Spring 2011, Math 276: Index Theory, Homework 1

Please submit by February 1 and contact Dmitri Pavlov (pavlov@math) for all questions about homework.

Problem 1: Differential operators. Suppose that V and W are smooth vector bundles over a smooth manifold M. Define the set of all differential operators $\mathrm{Diff}^{\le k}(V,W)$ of order at most k from V to W as follows: For $k<0$ we have $\mathrm{Diff}^{\le k}(V,W)=\{0\}$ and otherwise it consists of $\mathbf{C}$-linear maps $D\colon C^\infty(V)\to C^\infty(W)$ such that $Dm_V(a)-m_W(a)D\in \mathrm{Diff}^{\le k-1}(V,W)$ for all $a\in C^\infty(M)$. Here $m_V(a)$ denotes the operator $C^\infty(V)\to C^\infty(V)$ given by the multiplication by a and likewise for $m_W(a)$. In particular, $\mathrm{Diff}^{\le 0}(V,W)$ consists of all $C^\infty(M)$-linear morphisms $C^\infty(V)\to C^\infty(W)$, i.e., morphisms of vector bundles $V\to W$.

(a) Prove that the coordinate definition of a differential operator is equivalent to the one above, i.e., for any open set $U\subset \mathbf{R}^n$ and for any family of smooth functions $f_\alpha\in C^\infty(U)$ prove that every operator of the form $\sum_{|\alpha|\le k} f_\alpha\partial_\alpha$ is an element of $\mathrm{Diff}^{\le k}(U)$ and every element of $\mathrm{Diff}^{\le k}(U)$ can be represented in this form. Here α denotes multiindices and ∂_α denotes the corresponding composition of partial derivatives.

*(b) A *differential operator* is a morphism of sheaves of real vector spaces $C^\infty_{\mathrm{sheaf}}(V)\to C^\infty_{\mathrm{sheaf}}(W)$. Alternatively, a differential operator is a morphism of real vector spaces $D\colon C^\infty(V)\to C^\infty(W)$ that preserves supports of sections: $\mathrm{supp}(D(f))\subset\mathrm{supp}(f)$ for all $f\in C^\infty(V)$. Denote by $\mathrm{Diff}(V,W)$ the set of all differential operators from V to W and prove that $\mathrm{Diff}^{\le k}(V,W)\subset\mathrm{Diff}(V,W)$ for all k. Show that the canonical morphism $\mathrm{colim}_k\,\mathrm{Diff}^{\le k}(V,W)\to\mathrm{Diff}(V,W)$ given by the universal property of the colimit is an isomorphism. Here the colimit is taken in the category of sheaves (think of fiberwise colimit followed by sheafification).

Problem 2: Symbols of differential operators. Recall that in Problem 1 we constructed for an arbitrary smooth manifold M a category Diff of vector bundles and differential operators together with the filtration $\mathrm{Diff}^{\le k}(U,V)$. In this problem we study the associated graded category of this filtered category.

(a) In the notation of Problem 1 prove the following relations: $\mathrm{Diff}^{\le i}(V,W)\,\mathrm{Diff}^{\le j}(U,V)\subset\mathrm{Diff}^{\le i+j}(U,W)$ and $[\mathrm{Diff}^{\le i}(V,V),\mathrm{Diff}^{\le j}(V,V)]\subset\mathrm{Diff}^{\le i+j-1}(V,V)$ for all i and j. In particular, the category Diff is filtered by $\mathrm{Diff}^{\le k}$.

(b) Consider the category of symbols Symb over M, whose objects are vector bundles over M and morphisms from E to F are $\mathrm{Symb}(E,F):=\mathrm{Sym}(TM)\otimes_{C^\infty(M)}\mathrm{Hom}(E,F)$. Composition and identity morphisms are defined in a natural way. The category Symb admits a natural filtration $\mathrm{Symb}^{\le k}$ (polynomials of degree at most k). Construct a natural map of the associated graded categories of Symb and Diff: $\mathrm{Symb}^{\le k}(E,F)/\mathrm{Symb}^{\le k-1}(E,F)\to\mathrm{Diff}^{\le k}(E,F)/\mathrm{Diff}^{\le k-1}(E,F)$. (Hint: Use the fact that sections of TM are derivations of $C^\infty(M)$ and combine it with part (a).)

(c) Construct a natural map $\mathrm{Diff}^{\le k}(E,F)/\mathrm{Diff}^{\le k-1}(E,F)\to\mathrm{Symb}^{\le k}(E,F)/\mathrm{Symb}^{\le k-1}(E,F)$. (Hint: An element of $\mathrm{Symb}^k(E,F):=\mathrm{Symb}^{\le k}(E,F)/\mathrm{Symb}^{\le k-1}(E,F)$ can be constructed fiberwise. The fiber of $\mathrm{Symb}^k(E,F)$ at point x can be identified with $\mathrm{Hom}(C^\infty(E)m_x^k/C^\infty(E)m_x^{k+1},C^\infty(F)/C^\infty(F)m_x)$, where m_x is the ideal of functions in $C^\infty(M)$ that vanish at x. Elements of $\mathrm{Diff}^{\le k}(E,F)$ act on $C^\infty(E)$.)

(d) Prove that the two natural maps constructed in parts (b) and (c) are the mutual inverses of each other. In particular, they give an equivalence of the associated graded categories of Symb and Diff.

Problem 3: Connections and parallel transport. A connection on a smooth vector bundle V over a smooth manifold M is an $\mathbf{R}$-linear map $\nabla\colon C^\infty(V)\to\Omega^1(M)\otimes_{C^\infty(M)}C^\infty(V)=C^\infty(T^*M\otimes V)$ such that for all $f\in C^\infty(M)$ and $s\in C^\infty(V)$ we have $\nabla(fs)=f\nabla(s)+df\otimes s$.

(a) Prove that any connection is a differential operator of order 1. Prove that if ∇ is a connection and $A\in\mathrm{Hom}_{C^\infty(M)}(C^\infty(V),\Omega^1(M)\otimes_{C^\infty(M)}C^\infty(V))=\Omega^1(M,\mathrm{End}(V))$, then $\nabla+A$ is also a connection. Prove that the difference of any two connections is such a $C^\infty(M)$-linear map.

(b) Consider the trivial vector bundle $V=\mathbf{R}^n\times M\to M$. Prove that the de Rham differential induces a connection $d\otimes\mathrm{id}\colon C^\infty(M)\otimes\mathbf{R}^n\to\Omega^1(M)\otimes\mathbf{R}^n$ on V. Use part (a) to find an explicit expression for all connections on V and conclude that connections exist on any vector bundle.

(c) A section $s\in C^\infty(V)$ is called *parallel* if $\nabla(s)=0$. For $M=[0,1]$, prove that the restriction map r_x from the set of all parallel sections of V to the fiber over a point $x\in M$ is an isomorphism of vector spaces. The morphism $r_1r_0^{-1}$ is called the *parallel transport* from 0 to 1. Write down an explicit formula for the parallel transport from 0 to 1 in terms of the explicit description of connections over M obtained in part (b). Hint: $\Omega^1(M, \mathrm{End}(E))$ can be identified with $C^\infty(M, \mathrm{End}(V))$.

(d) Use part (c) to define parallel transport for an arbitrary vector bundle V equipped with a connection ∇ over a smooth manifold M along an arbitrary path $p\colon [0,1] \to M$. Hint: Parallel transport commutes with pull-backs of connections.