Spring 2011, Math 276: Index Theory, Homework 2

Please submit by February 8 and contact Dmitri Pavlov (pavlov@math) for all questions about homework.

Problem 4: The 6-term exact sequence. Suppose X is a compact Hausdorff space. In this problem we consider automorphisms of vector bundles over X. Two automorphism $a: E \to E$ and $b: F \to F$ are isomorphic if there is an isomorphism $c: E \to F$ such that $ca = bc$. Moreover, the sum of a and b is $a \oplus b: E \oplus F \to E \oplus F$. An automorphism $a: E \to E$ is *elementary* if it is homotopic to the identity in the space of all automorphisms of E.

(a) Prove that the quotient $\mathrm{K}^{-1}(X)$ of the commutative monoid of isomorphism classes of automorphisms of vector bundles over X by the submonoid of isomorphism classes of elementary automorphisms is a commutative group. Describe the negation map of this group. Prove that homotopic automorphisms represent the same element in this group. Is the converse of this statement true?

(b) Suppose $Y \subset X$ is closed. Define $\mathrm{K}^{-1}(X, Y)$ in exactly the same way as $\mathrm{K}^{-1}(X)$ except that all automorphisms $a: E \to E$ of a vector bundle E on X must restrict to the identity automorphism on Y and all homotopies must stay in the space of such automorphisms. Prove the analogues of the statements in part (a) for these relative groups.

(c) Define natural maps $\mathrm{K}^{-1}(Y) \to \mathrm{K}^0(X, Y)$ such that the following 6-term sequence is exact:

$$\mathrm{K}^{-1}(X, Y) \to \mathrm{K}^{-1}(X) \to \mathrm{K}^{-1}(Y) \to \mathrm{K}^0(X, Y) \to \mathrm{K}^0(X) \to \mathrm{K}^0(Y).$$

(d) Give an alternative definition of $\mathrm{K}^0(X, Y)$ in terms of chain complexes $0 \to E_0 \to E_1 \to \cdots \to E_k \to 0$ of vector bundles over X that are exact over Y. Hint: Combine E_i with the same parity of i together and choose hermitian inner products.

Problem 5: Computing $\mathrm{K}(X)$**.**

(a) Suppose that $Y \to X$ is a cofibration of compact Hausdorff spaces. Prove that $\mathrm{K}^i(X, Y) = \tilde{\mathrm{K}}^i(X/Y)$ for $i \in \{0, -1\}$. Here $\tilde{\mathrm{K}}^{-1} = \mathrm{K}^{-1}$.

(b) Suppose X is a compact Hausdorff space. Prove that $\mathrm{K}^{-1}(X) \cong \tilde{\mathrm{K}}(SX) \cong [X, \mathrm{GL}_\infty]$, where SX is the *unreduced suspension* of X, which is obtained from the space $X \times [0, 1]$ by collapsing $X \times \{0\}$ and $X \times \{1\}$ to points. Here GL_∞ is the colimit of groups $\mathrm{GL}(\mathbf{C}^n)$, where inclusions $\mathrm{GL}(\mathbf{C}^m) \to \mathrm{GL}(\mathbf{C}^n)$ are given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(c) Compute K^0 and K^{-1} for the complex n-dimensional projective spaces $\mathbf{CP}^n$, wedges $S^m \vee S^n$ and products $S^m \times S^n$ of spheres.

Problem 6: Sobolev spaces have well defined topologies and the second extreme.

(a) Suppose M is a compact d-dimensional smooth manifold. Recall that one way to define Sobolev spaces of M is to choose a partition of unity (ψ, U) indexed by a set I together with embeddings $\phi_i: U_i \to \mathbf{T}^d$ and equip $C^k(M)$ with the norm $f \in C^k(M) \mapsto \|f\|_k = \sum_{i \in I} \|(\psi_i f) \circ \phi_i^{-1}\|_k$, where the Sobolev norm on $C^k(\mathbf{T}^d)$ is $f \in C^k(\mathbf{T}^d) \mapsto \|f\|_k = \sum_{|r| \le k} \sup |\partial_r f|$. Prove that all norms on $C^k(M)$ induced by different choices of (ψ, U, ϕ) are equivalent to each other and hence define the same topology on $C^k(M)$.

(b) The second extreme: Consider an order 0 differential operator $D: C^\infty(M) \to C^\infty(M)$ on the trivial line bundle on a smooth compact manifold M given by the multiplication by a function $f \in C^\infty(M)$. Suppose that all zeros of f are isolated. Compute the kernel and the cokernel of D. Do the same for the extension of D to $\mathrm{L}^2(M)$: $\hat{D}: \mathrm{L}^2(M) \to \mathrm{L}^2(M)$. Discuss which of these operators are Fredholm and compute their index.