–Simons theory

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathrm{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -, - \rangle]$.

Example: the prequantum Chern–Simons theory

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathsf{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -, - \rangle]$.

Part I: $\mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_3^{\mathsf{B}_\nabla G}, \mathsf{B}^3\mathrm{U}(1)) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times)$

Example: the prequantum Chern–Simons theory

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathsf{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -, - \rangle]$.

Part I: $\mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_3^{\mathsf{B}_\nabla G}, \mathsf{B}^3\mathrm{U}(1)) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times)$

Part II: $(\mathsf{B}^3\mathrm{U}(1))_3^\times(\mathbf{R}^3 \times U \to U) \simeq \mathsf{B}^3\mathrm{C}^\infty(U, \mathrm{U}(1))$ (P.–Stolz–Teichner)

Example: the prequantum Chern–Simons theory

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathsf{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -, - \rangle]$.

Part I: $\mathbf{R}\operatorname{Map}(\mathfrak{Bord}_3^{\mathsf{B}_\nabla G}, \mathsf{B}^3\mathrm{U}(1)) \simeq \mathbf{R}\operatorname{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times)$

Part II: $(\mathsf{B}^3\mathrm{U}(1))_3^\times(\mathbf{R}^3 \times U \to U) \simeq \mathsf{B}^3\mathrm{C}^\infty(U, \mathrm{U}(1))$ (P.–Stolz–Teichner)

Poincaré: $\mathbf{R}\operatorname{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times) \simeq \mathbf{R}\operatorname{Map}(\mathsf{B}_\nabla G, \mathsf{B}^3\mathrm{C}^\infty(-, \mathrm{U}(1)))$

$$\simeq \mathbf{R}\operatorname{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{Z}) \times^h_{\mathbf{R}\operatorname{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{R})} \mathbf{R}\operatorname{Map}(\mathsf{B}_\nabla G, \Omega^4_{\mathrm{cl}})$$

Example: the prequantum Chern–Simons theory

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathsf{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -, - \rangle]$.

Part I: $\mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_3^{\mathsf{B}_\nabla G}, \mathsf{B}^3\mathrm{U}(1)) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times)$

Part II: $(\mathsf{B}^3\mathrm{U}(1))_3^\times(\mathbf{R}^3 \times U \to U) \simeq \mathsf{B}^3\mathrm{C}^\infty(U, \mathrm{U}(1))$ (P.–Stolz–Teichner)

Poincaré: $\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^3\mathrm{C}^\infty(-, \mathrm{U}(1)))$

$$\simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{Z}) \times^h_{\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{R})} \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \Omega^4_{\mathsf{cl}})$$

Freed–Hopkins: $\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \Omega^4_{\mathsf{cl}}) \simeq (\mathrm{Sym}^2\,\mathfrak{g}^*)^G \ni \langle -, - \rangle$

Example: the prequantum Chern–Simons theory

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -, - \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathsf{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -, - \rangle]$.

Part I: $\mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_3^{\mathsf{B}_\nabla G}, \mathsf{B}^3\mathrm{U}(1)) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times)$

Part II: $(\mathsf{B}^3\mathrm{U}(1))_3^\times(\mathbf{R}^3 \times U \to U) \simeq \mathsf{B}^3\mathrm{C}^\infty(U, \mathrm{U}(1))$ (P.–Stolz–Teichner)

Poincaré: $\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^3\mathrm{C}^\infty(-, \mathrm{U}(1)))$

$$\simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{Z}) \times^h_{\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{R})} \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \Omega^4_{\mathsf{cl}})$$

Freed–Hopkins: $\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \Omega^4_{\mathsf{cl}}) \simeq (\mathrm{Sym}^2\,\mathfrak{g}^*)^G \ni \langle -, - \rangle$

$\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{Z}) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}G, \mathrm{K}(\mathbf{Z}, 4)) \ni c$

Example: the prequantum Chern–Simons theory

- G: a Lie group with Lie algebra $\mathfrak{g}$;
- $\langle -,- \rangle$: a G-invariant polynomial of degree 2 on $\mathfrak{g}$;
- a level $c \in \mathrm{H}^4(\mathsf{B}G, \mathbf{Z})$ such that $[c]_{\mathbf{R}} = [\langle -,- \rangle]$.

Part I: $\mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_3^{\mathsf{B}_\nabla G}, \mathsf{B}^3\mathrm{U}(1)) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times)$

Part II: $(\mathsf{B}^3\mathrm{U}(1))_3^\times(\mathbf{R}^3 \times U \to U) \simeq \mathsf{B}^3\mathrm{C}^\infty(U, \mathrm{U}(1))$ (P.–Stolz–Teichner)

Poincaré: $\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, (\mathsf{B}^3\mathrm{U}(1))_3^\times) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^3\mathrm{C}^\infty(-, \mathrm{U}(1)))$

$$\simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{Z}) \times^h_{\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{R})} \mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \Omega^4_{\mathsf{cl}})$$

Freed–Hopkins: $\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \Omega^4_{\mathsf{cl}}) \simeq (\mathrm{Sym}^2\,\mathfrak{g}^*)^G \ni \langle -,- \rangle$

$\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{Z}) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}G, \mathrm{K}(\mathbf{Z}, 4)) \ni c$

$\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, \mathsf{B}^4\mathbf{R}) \simeq \mathbf{R}\,\mathrm{Map}(\mathsf{B}G, \mathrm{K}(\mathbf{R}, 4)) \ni [c]_{\mathbf{R}} \simeq [\langle -,- \rangle]$

Remark: Also works for higher dimensions and Lie ∞-groups!

Applications (current and future)

- Consequence of the GCH: smooth invertible FFTs are classified by the smooth Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)

Applications (current and future)

- Consequence of the GCH: smooth invertible FFTs are classified by the smooth Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)
- The Stolz–Teichner conjecture: concordance classes of extended FFTs have a classifying space. (Proof: Locality + the shape theorem (Berwick-Evans–Boavida de Brito–P.).
- Construction of power operations on the level of FFTs (extending Barthel–Berwick-Evans–Stapleton).

Applications (current and future)

- Consequence of the GCH: smooth invertible FFTs are classified by the smooth Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)
- The Stolz–Teichner conjecture: concordance classes of extended FFTs have a classifying space. (Proof: Locality + the shape theorem (Berwick-Evans–Boavida de Brito–P.).
- Construction of power operations on the level of FFTs (extending Barthel–Berwick-Evans–Stapleton).
- (G.) The Freed–Hopkins conjecture (Conjecture 8.37 in *Reflection positivity and invertible topological phases*)

Applications (current and future)

- Consequence of the GCH: smooth invertible FFTs are classified by the smooth Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)
- The Stolz–Teichner conjecture: concordance classes of extended FFTs have a classifying space. (Proof: Locality + the shape theorem (Berwick-Evans–Boavida de Brito–P.).
- Construction of power operations on the level of FFTs (extending Barthel–Berwick-Evans–Stapleton).
- (G.) The Freed–Hopkins conjecture (Conjecture 8.37 in *Reflection positivity and invertible topological phases*)
- Examples of nontopological FFTs: bundle $(d-1)$-gerbes with connection (P.–Stolz–Teichner)

Applications (current and future)

- Consequence of the GCH: smooth invertible FFTs are classified by the smooth Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)
- The Stolz–Teichner conjecture: concordance classes of extended FFTs have a classifying space. (Proof: Locality + the shape theorem (Berwick-Evans–Boavida de Brito–P.).
- Construction of power operations on the level of FFTs (extending Barthel–Berwick-Evans–Stapleton).
- (G.) The Freed–Hopkins conjecture (Conjecture 8.37 in *Reflection positivity and invertible topological phases*)
- Examples of nontopological FFTs: bundle $(d-1)$-gerbes with connection (P.–Stolz–Teichner)
- Quantization of functorial field theories.

Tools

Tools

- Higher category theory;

Tools

- Higher category theory;
- Sheaves and stacks (∞-sheaves);

Tools

- Higher category theory;
- Sheaves and stacks (∞-sheaves);
- Simplicial sets and simplicial presheaves;

Tools

- Higher category theory;
- Sheaves and stacks (∞-sheaves);
- Simplicial sets and simplicial presheaves;
- Elementary differential topology;

Tools

- Higher category theory;
- Sheaves and stacks (∞-sheaves);
- Simplicial sets and simplicial presheaves;
- Elementary differential topology;
- Elementary Morse theory;

Tools

- Higher category theory;
- Sheaves and stacks (∞-sheaves);
- Simplicial sets and simplicial presheaves;
- Elementary differential topology;
- Elementary Morse theory;
- Homotopy theory.

A preview of things to come...

Lecture 2: Definitions and examples
Lecture 3: Locality (Part I)
Lecture 4: The framed case (Part II)