Spring 2011, Math 276: Index Theory, Homework 4

Please submit by whatever date you deem appropriate.

Problem 9: Chern-Weil theory. Recall that a connection on a vector bundle E over a manifold M can be seen as a degree 1 endomorphism ∇ of the complex $\Omega(M) \otimes_{C^\infty(M)} C^\infty(E)$ that satisfies Leibniz identity. The endomorphism ∇^2 is $C^\infty(M)$-linear and therefore can be seen as an element of $\Omega^2(M) \otimes_{C^\infty(M)} \mathrm{End}(E)$, which we denote by $R(\nabla)$ and call the *curvature* of ∇. The nth power of $R(\nabla)$ can be defined as the element of $\Omega^{2n}(M, \mathrm{End}(E))$ that corresponds to ∇^{2n}. We equip $\Omega(M, \mathrm{End}(E))$ with the obvious trace and Lie bracket. Finally we observe that we can substitute $R(\nabla)$ into any $f \in \mathbf{k}[[x]]$ in the obvious way, where $\mathbf{k}$ is the field of coefficients (i.e., real or complex numbers).

(a) Prove that for any $f \in \mathbf{k}[[x]]$ the form $\mathrm{tr}(f(R(\nabla)))$ is closed. Prove that if ∇' is a different connection on E, then the difference $\mathrm{tr}(f(R(\nabla))) - \mathrm{tr}(f(R(\nabla')))$ is exact. Conclude that any $f \in \mathbf{k}[[x]]$ gives a cohomology class that does not depend on the choice of ∇. Hint: Connections form an affine space and any two connections can be connected by a path.

(b) Prove that the formal powers series $\exp(ix/2\pi) \in \mathbf{C}[[x]]$ gives the Chern character. Prove that the total Chern class is the exponent of the class given by the formal power series $\log(1 + ix/2\pi) \in \mathbf{C}[[x]]$. Prove that the total Pontryagin class is the exponent of the class given by the formal power series $\log((1 - (x/2\pi)^2)^{1/2}) \in \mathbf{R}[[x]]$. Is there a power series that gives the Euler class?

Problem 10: Chern characters and the topological index of operators on trivial line bundles.

(a) Define the *odd Chern character* as the composition of maps $\mathrm{K}^{\mathrm{odd}}(X) \to \tilde{\mathrm{K}}^{\mathrm{even}}(SX) \to \mathrm{H}^{\mathrm{even}}(SX) \to \mathrm{H}^{\mathrm{odd}}(X)$. Prove that Chern characters combine into a homomorphism of graded rings $\mathrm{K}^{\mathrm{even/odd}}(X) \to \mathrm{H}^{\mathrm{even/odd}}(X)$. Extend Chern character to the relative setting and prove that the diagram consisting of the long exact sequences for K-theory and ordinary cohomology connected by Chern character is commutative.

(b) Use part (a) to prove that the topological index of every elliptic operator from any trivial line bundle to itself vanishes.