These notes: dmitripavlov.org/fft

Functorial field theory

and the geometric cobordism hypothesis

Goals:

1) Cover the necessary prerequisites for GCH
2) Explain the GCH
3) Compute examples of FFTs using GCH.

Optional activities

1) Write up one or more lectures.
   ⇒ get a book if enough volunteers
   Example: Differential cohomology (Amabel / Debray / Haine)
2) Maintain an archive of video recordings
3) Write up solutions to exercises.

Ask questions! in chat / voice during the lectures!

Quantum field theory

Heisenberg
algebraic QFT
factorization algebras
mostly perturbative

Schrödinger
(nonperturbative)
Segal, Witten mid 1980s
CFT
later: topological QFT

Functorial field theory
- encodes both classical/prequantum data and its quantization
- is nonperturbative by its nature, but perturbative information can be extracted

Example: A spin $\frac{1}{2}$ particle propagating in a Riemannian manifold X (Quantum mechanics)

Step 1 out of 2: Prequantum FFT

Input data:

- X: Riemannian manifold
- S: a finite dimensional vector bundle with an action of $Cl(TX)$ (spinor bundle)

Output data: A prequantum FFT

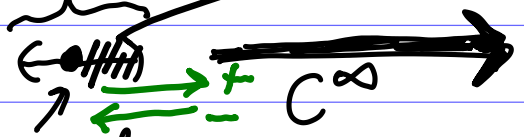
$$\mathrm{Bord}_1^{\mathrm{Riem}\times X} \xrightarrow{P} \mathrm{Vect}$$

symmetric monoidal functor

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1) $\mathrm{Bord}_1^{\mathrm{Riem}\times X}$: sym. mon category $\equiv \otimes$-cat.

Objects: collar — a bordism will appear here (for the source)

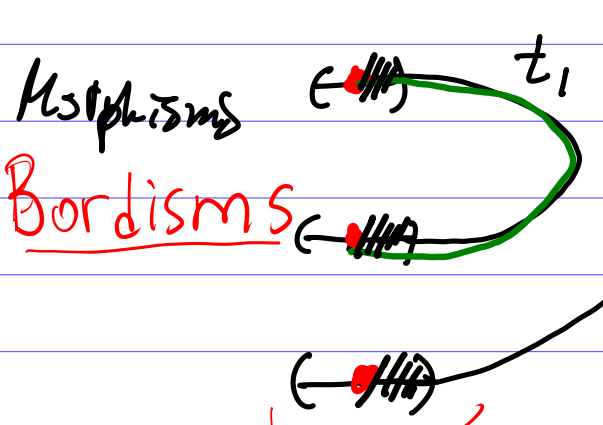


1-dim germ of a point with a Riemannian metric and orientation

More generally, a finite collection of points

Morphisms

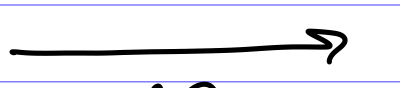
Bordisms



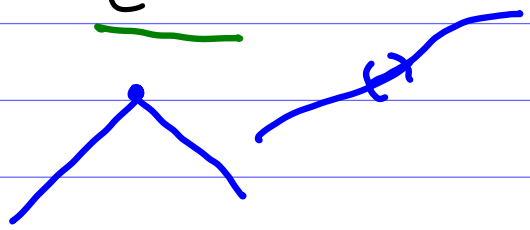
$V \xrightarrow{i} W$

$V \otimes W^* \to \mathbb{C}$

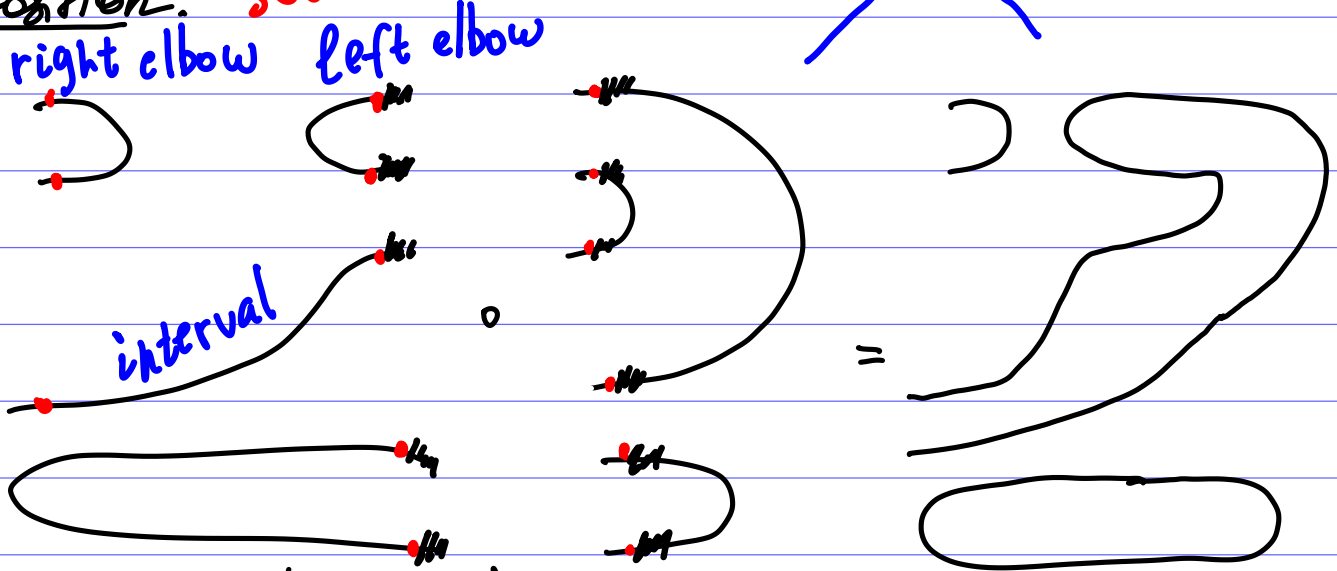
$\mathbb{C} \to V^* \otimes W$



C^∞



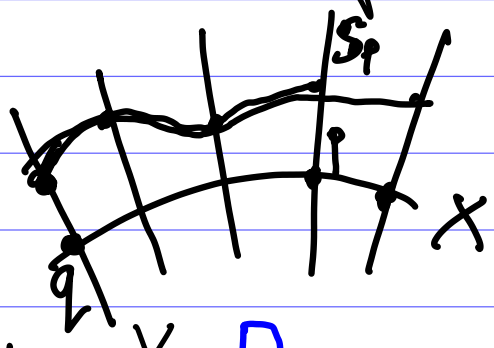
Composition:



$\otimes$ = disjoint union $\sqcup$

2) Vect: $\otimes$ - cat of vector spaces

Ob: vector spaces
Mor: linear maps
$\otimes$: tensor product.



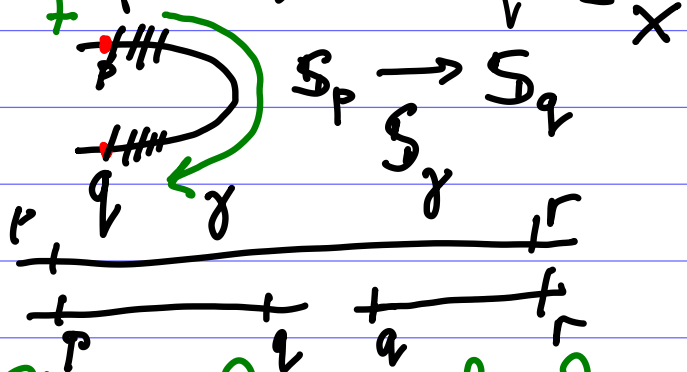
3) The functor $\text{Bord}_1^{\text{Riem}\times X} \xrightarrow{P} \text{Vect}$
(prequantum functorial field theory)

$+ \to \xrightarrow{c^\infty} X \longmapsto S_p$ (fiber over p)

$- \to \xrightarrow{q} X \longmapsto S_q^* = \text{Hom}(S_q, \mathbb{C})$

$p, q \longmapsto S_p \otimes S_q$

$\gamma: p \to q \longmapsto S_p \xrightarrow{\|_\gamma} S_q$ (parallel transport)



$S_p \to S_q$
S_γ

$S_p \otimes S_q^* \to \mathbb{C}$
$v, f \longmapsto f(S_\gamma(v))$



Step 2 out of 2: Quantum FFT:

no map to X

$\text{Bord}_1^{\text{Riem}} \xrightarrow{Q} \text{Vect}$

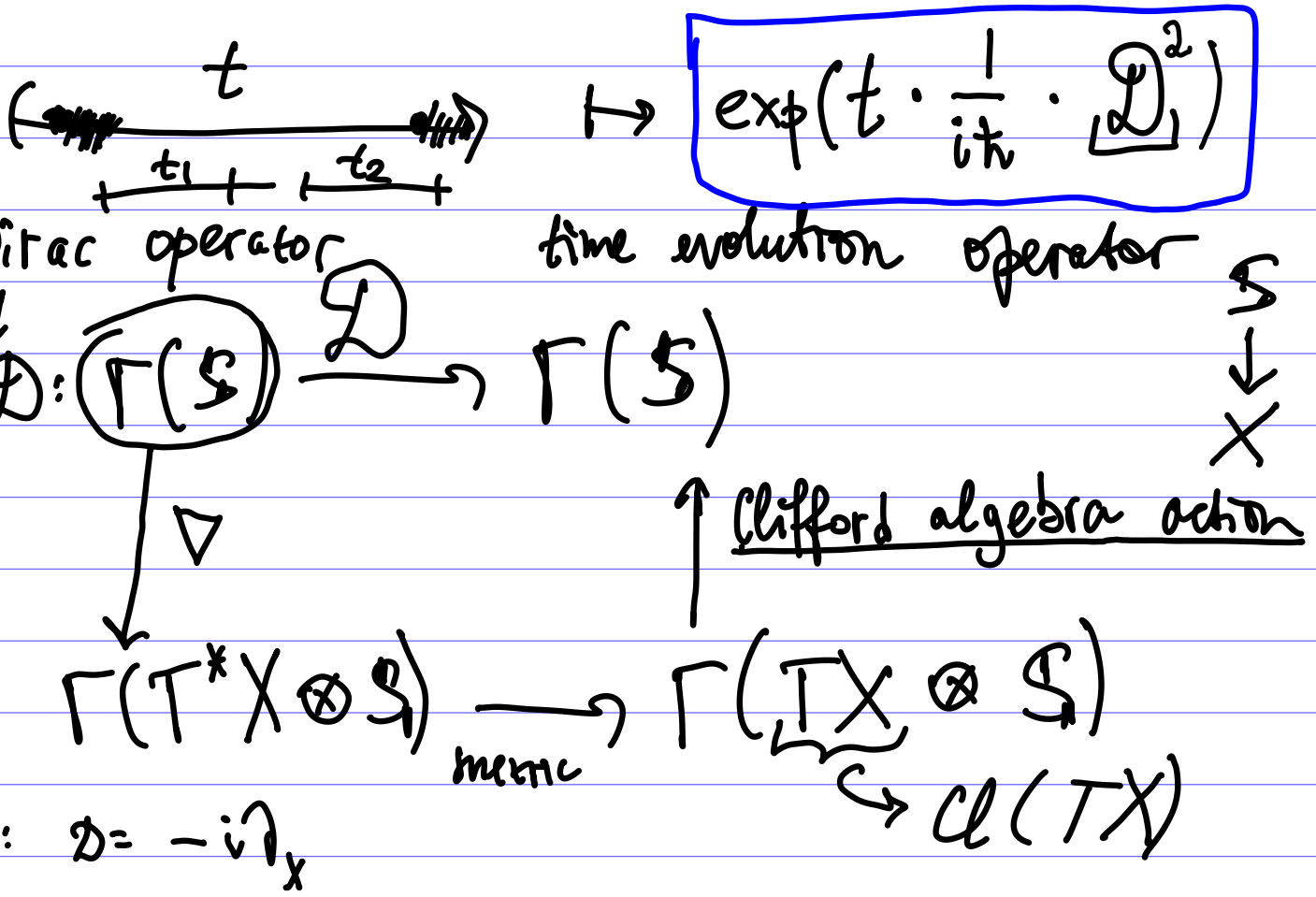
$+ \to \longmapsto \Gamma(S)$

Sections of the spinor bundle (smooth or distributional)

bra vectors

$- \leftarrow$

ket vectors



How to pass from Step 1 to Step 2?

(Quantization ≡ pushforward)

The third way: path integral

(Feynman, 1949)

$$Q(\leftrightarrow) = \prod_{\leftrightarrow \to X} P(p)$$

p

= Smooth sections of S

$Q\left(\xrightarrow{t}\right)$ = linear operator

$Q(\cdot) \longrightarrow Q(\cdot)$

$\Gamma(S) \xrightarrow{U_t} \Gamma(S)$

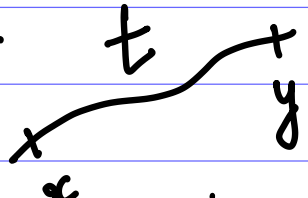
integral kernel

$U_t(x, y)$ $x, y \in X$

$= \int_{\gamma: x \to y} P(\gamma)\, dW$

Feynman–Kac formula

Wiener measure



Don't know how to define rigorously for bordisms dim > 1.

The new way: Kan extensions

$$\begin{array}{ccc} \mathrm{Bord}_1^{\mathrm{Riem}} \times X & \xrightarrow{P} & \mathrm{Vect} \\ \downarrow \Pi & \nearrow Q = \mathrm{Ran}_{\Pi} P & \\ \mathrm{Bord}_1^{\mathrm{Riem}} & & \end{array}$$

discard the map to X

What enables this construction?

The geometric cobordism hypothesis

$$\mathrm{FFT}_{d,S,V} = \mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^S, V)$$

$$\simeq \downarrow \text{evaluate at pt}$$

$$\mathrm{Map}(S, (V^{fd})^X)$$

Instead of doing ∞-dim path $\int$ need to do finite dim constructions!

$$P \in \mathrm{FFT}_{1,\mathrm{Riem}\times X,\mathrm{Vect}} \xrightarrow{GCH}_{\simeq} \mathrm{Map}(\mathrm{Riem}\times X, \mathrm{Vect}^{\times})$$

∞-dim path integral $\downarrow$

finite-dim $\swarrow$

take sections over X $\downarrow$

$$Q \in \mathrm{FFT}_{1,\mathrm{Riem},\mathrm{Vect}} \xrightarrow{GCH}_{\simeq} \mathrm{Map}(\mathrm{Riem}, \mathrm{Vect}^{\times})$$

What's left out?

A (geometric) FFT of dimension d is

- a smooth symmetric monoidal (∞,d)-functor

$$\mathrm{Bord}_d^S \longrightarrow V$$

Need to define

a) Smooth sym mon (∞,d)-cat

b) .... cat of bordisms with geometric structure S

c) GCH

a) Higher category theory $\subset$ homotopy theory

Homotopy theory
- Simplicial sets
- Model categories
- Homotopy (co) limits
- Segal's Γ-objects & CSS
- Simplicial presheaves & descent; diff. spaces

Higher category theory

Geometric structures & bordism categories

→ Locality of FFTs & GCH

tikzit.github.io
TikZiT

Examples: 1-dim, d-dim.

q.uiver.app

dmitripavlov.org / git

dmitripavlov.org/fft

Oct 15

Oct 22

Oct 29: no lecture

Nov 5: 10 am UTC-06 (daylight saving time)

Topic 1: Elementary homotopy theory

Functor of points
Simplicial sets & ∞-groupoids
Relative categories as $(\infty,1)$-cat
Model structures

dmitripavlov.org/writing

Lecture 2

Functor of points (Grothendieck)

A foundation for most constructions in the lectures to follow

dmitripavlov.org/fft.pdf

For example:

- simplicial sets
- complete Segal spaces
- Γ-objects
- simplicial presheaves and stacks
- diffeological spaces and smooth sets
- various flavors of higher categories

0) Want to define a "big" category $\mathcal{C}$

Examples:

- Geometry: smooth sets, stacks, derived stacks, ...
- Algebra: groups, rings, modules, E_∞-rings, ...

1) Identify a "simple" full subcategory $\mathcal{S} \subset \mathcal{C}$

Examples

$\mathcal{S}$	$\mathcal{C}$
simplices	simplicial sets
free groups	groups
free modules	modules
free $\mathrm{Alg}_{\mathbb{R}} \simeq (\mathbb{R}^n, \text{polynom})^{op}$	commutative real algebras
affine schemes $\simeq \mathrm{CRing}^{op}$	schemes; algebraic spaces
$(\mathbb{R}^n, C^\infty\text{-maps})^{op}$	C^∞-rings
$\mathrm{Cart} = (\mathbb{R}^n, C^\infty\text{-maps})$	smooth sets
$(\mathrm{aff\,Sch}) \times (\mathrm{simplices})$	stacks $\simeq \infty$-sheaves
simplicial algebras	derived schemes/stacks

Idea: $\mathcal{S}$ can be constructed first

$\mathcal{C}$ can be reconstructed from $\mathcal{S}$

2) $\mathcal{C}$ is a full subcategory of

$PSh(\mathcal{S}) = Fun(\mathcal{S}^{op}, Set)$

think: ∞-dim manifolds

Embedding of $\mathcal{C}$: (the restricted Yoneda emb.)

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & P\mathcal{S}h(\mathcal{S}) = Fun(\mathcal{S}^{op}, Set) \\ \| & & \| \\ X & \mapsto & \mathcal{C}(-,X) \in \mathcal{S} \to Set \end{array}$$

$\mathcal{C}(-,X)$ is the functor of points of X (should contain all info about X)

"point": $P \in \mathcal{S}$ (probe)

$$\begin{array}{ccc} P & & \mathcal{C}(P,X) \\ g \downarrow & \mapsto & \uparrow (-)\circ g \\ P' & & \mathcal{C}(P',X) \end{array}$$

$X, X' \in \mathcal{C}$

$$\begin{array}{ccc} X & & \mathcal{C}(P,X) \\ h \downarrow & \mapsto & \downarrow h\circ(-) \\ X' & & \mathcal{C}(P,X') \end{array}$$

What if we only know $\mathcal{S}$, not $\mathcal{C}$? $\equiv$

The embedding $\mathcal{S} \hookrightarrow PSh(\mathcal{S})$ still makes sense!

$X \mapsto \mathcal{S}(-,X)$ $(X \in \mathcal{S})$

$\Rightarrow$ The Yoneda embedding.

2a) 2) tells us how to think of objects in $PSh(\mathcal{S})$ ($\mathcal{S} = Cart$) even if we don't know $\mathcal{C}$.

$F: \mathcal{S}^{op} \longrightarrow Set$ $\quad F \in PSh(\mathcal{S})$

Think of F as a "generalized $\mathcal{S}$-object" (generalized manifold)

Given $P \in \mathcal{S}$:

$$F(P) = \{ \text{maps } P \xrightarrow{C^\infty} F \}$$

given $f: P \to P'$

$$F(f): \{ \text{maps } P' \to F \}$$

$$\downarrow (-) \circ f$$

$$\{ \text{maps } P \longrightarrow F \}$$

Yoneda lemma: If $F = y_T$, $T \in \mathcal{S}$ then $F(S) \cong \mathcal{S}(S, T)$

3) $\mathcal{C}$ is a full subcategory of $PSh(\mathcal{S})$.

Which full subcategory?

In practice:

Some colimits in $\mathcal{S}$ must be preserved by the embedding $\mathcal{S} \hookrightarrow \mathcal{C}$.

Typically, they are not preserved by $y: \mathcal{S} \hookrightarrow PSh(\mathcal{S})$

Solution Restrict to the full subcategory of $PSh(\mathcal{S})$ comprising objects that "perceive" some cocones in the image of $\mathcal{S} \hookrightarrow PSh(\mathcal{S})$ as colimit cocones.

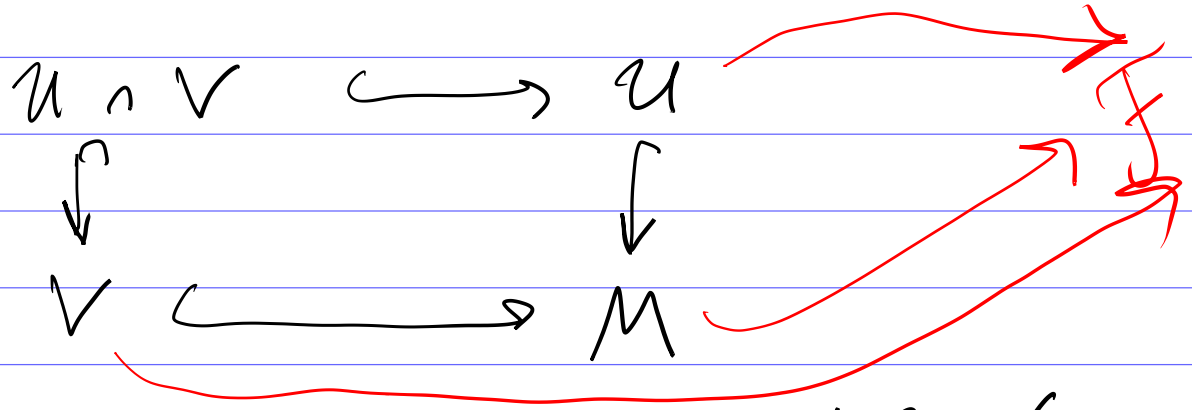
Example Smooth sets

Cart: Ob: $\mathbb{R}^n$, $n \geq 0$

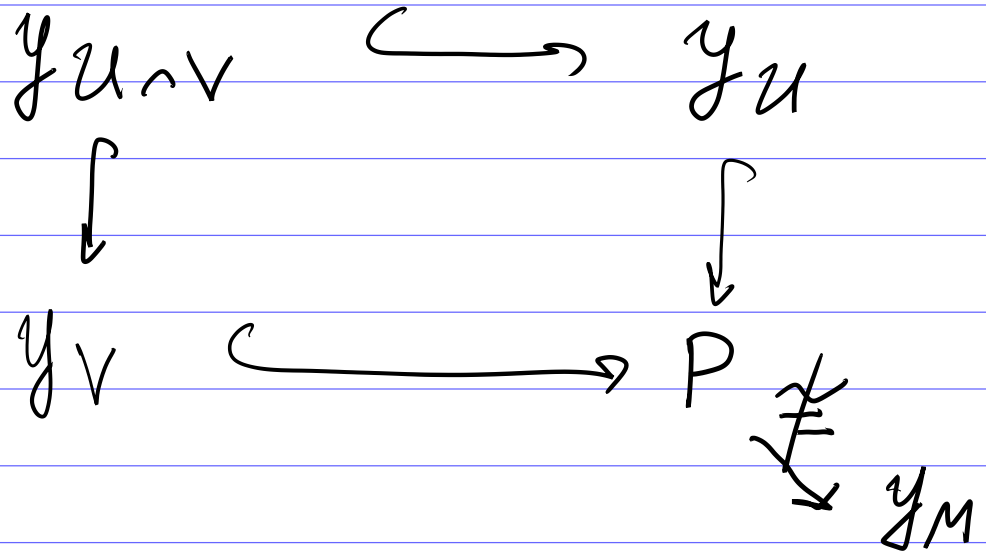
Mor: C^∞-maps $\mathbb{R}^m \longrightarrow \mathbb{R}^n$.

$M = U \cup V$;

pushout in Cart



not a pushout in PSh(Cart)



Explanation There are smooth maps $S \longrightarrow M$ that do not factor through U or V

$\Rightarrow$ do not factor through P.

Solution Require $\mathcal{F}$ to "perceive" the square as a pushout square:

$$\begin{array}{ccc} \mathcal{F}(U \cap V) & \longleftarrow & \mathcal{F}(U) \\ \uparrow & & \uparrow \\ \mathcal{F}(V) & \longleftarrow & \mathcal{F}(M) \end{array}$$

$\mathcal{F}$ is a sheaf

$$\mathcal{C} = Sh(Cart) \hookrightarrow PSh(Cart)$$

Example: M, N manifolds

What is $C^\infty(M, N)$?

$Hom(M, N)(S) = C^\infty(S \times M, N)$

points in $C^\infty(M,N)$ = Smooth maps $M \to N$

Traditional approach to smooth structure on $C^\infty(M,N)$:
topological vector spaces (Frechet spaces)
$\Rightarrow$ Difficult!

Initial observation:

hom-product adjunction for Set
"currying"

maps of sets

$$\{M \times P \to N\} \cong \{P \to N^M\}$$

$$N^M = \{f : M \to N\}.$$

$$g : M \times P \to N \longmapsto h : P \to N^M$$
$$p \in P \quad h(p) : M \to N$$
$$h(p)(m) = g(p,m)$$

$M, N \in Man \qquad \mathcal{S} = Cart \quad (\mathbb{R}^n, C^\infty)$

$N^M \in PSh(Cart)$

$N^M : Cart^{op} \longrightarrow Set$

$N^M(P) = C^\infty(M \times P, N)$

$P \xrightarrow{f} P'$

$N^M(f): N^M(P') \longrightarrow N^M(P)$

$\| \qquad\qquad \|$

$C^\infty(M \times P', N) \longrightarrow C^\infty(M \times P, N)$

$h \longmapsto h \circ (id_M \times f).$

N^M contains the entire totality of info about the smooth structure!

$P = \mathbb{R}^0 = \text{point}$

$$N^M(P) = C^\infty(M \times P, N)$$

$$\| \qquad\qquad \| \wr$$

$$P \to N^M \qquad C^\infty(M, N)$$

points in N^M are precisely smooth maps $M \to N$

Can do differential geometry in this setting!

Example: Differential forms.

What is a differential form on N^M?

Answer: $N^M \longrightarrow \Omega^n$

$n \geq 0$ $\Omega^n : \mathrm{Cart}^{op} \to \mathrm{Set}$

$P \in \mathrm{Cart}$: $\Omega^n P = \{$differential n-forms on $P\}$

$P \xrightarrow{f} P'$ $\Omega^n P' \xrightarrow{f^*} \Omega^n P$

pullback of differential forms.

Def Given $\mathcal{F} \in PSh(\mathrm{Cart})$, a differential n-form on $\mathcal{F}$ is a morphism (in $PSh(\mathrm{Cart})$)

$$\mathcal{F} \longrightarrow \Omega^n.$$

Remark Suppose

$$\mathcal{F} = \mathrm{Cart}(-, T)$$

is the representable presheaf of some $T \in \mathrm{Cart}$.

The Yoneda lemma:

$$\{\mathcal{F} \longrightarrow \Omega^n\} \cong \Omega^n(T)$$

What is $N^M \xrightarrow{\omega} \Omega^n$ concretely?

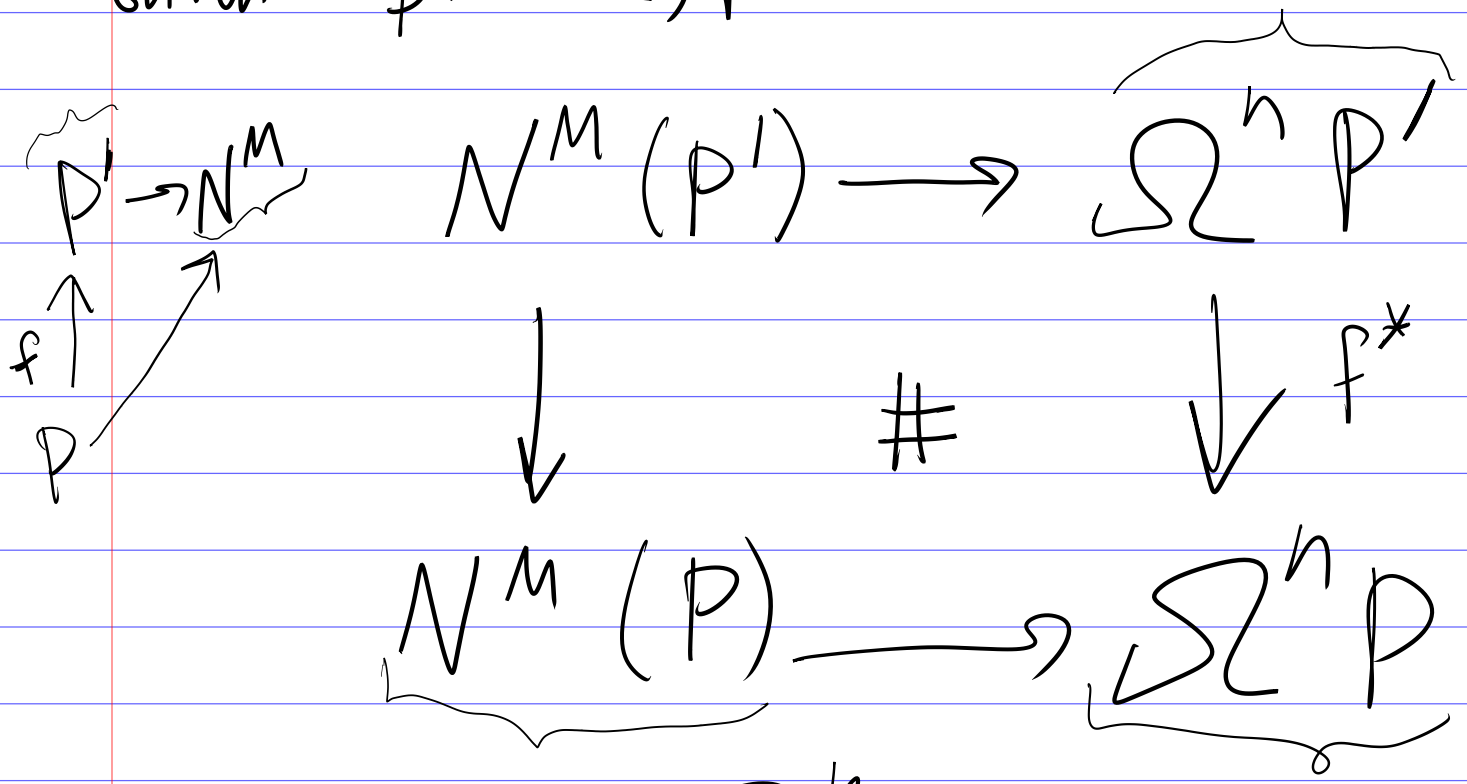
Given $P \cong \mathbb{R}^p$:

$$\omega(P): N^M(P) \longrightarrow \Omega^n P$$

$$\| \qquad\qquad \|$$

$$\widetilde{P \to N^M} \qquad C^\infty(P \times M, N)$$

Given $P \xrightarrow{f} P'$



Remark Ω^n is an object of $\mathrm{PSh}(\mathrm{Cart})$

What is a differential m-form on Ω^n?

Answer: $\Omega^n \longrightarrow \Omega^m$

Concretely: $P \cong \mathbb{R}^p$: $\Omega^n(P) \to \Omega^m(P)$ compatible with pullbacks

Examples: $\omega \longmapsto \lambda \cdot \omega$ $(m=n)$ $\lambda \in \mathbb{R}$

$\omega \longmapsto \lambda \cdot d\omega$ $(m = n+1)$ $\lambda \in \mathbb{R}$

Theorem (Palais) $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, $m = n = 0$

$$PSh(Cart)(\Omega^n, \Omega^m) = \begin{cases} \mathbb{R}, & m = n \\ \mathbb{R}, & m = n+1 \\ \{0\}, & \text{otherwise} \end{cases}$$

$$\Omega^n \xrightarrow{d} \Omega^{n+1}$$

Def Given $\mathcal{F} \in PSh(Cart)$, the de Rham complex of $\mathcal{F}$ is $PSh(Cart)(\mathcal{F}, d)$

$$PSh(Cart)(\mathcal{F}, \Omega^0) \to PSh(Cart)(\mathcal{F}, \Omega^1) \to \cdots$$

What is Ω^n as a space?

$\Omega^n(\mathbb{R}^0) = \{0\}$ $\quad n > 0$

$\Omega^0(\mathbb{R}^0) = \mathbb{R}$ $\quad n = 0$

$\Omega^0 \cong \mathbb{R} := y_{\mathbb{R}} = \mathrm{Cart}(-, \mathbb{R})$

A differential 0-form on F is a map $F \to \Omega^0 \cong \mathbb{R}$,

i.e., a smooth function on F.

If $n > 0$: Ω^n has a single point,

but $\Omega^n \neq \mathbb{R}^0 = \mathrm{Cart}(-, \mathbb{R}^0)$

$\Omega^n(\mathbb{R}^n) \neq \mathrm{Cart}(\mathbb{R}^n, \mathbb{R}^0)$

Presheaves on Cart are not determined by their sets of points,

even if we equip them with some additional structure

Remark: Calculus of variations secretly uses functor of points

$$\gamma : [0,1] \to \mathbb{R}^n$$

$$\gamma(0) = x, \quad \gamma(1) = y$$

$$\gamma \mapsto \int_0^1 \|\gamma'(t)\|^2 \, dt = S(\gamma)$$

$$\delta : [0,1] \to \mathbb{R}^n$$

$$\delta(0) = \delta(1) = 0$$

$$\mathbb{R} \ni t \mapsto S(\gamma + t \cdot \delta)$$

Exercise Formalize this definition using functor of points.

Move the class from 10 am to 9 am?
(UTC−06)

These notes: dmitri pavlov.org/fft.pdf

Next time:

Sunday, November 12

New time: 9 am UTC−06

Lecture 3: Functor of points redux

Last time: study space by launching probes into it (launch) site

$$\underbrace{\mathrm{Cart}}^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$$

probes originate here ↑ results

$F(U)$ = "the set of maps $U \to F$"

$g: U \xrightarrow{C^\infty} V \qquad U, V \in \mathrm{Cart}$

$\downarrow$

$F(g): F(V) \to F(U)$

"a map $V \xrightarrow{h} F$"

$\overset{F(g)}{\mapsto}$ "$U \xrightarrow{g} V \xrightarrow{h} F$"

Example $\mathrm{Fun}(\mathrm{Cart}^{\mathrm{op}}, \mathrm{Set})$ contains smooth sets, diffeological spaces as full subcategories

Any smooth manifold M can be reconstructed from its functor of points F_M

$$F_M : \mathrm{Cart}^{op} \longrightarrow \mathrm{Set}$$
$$U \longmapsto C^\infty(U, M).$$

The underlying set is $F_M(\mathbb{R}^0) = X$

The algebra of smooth functions on X is reconstructed by evaluating morphisms $F_M \xrightarrow{h} F_{\mathbb{R}}$ on $\mathbb{R}^0$.

$$h_{\mathbb{R}^0} : X \longrightarrow \mathbb{R}$$

A subset $U \subset X$ is open if $\exists h$: $U = (h_{\mathbb{R}^0})^{-1}(\mathbb{R} \setminus \{0\})$.

Thm The functor
$$M \longmapsto F_M$$
$$\mathrm{Man} \longrightarrow \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{Set})$$
is fully faithful.

$M, N \in \mathrm{Man}$

$$C^\infty(M,N) \xrightarrow{\cong} \mathrm{Nat}(\mathcal{F}_M, \mathcal{F}_N)$$

Yoneda Lemma: $U \in \mathrm{Cart}$ $\quad G : \mathrm{Cart}^{op} \to \mathrm{Set}$

$$G(U) \xrightarrow{\cong} \mathrm{Nat}(\mathcal{F}_U, G)$$

Advantages of smooth sets

- Can immediately define and work with objects like $C^\infty(M,N)$, Ω^n, etc.
- Free of technicalities arising from formalisms based on topological vector spaces.

Other similar examples:

- schemes, algebraic spaces (replace Cart by affine schemes)
- stacks
- derived stacks

To be developed later: gluing properties

An algebraic example:

groups, rings, modules, algebras

commutative real algebras $\mathrm{Alg}_{/\mathbb{R}}$

Study $A \in CAlg_{\mathbb{R}}$ by launching probes into it.

Probes: free commutative real algebras on finitely many generators:

$P = \mathbb{R}[x_1, \dots, x_n]$. (= polynomials)

$CAlg_{\mathbb{R}}(P, A) = (U(A))^n \cong \underbrace{A \times \cdots \times A}_{n}$

$U(A)$ = the underlying set of A

$FCAlg_{\mathbb{R}}^{op} \xrightarrow{G} Set$

Morphisms of probes

$\mathbb{R}[y_1, \dots, y_n] \xrightarrow{f} \mathbb{R}[x_1, \dots, x_m]$

n polynomials in m variables

$r_j = f(y_j)$ $\quad r_j(x_1, \dots, x_m)$ $\quad 1 \le j \le n$

$\Rightarrow f$ yields a map $A^m \xrightarrow{G(f)} A^n$

$\| G(\dots) \qquad \| G(\dots)$

$$\mathbb{R}[y_1, \dots, y_n] \xrightarrow{f} \mathbb{R}[x_1, \dots, x_m] \xrightarrow{p} A$$

$$\boxed{p(x_i) = a_i \in A \quad 1 \le i \le m}$$

$$(p \circ f)(y_j) = p(f(y_j)) = p(r_j)$$

$$= \tilde{p}(r_j(x_1, \dots, x_m))$$

$$= r_j(p(x_1), \dots, p(x_m))$$

$$= r_j(a_1, \dots, a_m).$$

Examples $n = 1$ (a single polynomial r_1)

$$\mathbb{R}[y_1] \longrightarrow \mathbb{R}[x_1, \dots, x_m]$$

$$y_1 \longmapsto r_1(x_1, \dots, x_m)$$

$$A \longleftarrow A^m$$

a) $n = 2 \qquad r(x_1, x_2) = x_1 + x_2$

$$A \xleftarrow{+} A^2$$

$$a_1 + a_2 \longleftarrow\!\shortmid a_1, a_2$$

b) $n = 2 \qquad r(x_1, x_2) = x_1 \cdot x_2$

$$A \xleftarrow{\cdot} A^2$$

$$a_1 \cdot a_2 \longleftarrow\!\shortmid a_1, a_2$$

c) $n = 1 \qquad r(x_1) = -x_1$

$$A \longleftarrow A$$

$$-a_1 \longleftarrow\!\shortmid a_1$$

d) $n = 1 \qquad r(x_1) = c \cdot x_1, \; c \in \mathbb{R}$

$$A \longleftarrow A$$

$$c \cdot a_1 \longleftarrow\!\shortmid a_1$$

e) $n = 0, \qquad r = \overline{c}, \; c \in \mathbb{R}$

$$A \longleftarrow A^0 = \{()\}$$

$$c \longleftarrow\!\shortmid ()$$

$$c \cdot \mathbb{1} \qquad \mathbb{1} \in A$$

Proposition A commutative real algebra A can be reconstructed from its functor of points

$$FCAlg_{\mathbb{R}}^{op} \xrightarrow{F_A} Set$$

$$B \longmapsto CAlg_{\mathbb{R}}(B, A)$$

The underlying set is $F(\mathbb{R}[y]) = X$

The addition operation is

$$F(\mathbb{R}[y] \longrightarrow \mathbb{R}[x_1, x_2]) : X \times X \xrightarrow{+} X$$

$$y \longmapsto x_1 + x_2$$

etc. $\qquad g \longmapsto h \circ g$

$$F_h(P) : CAlg_{\mathbb{R}}(P, A_1) \to CAlg(P, A_2)$$

Likewise, any homomorphism $A_1 \xrightarrow{h} A_2$ can be reconstructed from $F_h : F_{A_1} \longrightarrow F_{A_2}$

The underlying map of sets is $F_h(\mathbb{R}[y])$.

Not every functor $FCAlg_R^{op} \xrightarrow{F} Set$ comes from an algebra.

An obvious necessary condition:
F must preserve finite products (up to an isomorphism)

Finite products in $FCAlg_R^{op}$
$=$ finite coproducts in $FCAlg_R$
$=$ tensor products of algebras

$$\star = CAlg_R(R[x_1,\dots,x_m] \otimes R[y_1,\dots,y_n], A)$$
$$\cong CAlg_R(R[x_1,\dots,x_m], A) \times CAlg_R(R[y_1,\dots,y_n], A)$$
$$\cong A^m \times A^n.$$

$$\star = CAlg(R[z_1,\dots,z_{m+n}], A) \cong A^{m+n}$$

Theorem $CAlg_R \longrightarrow Fun^{\times}(FCAlg_R^{op}, Set)$
$\times$ = preserves finite product
is an equivalence of categories!

$$\mathbb{R}[x_1, \dots, x_m] \otimes \mathbb{R}[y_1, \dots y_n]$$

0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{R}\langle x_1, \dots, x_m \rangle$	$\mathbb{R}\langle y_1, \dots, y_n \rangle$
2	$\mathbb{R}\langle x_i\, x_j \rangle$ $1 \le i \le j \le m$	$\mathbb{R}\langle y_i\, y_j \rangle$ $1 \le i \le j \le n$

$$(A \otimes B)_r = \bigoplus_{p+q=r} A_p \otimes B_q$$

0: $\mathbb{R} \otimes \mathbb{R} = \mathbb{R}$

1: $\left(\mathbb{R}\langle x_1, \dots, x_m \rangle \otimes \mathbb{R}\right) \oplus \left(\mathbb{R} \otimes \mathbb{R}\langle y_j \rangle\right)$

$= \mathbb{R}\langle x_1, \dots, x_m \rangle \oplus \mathbb{R}\langle y_1, \dots, y_n \rangle$

$= \mathbb{R}\langle x_1, \dots, x_m, y_1, \dots, y_n \rangle$

2: $\mathbb{R}\langle x_i\, x_j \rangle \oplus \left(\mathbb{R}\langle x_i \rangle \otimes \mathbb{R}\langle y_j \rangle\right) \oplus \mathbb{R}\langle y_i y_j \rangle$

$\mathbb{R}\langle x_i\, x_j,\ x_i\, y_j,\ y_i\, y_j \rangle$.

A geometric point of view

on $\mathrm{FCAlg}_{\mathbb{R}}^{op}$

Proposition $\mathrm{FCAlg}_{\mathbb{R}}^{op} \simeq \mathrm{CartPoly}$

CartPoly: Ob: $\mathbb{R}^n$, $n \geq 0$

Mor: $\mathbb{R}^m \to \mathbb{R}^n$ polynomial maps

(n polynomials in m variables).

What happens if we replace

$\mathrm{CartPoly} \rightsquigarrow \mathrm{Cart}$?

C^∞-maps

Definition The category of C^∞-rings is the category of finite product-pres functors $\mathrm{Cart} \longrightarrow \mathrm{Set}$

$C^\infty\mathrm{Ring} := \mathrm{Fun}^{\times}(\mathrm{Cart}, \mathrm{Set})$

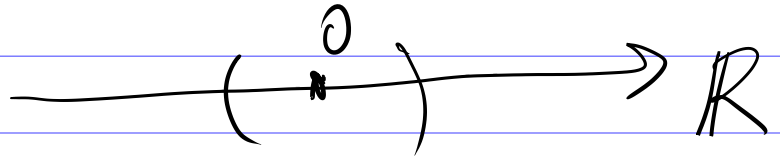
$\mathbb{R}[t]/(t^{n+1})$ nth order infinitesimal neighborhood

$\mathbb{R}[[\varepsilon]]$ formal neighborhood

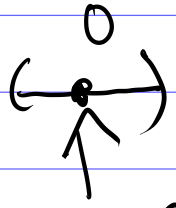
$C^\infty\mathbb{R}/I$ germy neighborhood

$I = \{f : \mathbb{R} \to \mathbb{R} \mid \exists \varepsilon > 0 : f|_{(-\varepsilon, \varepsilon)} = 0\}$

$\mathbb{R}[x]$ $C^\infty(\mathbb{R})$



$C^\infty\mathbb{R}/I$



germ

$\cup$

$(-\infty)$ formal neighborhood

$\cup$

$(-\bullet)$ order k neighbor

November 12
Lecture 4

More on the functor of points

Last time:

Smooth set:
an object of $Fun(Cart^{op}, Set)$
+ gluing condition

interpretation: Given $F: Cart^{op} \to Set$,
$F(\mathbb{R}^n)$ is the set of C^∞ maps $\mathbb{R}^n \to F$
$F(\mathbb{R}^m \xrightarrow{h} \mathbb{R}^n): F(\mathbb{R}^n) \longrightarrow F(\mathbb{R}^m)$
$\mathbb{R}^n \to F \longmapsto \mathbb{R}^m \xrightarrow{h} \mathbb{R}^n \to F$

C^∞-ring: an object of $Fun(Cart, Set)$
+ preservation of finite products

interpretation: Given $A: Cart \to Set$,
$A(\mathbb{R}^n)$ is the set of C^∞-maps
$Spec\ A \longrightarrow \mathbb{R}^n$
$A(\mathbb{R}^m \xrightarrow{h} \mathbb{R}^n): A(\mathbb{R}^m) \to A(\mathbb{R}^n)$
$Spec\ A \to \mathbb{R}^m \longmapsto Spec\ A \to \mathbb{R}^m \xrightarrow{h} \mathbb{R}^n$

(*) formally true by the Yoneda lemma

What is Spec?

Recall: Def $AffSch := CRing^{op}$
Def $C^\infty\text{-}AffSch := C^\infty Ring^{op}$
What kind of objects does C^∞-AffSch contain?

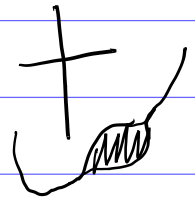
a) C^∞-manifolds $\xhookrightarrow{\text{fully faithful}}$ C^∞-Aff Sch
$M \longmapsto C^\infty M$
$M \xrightarrow{h} N \longmapsto C^\infty M \xleftarrow{(-)\circ h} C^\infty N$

(*) $M = Spec\ C^\infty M$
$(C^\infty M)(\mathbb{R}^n) = C^\infty(M, \mathbb{R}^n)$

b) locally closed subsets of $\mathbb{R}^n$ $F \cap U$

$M \subset \mathbb{R}^n \longmapsto C^\infty M = \{ f : M \to \mathbb{R} \mid f = g|_M, \ g : \mathbb{R}^n \to \mathbb{R} \}$

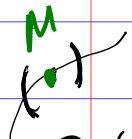
fully faithful



c) germs of subsets $\simeq$ C^∞-rings

$M \subset \mathbb{R}^n \longmapsto C^\infty_{germ}(M) = C^\infty \mathbb{R}^n / I$

$I = \{ f : \mathbb{R}^n \to \mathbb{R} \mid f^{-1}\{0\}$ is a neighb of $M \}$



$\bigcap_{\varepsilon > 0} (-\varepsilon, \varepsilon)$ = the germ of $0 \in \mathbb{R}$.

$\exp(x^{-2})$: vanishes on the formal neighbor but not on the germ

d) formal neighborhood: all derivatives of f at M vanish

$0 \in \mathbb{R} \quad \mathbb{R}[[\varepsilon]] \quad a_n \in \mathbb{R} \quad \sum_{n \geq 0} a_n \varepsilon^n$

e) finite-order neighb.: ...

$0 \in \mathbb{R} \quad \mathbb{R}[\varepsilon]/(\varepsilon^{n+1})$ nth order neigh

Can do differential geometry on a) - e)!

Algebraic geometry of affine schemes
= commutative algebra

Differential geometry of affine C^∞-schemes
= commutative algebra of C^∞-rings

Moerdijk - Reyes, Models for smooth infinitesimal analysis

In a commutative real algebra A:
for any polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$
can evaluate f on some $a_1, \ldots, a_n \in A$.
$\Rightarrow$ get $A^f : A^n \to A$ $\quad f(a_1, \ldots, a_n)$
A^f satisfy "associativity": $f(g_1(a_{11}, \ldots, a_{1n}), \ldots, g_m(a_{m1}, \ldots)) = (f(g_1, \ldots, g_m))(a_{11}, \ldots, a_{mn})$.

Morphisms of C^∞-rings

$A, B : \mathrm{Cart} \to \mathrm{Set}$

$$A(\mathbb{R}^n) = C^\infty(\operatorname{Spec} A, \mathbb{R}^n)$$

A morphism $A \xrightarrow{f} B$ is a natural transformation

$f_{\mathbb{R}^n} : A(\mathbb{R}^n) \to B(\mathbb{R}^n)$

$\mathbb{R}^m \xrightarrow{h} \mathbb{R}^n$

$$\begin{array}{ccc} A(\mathbb{R}^m) & \longrightarrow & A(\mathbb{R}^n) \\ \downarrow & & \downarrow \\ B(\mathbb{R}^n) & \longrightarrow & B(\mathbb{R}^m) \end{array}$$

$f_{\mathbb{R}^n} : C^\infty(\operatorname{Spec} A, \mathbb{R}^n) \to C^\infty(\operatorname{Spec} B, \mathbb{R}^n)$

$(-) \circ \operatorname{Spec} f$

$\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A$

$$\begin{array}{ccc} C^\infty(\operatorname{Spec} A, \mathbb{R}^m) & \xrightarrow{h\circ(-)} & C^\infty(\operatorname{Spec} A, \mathbb{R}^n) \\ \downarrow (-)\circ\operatorname{Spec} f & & \downarrow (-)\circ\operatorname{Spec} f \\ C^\infty(\operatorname{Spec} B, \mathbb{R}^m) & \to & C^\infty(\operatorname{Spec} B, \mathbb{R}^n) \end{array}$$

$A(\mathbb{R}^1)$ is the underlying set

$f_{\mathbb{R}^1}$ is the underlying map of sets

$h \circ (-)$

$A(\mathbb{R}^n) = A(\mathbb{R}^1) \times \cdots$

$f_{\mathbb{R}^n} = f_{\mathbb{R}^1} \times \cdots$

The universal property of $C^\infty\mathbb{R}^n$

$$\mathrm{Alg}_{\mathbb{R}}(\mathbb{R}[x_1,\dots,x_n], A) \xrightarrow{\cong} A^n$$

$$f \longmapsto (f(x_1),\dots,f(x_n))$$

$$(h \mapsto h(a_1,\dots,a_n)) \longmapsto (a_1,\dots,a_n)$$

$$C^\infty\mathrm{Ring}(\mathbb{R}\{x_1,\dots,x_n\}, A) \xrightarrow{\cong} A^n$$

$h \in C^\infty(\mathbb{R}^n)$

$$f \longmapsto (f(x_1),\dots,f(x_n))$$

$$(h \mapsto h(a_1,\dots,a_n)) \longmapsto (a_1,\dots,a_n)$$

What is $f(x_i)$?

$$f(x_i) = (f_{\mathbb{R}^1})(x_i)$$

$$x_i \in (\mathbb{R}\{x_1,\dots,x_n\})(\mathbb{R}^1)$$

$$= (C^\infty(\mathbb{R}^n))(\mathbb{R}^1)$$

$$\overset{\text{def}}{=} C^\infty(\mathbb{R}^n, \mathbb{R}^1)$$

x_i is the ith projection

Limits and colimits of C^∞-rings

$\mathrm{Cart} \xrightarrow{\mathcal{A}} \mathrm{Set}$ preserves finite products.

Products: $\left(\prod_i \mathcal{A}_i\right)(\mathbb{R}^n) = \prod_i \mathcal{A}_i(\mathbb{R}^n)$

Limits: $\left(\lim_{i \in I} \mathcal{A}_i\right)(\mathbb{R}^n) = \lim_{i \in I} \mathcal{A}_i(\mathbb{R}^n)$

Limits commute with finite products

$(P_1 \sqcup P_2) \times (P_1 \sqcup P_2) \ncong P_1 \times P_1 \sqcup P_2 \times P_2$

Coproducts do not commute w/ finite prod

Sifted colimits commute w/ finite prod

include: filtered colimits
reflexive coequalizers ↙ quotients
colimits over Δ^{op} ↘ coproduct

Practical example: $\underbrace{(C^\infty(\mathbb{R}^m)/I)}_{} \otimes \underbrace{(C^\infty(\mathbb{R}^n)/J)}_{}$

$= (\underbrace{C^\infty\mathbb{R}^m}_{} \otimes \underbrace{C^\infty\mathbb{R}^n}_{})/(I, J)$

$\cong (C^\infty \mathbb{R}^{m+n})/(I, J)$

Coproducts are not sifted.

In a C^∞-ring: f is any smooth function.

Thus, C^∞-rings axiomatize smooth functional calculus

$CAlg_{\mathbb{R}}$	C^∞Ring	C^∞AffSch
$\mathbb{R}[x_1,\dots,x_n]$ = polynomial functions on $\mathbb{R}^n$	$\mathbb{R}\{x_1,\dots,x_n\}$ = $C^\infty(\mathbb{R}^n)$	$C^\infty\mathbb{R}^n \to A$, $f(x_i)\in A$, $g\in C^\infty\mathbb{R}^n$, $f(g)=g(f(x_1),\dots,f(x_n))$ — the universal property of $C^\infty\mathbb{R}^n$
$\mathbb{R}[x_1,\dots,x_n] = \mathbb{R}[x_1]\otimes_{\mathbb{R}}\mathbb{R}[x_2]\otimes\cdots$	$\mathbb{R}\{x_1\}\otimes\mathbb{R}\{x_2\}\otimes\cdots$ ($f(x)g(y)$)	
A/I	A/I	
$A\otimes B$	$A\underset{C^\infty}{\otimes} B$ ~ categorical coproduct	Spec $A\times$ Spec B
$A[x] = A\otimes\mathbb{R}[x]$	$A\{x\} = A\otimes_{C^\infty} C^\infty(\mathbb{R})$	Spec $A\times\mathbb{R}$
$A[s^{-1}] = A[x]/(xs-1)$	$A\{s^{-1}\} = A\{s\}/(xs-1)$	Spec $A\setminus Z(s)$, $Z(s)=s^{-1}\{0\}$
Kähler differentials $\Omega^* A = A[dA]/\mathcal{L}$	$\Omega^1 A = A\{dA\}/\mathcal{L}$	$\mathcal{L}$: Hom$(\mathbb{R}^{0\mid 1}, \text{Spec } A)$
$\mathcal{L}$: $d(f(a_1,\dots,a_n)) - \sum_i \frac{\partial f}{\partial x_i}(a_1,\dots,a_n)\cdot da_i$	f is a smooth function	
f is a polynomial function		
$d(a_1a_2) = da_1\cdot a_2 + a_1 da_2$		
	$\Omega^* C^\infty$	

The de Rham complex / CDGA

$CDGA_{\mathbb{R}}$: $A_n \in Vect_{\mathbb{R}}$ $\quad n \geq 0$

$A_m \otimes_{\mathbb{R}} A_n \to A_{m+n}$

$x \cdot y = (-1)^{m \cdot n}\, y \cdot x$ Koszul sign rule

$d: A_n \to A_{n+1}$

$d(xy) = dx \cdot y + (-1)^{|x|}\, x\, dy$

$$CAlg_{\mathbb{R}} \underset{(-)_0}{\overset{\Omega}{\rightleftarrows}} CDGA_{\mathbb{R}}$$

$A_0 \leftarrow A$ (via $(-)_0$)

$\Omega(\mathbb{R}[x_1, \dots, x_n]) = \mathbb{R}[x_1, \dots, x_n, dx_1, \dots, dx_n]$ (0 over x_i, 1 over dx_i)

differential graded C^∞-ring

$DGC^\infty Ring$: 1) $A \in CDGA_{\mathbb{R}}$

2) $A_0 \in C^\infty Ring$

3) $d: A_0 \to A_1$ a C^∞-derivation:

$$d(f(a_1, \dots, a_n)) = \sum_i \frac{\partial f}{\partial x_i}(a_1, \dots, a_n)\, da_i$$

$$C^\infty Ring \underset{(-)_0}{\overset{\Omega}{\rightleftarrows}} DGC^\infty Ring \quad (\Omega \dashv (-)_0)$$

Poincaré lemma is formal!

$\Omega^* C^\infty \mathbb{R}^n = \Omega^*(\mathbb{R}^n)$

$\uparrow \simeq$ Poincaré

$\mathbb{R}[0]$

$$\begin{array}{ccc}
\mathrm{Vect}_{\mathbb{R}} & \longleftarrow & C^\infty\mathrm{Ring} \\
\uparrow (-)_0 & \# & \uparrow (-)_0 \\
\mathrm{coCh}_{\mathbb{R}} & \longleftarrow & \mathrm{DG}\, C^\infty\mathrm{Ring} \\
\mathbb{R}^n & V \longmapsto & C^\infty(V^*) \\
\uparrow \mathrm{Vect}_{\mathbb{R}} & \longrightarrow & C^\infty\mathrm{Ring}? \\
\downarrow & \# & \downarrow \Omega \\
\mathrm{coCh}_{\mathbb{R}} & \xrightarrow[L]{} & \mathrm{DG}\, C^\infty\mathrm{Ring}
\end{array}$$

$C^\infty\mathbb{R}^n$

$\Omega(C^\infty\mathbb{R}^n)$

$\mathbb{R}^n \xrightarrow[\mathrm{id}]{} \mathbb{R}^n$

$0 \qquad 1$

$\simeq 0$

$L(\mathbb{R}^n \xrightarrow{\mathrm{id}} \mathbb{R}^n)$

$\simeq L(0) = \Omega C^\infty(\mathbb{R}^0)$

$= \Omega\mathbb{R}$

$= \mathbb{R}[0]$

Lecture 5 November 18, 2023

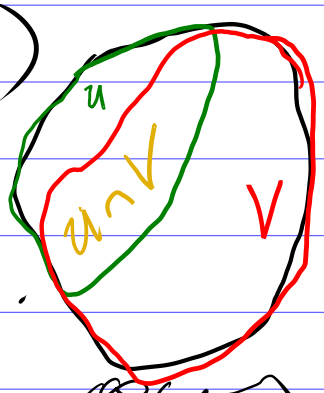
Sheaves and stacks

In functorial field theory:

- encode smooth families of bordisms
- encode geometric structures

Recall: a smooth set is $\mathcal{F}: \mathrm{Cart}^{op} \to \mathrm{Set}$

$U \in \mathrm{Cart}$: $\mathcal{F}(U) = C^\infty(U, \mathcal{F})$



Suppose $X \in \mathrm{Cart}$, $U, V \subset X$

$U, V, U \cap V \in \mathrm{Cart}$.

Want: $C^\infty(X, \mathcal{F}) \xrightarrow{\star} C^\infty(U, \mathcal{F}) \times C^\infty(V, \mathcal{F})$ should be an iso.

$C^\infty(U \cap V, \mathcal{F})$

compatible family

Counterexample:

$$\mathcal{F}(X) = \{ f : X \to \mathbb{R} \mid f \geq 0 \text{ or } f \leq 0 \}.$$

Solution Require $\star$ to be an iso as part of def of a smooth set.

Def $\mathcal{F}: \mathrm{Cart}^{op} \longrightarrow \mathrm{Set}$ is a **sheaf** if

$\forall X \in \mathrm{Cart}$ $\forall \{U_i\}_{i \in I}$ $U_i \subset X$, $U_i \in \mathrm{Cart}$

$U_i \cap U_j \in \mathrm{Cart}$

(= good cover)

the map ✡ is an equalizer.

(gluing property, descent property)

$$\begin{array}{ccccc} \mathcal{F}(X) & \xrightarrow{\text{✡}} & \prod_{i \in I} \mathcal{F}(U_i) & \rightrightarrows & \prod_{j,k\ (j<k)} \mathcal{F}(U_j \cap U_k) \\ & & \downarrow & & \downarrow \\ & & \mathcal{F}(U_k) & \longrightarrow & \mathcal{F}(U_j \cap U_k) \end{array}$$

$\mathcal{F}(U_j) \longrightarrow \mathcal{F}(U_j \cap U_k)$ (#, $\rho_{j,k}$)

equalizer: $E \to \mathcal{F}(X) \rightrightarrows \prod \mathcal{F}(U_i)$

$\{f_i : U_i \to \mathcal{F}\}_i$

$f_j|_{U_j \cap U_k} = f_k|_{U_j \cap U_k}$

$I = \{1, 2\}$

$$F(X) \longrightarrow F(U_1) \times F(U_2) \rightrightarrows F(U_1 \cap U_2)$$

$$(f_1, f_2) \longmapsto (f_1|_{U_1 \cap U_2})$$
$$\longmapsto (f_2|_{U_1 \cap U_2})$$

$f_1 \in F(U_1)$

$f_2 \in F(U_2)$

Examples a) $M \in \mathrm{Man}$

$U \mapsto C^\infty(U, M)$

b) $M, N \in \mathrm{Man}$

$\mathrm{Hom}(M, N)$

$U \mapsto C^\infty(U \times M, N)$

$C^\infty(X \times M, N) \longrightarrow C^\infty(U \times M, N) \times C^\infty(V \times M, N)$

$\{U \times M, V \times M\}$ open cover of $X \times M$ $\quad C^\infty((U \cap V) \times M, N)$

c) $\Omega^n \quad U \mapsto \Omega^n U$

also: Ω^n closed

d) Riem +degenerate $U \mapsto \mathrm{Riem}_U$

$= \Gamma(\mathrm{Sym}^2 T^* U)$



Nonexamples a) $U \mapsto C^\infty_{\text{bounded}}(U, \mathbb{R})$ satisfies the gluing property for covers with 2 elements, but not for infinite covers

b) $U \mapsto \{f : U \to \mathbb{R} \mid \mathrm{Supp}\, f \text{ is finite}\}$

$\mathbb{C}^\times$ $\frac{dz}{z}$ c) Ω^n_{exact} (on Man, not Cart)

C^X $\quad \frac{dz}{z} \quad z = x + iy$

$dz = dx + i\,dy$

$$d\left(\frac{dz}{z}\right) = d\left(\frac{dx}{x+iy} + i\frac{dy}{x+iy}\right)$$

$$= d\left(\frac{1}{x+iy}\right) \wedge dx + i\,d\left(\frac{1}{x+iy}\right) \wedge dy$$

$$= \left(\frac{\partial}{\partial x}\left(\frac{1}{x+iy}\right)dx + \frac{\partial}{\partial y}\left(\frac{1}{x+iy}\right)dy\right) \wedge dx$$

$$+ i(\cdots) \wedge dy$$

$$= \left(-\frac{\partial}{\partial y}\left(\frac{1}{x+iy}\right) + i\frac{\partial}{\partial x}\left(\frac{1}{x+iy}\right)\right) dx \wedge dy$$

$$= \left(-(-1)\cdot\frac{1}{(x+iy)^2}\cdot i\right.$$

$$\left. + i(-1)\cdot\frac{1}{(x+iy)^2}\right) dx \wedge dy = 0$$

$\frac{dz}{z}$ is not exact on $\mathbb{C}^\times$

$$\int_{S^1} \frac{dz}{z} \qquad z = \exp(it), \quad 0 \le t \le 2\pi$$

$$\| $$

$$\int_0^{2\pi} \frac{i \cdot \exp(it)\, dt}{\exp(it)} = 2\pi i.$$

If $U \subset \mathbb{C}^\times$, $U \cong \mathbb{R}^2$,

then $\left(\frac{dz}{z}\right)\Big|_U$ is exact by the Poincaré lemma

$\mathbb{C}^\times = U \cup V$

$U = \mathbb{C} \setminus (-\infty, 0] \cong \mathbb{R}^2$ — U

$V = \mathbb{C} \setminus [0, \infty) \cong \mathbb{R}^2$ — V

$U \cap V = \mathbb{C} \setminus \mathbb{R} \cong \mathbb{R}^2 \sqcup \mathbb{R}^2$

What do sheaves on Cart have to do with sheaf cohomology

Two important dichotomies

Morphisms in the site:

- arbitrary maps, e.g., Cart, AffSch
  ↳ sheaves are "generalized spaces", e.g., smooth sets, algebraic spaces

gros topos ∪

- open embeddings, e.g., $\mathrm{Emb}_{\mathrm{AffSch}}$, open imm.;
  ↳ sheaves are "generalized manifolds": e.g., manifolds, orbifolds, etale stacks, schemes.

petit topos

Objects in the site:

absolute

- "spaces", e.g., Cart, AffSch, Embd, ...

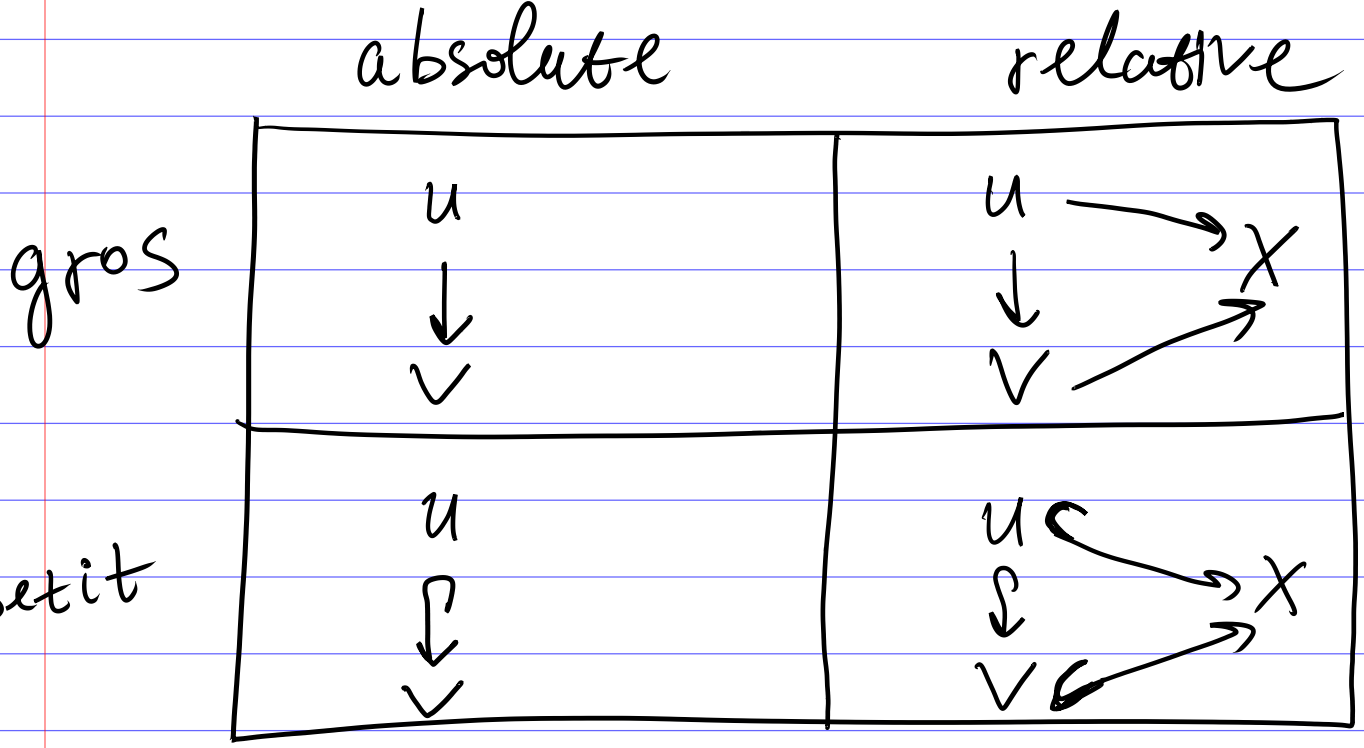
↳ sheaves are "generalized spaces/manifol

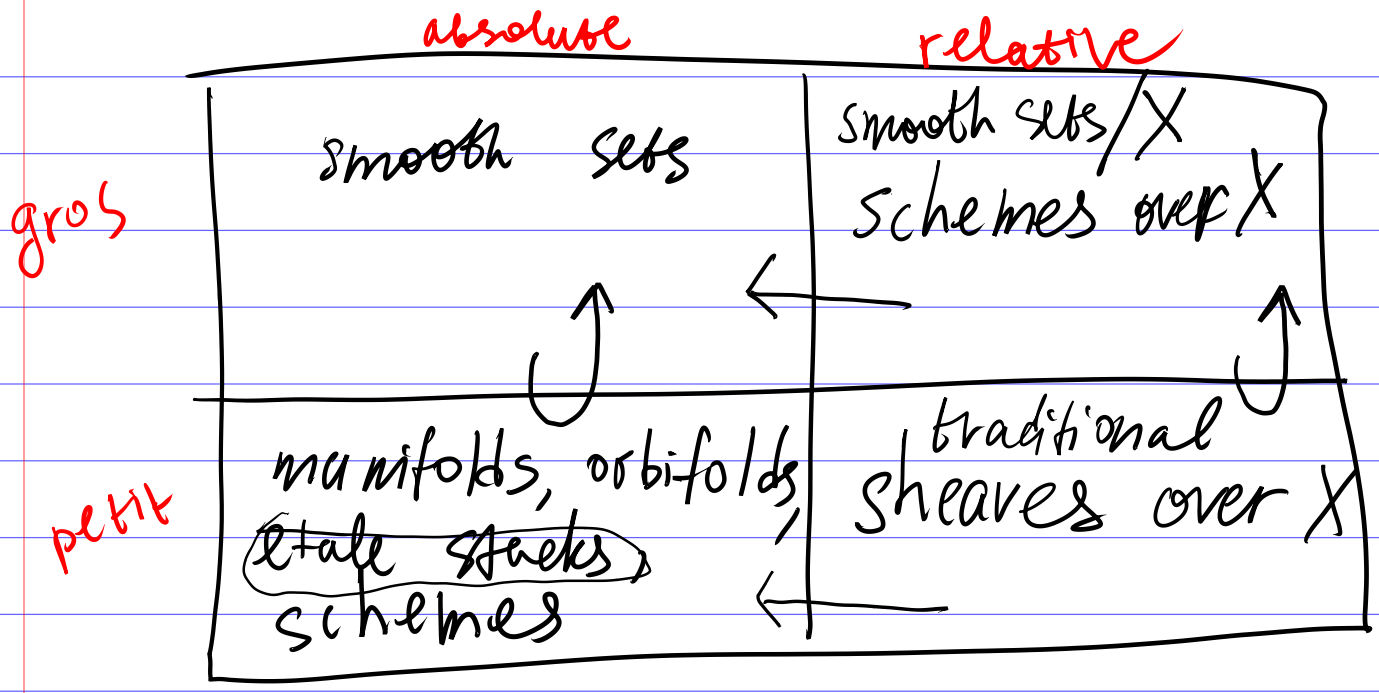
relative

- "spaces w/ a map to X"

X: an object in the site,
more generally: a sheaf on the site

↳ sheaves are generalized spaces with a map to X
(respectively: generalized manifolds w/ étale map to X).



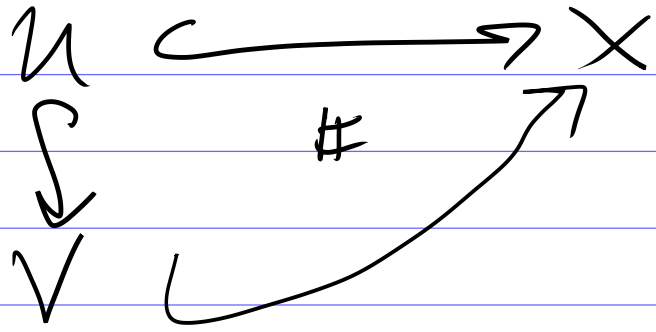


Traditional sheaves over $X \in \mathrm{Top}$:

site: $\mathrm{Open}(X)$ = poset of open subsets of X

Objects: $U \xrightarrow{\text{open}} X$

Morphisms:



What is "étale" about a traditional sheaf

A: Étale space / displayed space

$$Sh(X) \underset{}{\overset{\simeq}{\rightleftarrows}} Et/X$$

$$\| \qquad\qquad \mathcal{E} \xrightarrow{\text{local homeo}} X$$

$$\mathrm{Fun}_{Sh}(\mathrm{Open}(X)^{op}, Set) \qquad \mathcal{E} \xrightarrow{l.h.} \mathcal{E}' \xrightarrow{\text{local homeo}} X$$

$$\mathcal{F}: \mathrm{Open}(X)^{op} \to Set \qquad \mathcal{E} \xrightarrow{p} X$$

$$\mathcal{F}\left(\begin{array}{c} U \\ \downarrow \\ X \end{array}\right) = \left\{ \begin{array}{ccc} U & \to & \mathcal{E} \\ & \searrow^{\#} \swarrow & \\ & X & \end{array} \right\}$$

$$= \{ \text{sections of } p \text{ over } U \}$$

non-paracompact
non-Hausdorff

$$\mathcal{F} \longmapsto \text{étalé space of } \mathcal{F}$$

points are stalks of $\mathcal{F}$

petit $\hookrightarrow$ gros

$$Sh(\text{petit}) \underset{R}{\overset{L}{\rightleftarrows}} Sh(\text{gros})$$

The image of L
$=$ the subcategory
of etale stacks and etale maps

relative/X $\longrightarrow$ absolute

$U \to X \longmapsto U$

$$Sh(\text{relative}/X) \underset{(-)\times X}{\overset{L}{\rightleftarrows}} Sh(\text{absolute})$$

L: forget the map to X

$(-)\times X$: $\downarrow$ X

dmitripavlov.org/fft

Videos + notes

November 27

Lecture 6

Sheaves of groupoids and ∞-groupoids

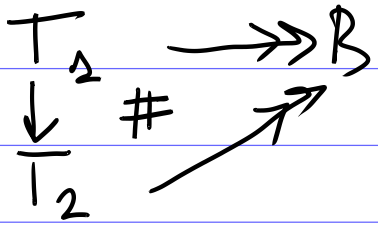
Def A vector bundle over $B \in \mathrm{Man}$ is a **module object** in Subm/B over the **ring object** R_B

Subm/B: Ob: $T \twoheadrightarrow B$ submersion (tangent map is surjective)

$A \times_B T \to T$, $A \to B$ (C^∞), $T \twoheadrightarrow B$ submersion

$\subset$ a B-indexed smooth family of smooth man.)

Mor:



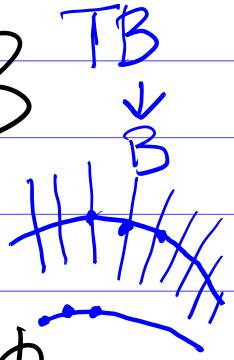
In Subm/B, colimits and connected limits are computed at the level of total sp.

$$\operatorname*{colim}_{i \in I} (T_i \twoheadrightarrow B) = \left(\operatorname*{colim}_{i \in I} T_i\right) \twoheadrightarrow B$$ (if exists!)

Products = fiber products over B

Example $\underbrace{\mathbb{R}_B}$ ring object in Subm/B

$\mathbb{R}\times B \longrightarrow B$



add, multiplication: $\mathbb{R}\times\mathbb{R}\times B \xrightarrow{+,\cdot} B$

$V\times B \to B$ trivial vector bundle

Summary A vector bundle over B is a smooth family of vector spaces indexed by B.

Question Can this be expressed using the language of sheaves?

$B \longrightarrow \mathrm{Vect}$

B: smooth set

~~smooth set (?)~~ smooth category/groupoid

$C^\infty(B, \mathrm{Vect})$

✗ = • the set of vector bundles over B (isomorphic bundles are different elements!)

✗ • the set of iso classes of v. bun. $/B$ (every vector bundle is locally trivial!) knows $\mathrm{Aut}(V)$

✓ • the category/groupoid of vector bundles $/B$

Vect cannot be a smooth set because $C^\infty(B, \mathrm{Vect})$ is a category!

Def A smooth category is

$$\mathcal{F} : \mathrm{Cart}^{op} \longrightarrow \underbrace{\mathrm{Cat}}_{\text{small categories and functors}}$$

such that the gluing condition [...].

$$\mathcal{F}(U \cup V) \longrightarrow \mathcal{F}(U) \times \mathcal{F}(V) \rightrightarrows \mathcal{F}(U \cap V)$$

is an equalizer fork (?)

Example $\mathrm{Vect} : \mathrm{Cart}^{op} \to \mathrm{Cat}$

$S \in \mathrm{Cart}$ $\mathrm{Vect}(S)$ = the cat. of vect. bun/S

Over a cartesian space S, all vector bundles are trivial

$V_1 \times S \to S$, $V_2 \times S \to S$

$V_1 \to V_2 : f \in C^\infty(S, \mathrm{Hom}(V_1, V_2))$

$S_1 \xrightarrow{h} S_2$ $\mathrm{Vect}(S_2) \to \mathrm{Vect}(S_1)$ ($\mathrm{Vect}(h)$)

$\begin{cases} V \mapsto V \\ f \mapsto f \circ h \end{cases}$

$\mathrm{Vect}: \mathrm{Cart}^{op} \longrightarrow \mathrm{Grpd}$

$S \in \mathrm{Cart}: \quad \mathrm{Vect}(S) \in \mathrm{Grpd}.$

Ob: finite-dimensional vector spaces

$\mathrm{Vect}(S)(V_1, V_2)$

$:= C^\infty(S, \mathrm{Hom}(V_1, V_2))$ ($\mathrm{Hom}(V_1, V_2) \in \mathrm{Cart}$)

$S_1 \xrightarrow{h} S_2 \qquad \mathrm{Vect}(S_2) \longrightarrow \mathrm{Vect}(S_1)$

Ob: id

$\mathrm{Vect}(S_2)(V_1, V_2) \to \mathrm{Vect}(S_1)(V_1, V_2)$

$f: S \to \mathrm{Hom}(V_1, V_2) \longmapsto f \circ h.$

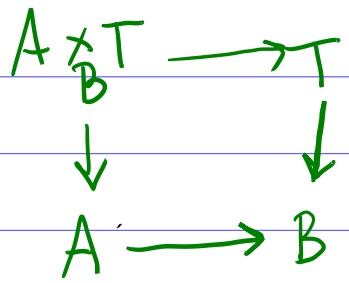
$\mathcal{Y}_{U_j \cap U_k} \longrightarrow \mathrm{Mor}(\mathrm{Vect})$

$\mathrm{Mor}(\mathrm{Vect})(U_j \cap U_k)$

$= \mathrm{Mor}(\mathrm{Vect}(U_j \cap U_k))$

$= \coprod_{V_1, V_2} C^\infty(S, \mathrm{Hom}(V_1, V_2))$

Multiple defects!



Defect 1: $M \in \mathrm{Man}$

$C^\infty(M, \mathrm{Vect})$ = the category of trivial vector bundles over M.

(Want: all vector bundles.)

Defect 2 $\mathrm{Vect}'(S)$ = category of vector bundles over S.

The inclusion $\mathrm{Vect} \longrightarrow \mathrm{Vect}'$

$V \longmapsto (V \times S \to S)$

is an objectwise equivalence but not an isomorphism.

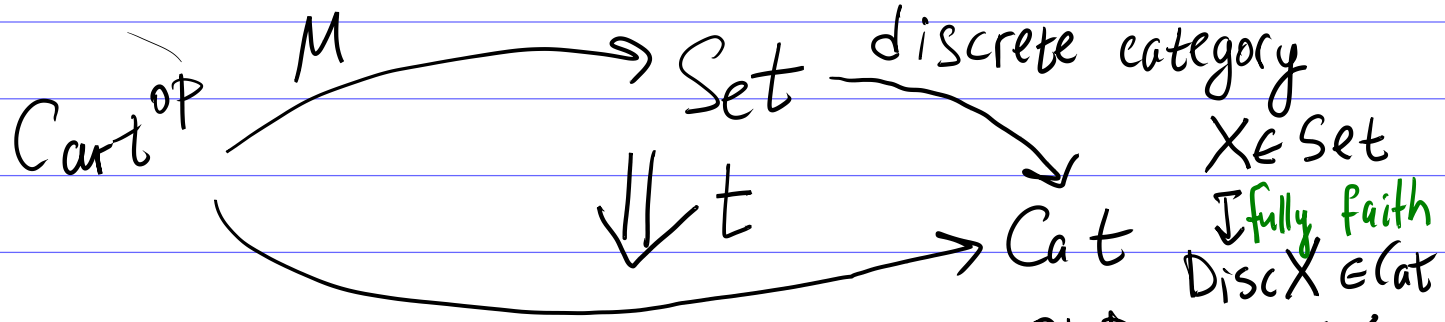
Our formalism has no way of talking about equivalences of presheaves

How to compute $C^\infty(M, Vect)$, $M \in Man$?

$\hookrightarrow C^\infty Set = Fun(Cart^{op}, Set)$

$S \longmapsto C^\infty(S, M)$

What is a morphism $M \xrightarrow{t} Vect$?



$S \in Cart$: $t_S : (Disc\, M)(S) \to Vect(S)$

discrete cat

$S_1 \xrightarrow{h} S_2$

$Ob \quad \{S \xrightarrow{f} M\} \longmapsto Ob\, Vect(S)$

$f \longmapsto V_f \in Vect$ (vector spaces)

$f \longmapsto V_f$, $h \longmapsto V_h$

$f \longmapsto f \circ h$

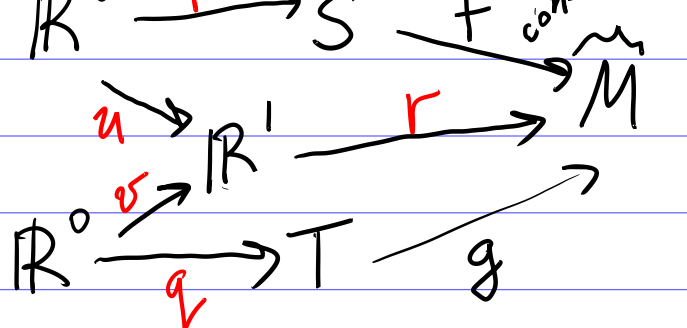
$M(S_2) \to M(S_1)$

t_{S_2} #

$Vect(S_2) \xrightarrow{Vect(h)} Vect(S_1)$

$V_f \mapsto V_{f \circ h} = V_f$

$V_f = V_g$



$$V_f = V_{f\circ p} = V_{r\circ u} = V_r$$

$$= V_{r\circ v} = V_{g\circ q} = V_g$$

t yields a vector space V

(i.e., the trivial vector bundle $\begin{array}{c} V\times M \\ \downarrow \\ M \end{array}$).

There is no place for transition functions!

Instead of $V_f = V_{f\circ h}$,

we want $V_f \xrightarrow[\star]{\cong} V_{f\circ h}$.

Solution (Grothendieck, 1960s)

Incorporate isomorphisms like $\star$ into natural transformations $\Rightarrow$ stacks!

Solution 1 Smooth categories

form a 2-category / 2-groupoid

⇝ Same problem repeats for n-groupoids

Solution 2 Smooth ∞-groupoids

form an ∞-groupoid.

electromagnetic field = principal $\mathcal{U}(1)$-bundle w/ ∇

(groupoid)

= hermitian line bundle w/ ∇

B-field = bundle gerbe w/ ∇

(2-groupoid)

SUGRA C-field = bundle 2-gerbe w/ ∇

(C = categorified line bundle w/ ∇)

(3-groupoid)

Ramond–Ramond field = cocycle in diff K-theory

(∞-groupoid)

Lecture 7: December 3

Homotopy theory

"Homotopy theory studies isomorphisms between isomorphisms."

"Homotopy theory studies ∞-categorical structures."

Higher category: has k-morphisms for $k \geq 2$

∞-groupoid: k-mor are invertible $k \geq 1$

$(\infty, 1)$-category: k-mor are invertible $k \geq 2$

Equivalent models for $(\infty, 1)$-categories:

1) Relative categories (C, W) (model categories)
2) Simplicial categories (=enriched in sSet)
3) Complete Segal spaces
4) Quasicategories (="∞-categories")

Motivation:

Recall: want a presheaf Vect such that the groupoid of maps

$$X \longrightarrow \mathrm{Vect}$$
$$\uparrow$$
$$\mathrm{Man}$$

is the groupoid of vector bundles over X

If $X \in \mathrm{Cart}$ is a representable presheaf, the groupoid of maps $X \to \mathrm{Vect}$ is equivalent to $\mathrm{Vect}(X)$.
This determines Vect uniquely as a presheaf of groupoids.

Defect Maps $X \to \mathrm{Vect}$ are trivial vector bundles over X (they ignore morphisms in X)

What we really want: $\{U_i\}_{i \in I}$ good open cover of X.

$\mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{Grpd})$
$\simeq$
$\mathrm{Grpd}(\mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{Set}))$

X — trivial vect bun → Vect

$\check{C}U$ — inverted formally → Vect

$\check{C}U$ coycle data for a vector bundle/(X vector bundles)

Call

Here $\check{C}U$ is a presheaf of groupoids:
(Alexandroff, 1928) Čech nerve of $\{U_i\}_{i \in I}$

$\mathcal{C}\mathcal{U}$: smooth set of objects $U_i \in \mathrm{Cart}$

$U_j \cap U_k \in \mathrm{Cart}$

Ob: (x,i) $x \in X$ $x \in U_i$ is $\coprod_{i \in I} U_i$ $y_{U_i} = \mathrm{Cart}(-, U_i)$

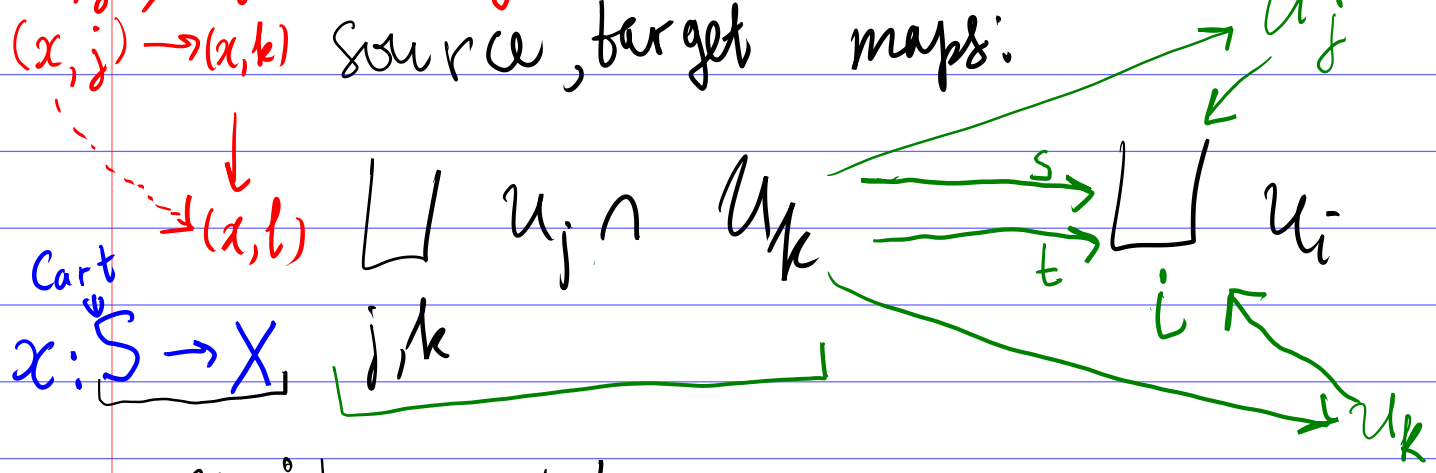
smooth set of morphisms of

Mor: $(x,j,k): (x,j) \to (x,k)$ $x \in X$ $x \in U_j \cap U_k$ $\coprod_{j,k} U_j \cap U_k$

$s(x,j,k) = (x,j)$ $t(x,j,k) = (x,k)$

$(x,j) \not\to (y,k)$ if $x \neq y$

$(x,j) \to (x,k)$ source, target maps:



$S = U_j \cap U_k$ identity: $\coprod_{i \in I} U_i \longrightarrow \coprod_{j,k} U_j \cap U_k$ $j = k = i$

composition:

$$\left(\coprod_{j,k} U_j \cap U_k\right) \times_{\coprod_k U_k} \left(\coprod_{k,l} U_k \cap U_l\right) \to \coprod_{j,l} U_j \cap U_l$$

What is a morphism $\check{C}\mathcal{U} \longrightarrow \mathrm{Vect}$?

Objects: $\coprod_{i \in I} U_i \longrightarrow \mathrm{Ob}(\mathrm{Vect})$

i.e., $V_i \in \mathrm{Vect}(U_i)$

(a vector space)

(encodes a trivial vect bundle U_i w/ fiber V_i)

Morphisms: $\coprod_{j,k} U_j \cap U_k \longrightarrow \mathrm{Mor}(\mathrm{Vect})$

(compatible w/ source, target)

$$t_{j,k} \in C^\infty(U_j \cap U_k, \mathrm{Hom}(V_j, V_k))$$

transition function

Preserves identities: $t_{i,i} = \mathrm{const}_{\mathrm{id}_{V_i}}$ (normalization)

Preserves composition: $t_{k,\ell} \circ t_{j,k} = t_{j,\ell}$ (pointwise composition; cocycle condition)

(restricted to $U_j \cap U_k \cap U_\ell$)

$\rightsquigarrow$ cocycle description of vector bundles.

Main takeaway: a "morphism" $X \to \mathrm{Vect}$ is a zigzag

$X \longleftarrow \mathcal{U} \longrightarrow \mathrm{Vect}$.

Cocycles for vector bundles

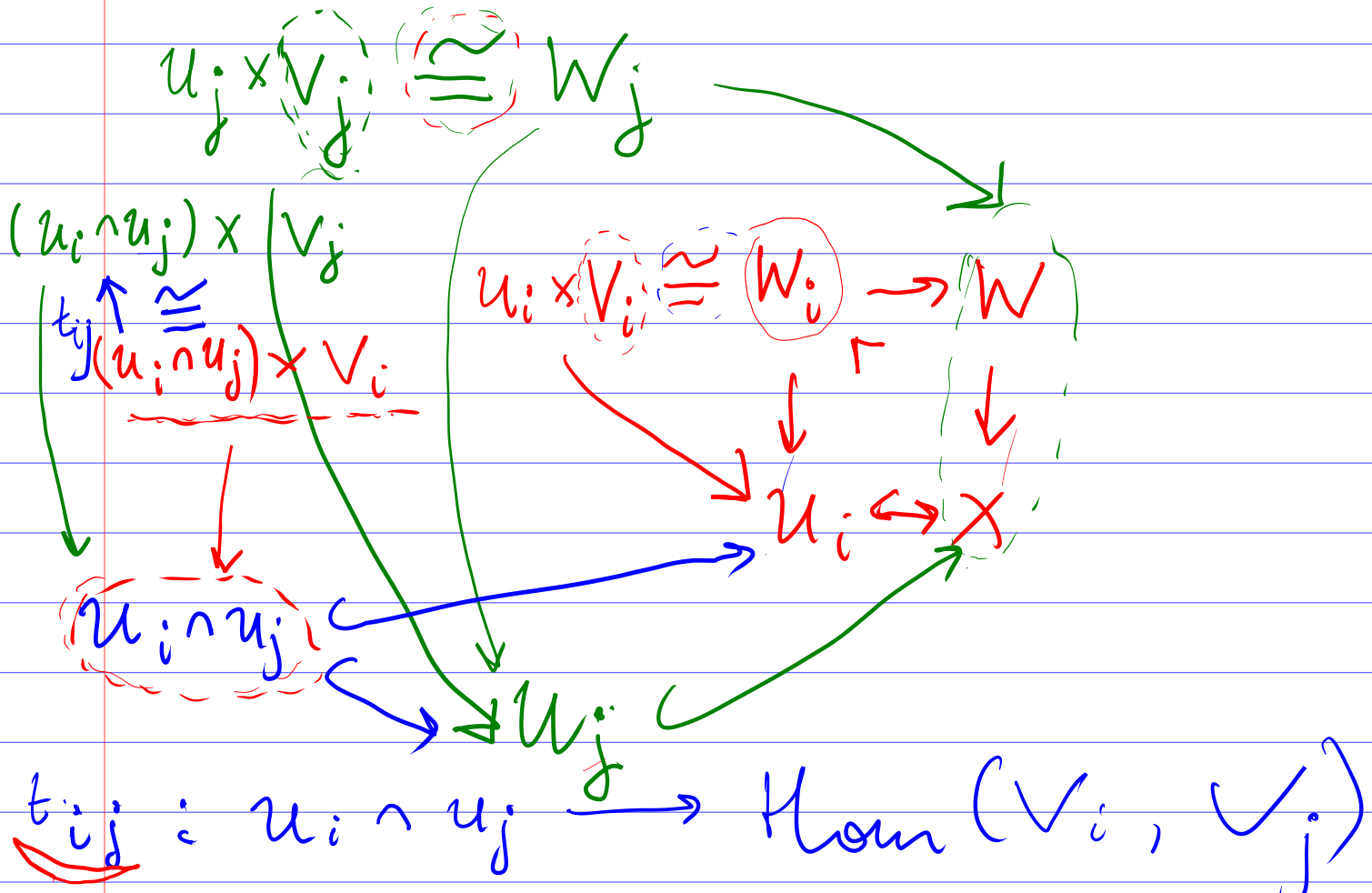
W

$\downarrow$

X

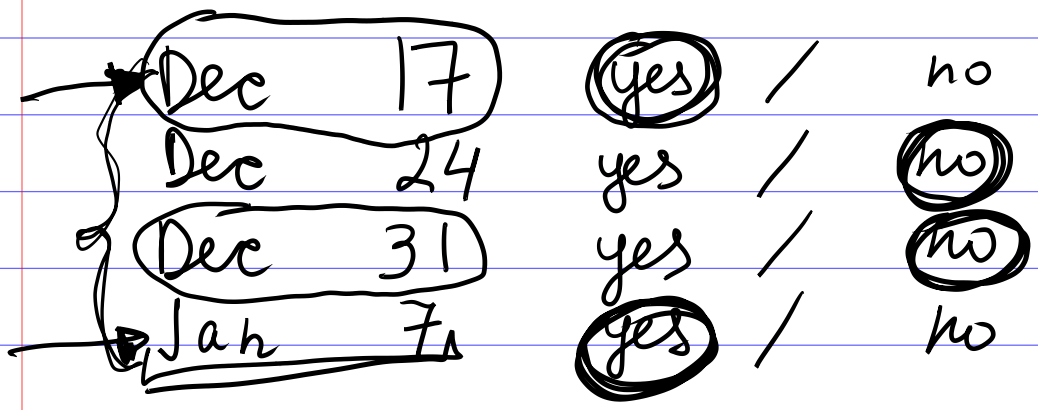
$\exists \{U_i\}_{i \in I}$

How to motivate t_{ij} and



Lecture 8: December 10.

Next lectures



Last time: $\mathrm{Vect} : \mathrm{Cart}^{op} \to \mathrm{Grpd}$

$X \in \mathrm{Man}$, $\{U_i\}_{i \in I}$ good open cover of X

$X \to \mathrm{Vect}$: trivial vector bundles / X

$\check{C}U \to \mathrm{Vect}$: Čech cocycles for vbun/X

What we really want:

$$\check{C}U \xrightarrow{L} X \quad \text{is invertible, so that}$$

$$X \xleftarrow{L} \check{C}U \longrightarrow \mathrm{Vect}$$

L^{-1}

allows us to present arbitrary vbun/X

Defect: L is not invertible

Suppose $\{U, V\}$ is a good open cover of X

X is connected

$\check{C}\{U, V\}$ $\qquad$ $\check{C}\{U, V\} \xrightarrow{\iota} X$

Ob: $U \sqcup V \longrightarrow X$

Mor: $U \sqcup V \sqcup (U \cap V) \sqcup (V \cap U) \longrightarrow X$

ι does not have an inverse

ι^{-1}: Ob: $X \xrightarrow{\iota^{-1}} \underbrace{U} \sqcup \underbrace{V}$

$u \in U \setminus V$

$v \in V \setminus U$

Remark: ι does have an inverse locally

$$\begin{array}{ccc} \check{C}\{\tilde{U}, U \cap V\} & \to & \check{C}\{U, V\} \\ \downarrow & \lrcorner & \downarrow \iota \\ \underbrace{U} & \longrightarrow & X \end{array}$$

Also: the underlying map of groupoids of ι is an equivalence of groupoids

Def A relative category is $(\mathcal{C}, W)$,

where $\mathcal{C}$ is a category,

$W \subset \mathcal{C}$ (weak equivalences) class of morphisms closed under $\circ$.

Example $\mathcal{C} = \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{Grpd})$,

W = stalkwise equivalences = local equivalences (Diff cohomology)

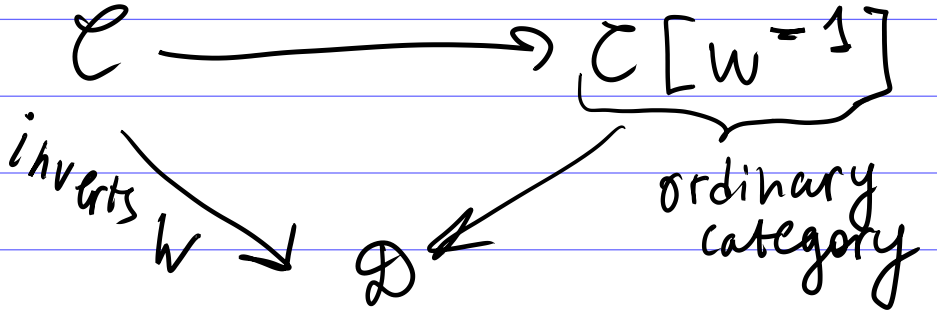
§3.4 (Amabel–Debray–Haine)

$f: F \xrightarrow{\sim} G$ if $\forall n \geq 0$: $\operatorname{colim}_{\varepsilon \to 0} F(B^n_\varepsilon) \xrightarrow{\sim} \operatorname{colim}_{\varepsilon \to 0} G(B^n_\varepsilon)$

n-dimensional stalk of f

Traditional approach: Study $\mathcal{C}[W^{-1}]$:

category of fractions = homotopy category



(Whitehead, 1946)

"New" / "Modern" approach (Quillen 1967)

$(\infty,1)$-category of fractions = homotopy $(\infty,1)$-category.

Example $\mathcal{C} = \underline{Fun(Open(X)^{op}, Set)}$,
$\quad\quad\quad

$$\operatorname{colim}_{U \in \mathrm{Open}(X),\, x \in U} \mathcal{F}(U) = \left(\bigsqcup_{U \in \mathrm{Open}(X),\, x \in U} \mathcal{F}(U) \right) / \sim$$

$U \to V$

$U \supset V \ni x$

$\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

ψ

s, t

$s \in \mathcal{F}(U)$ $\quad s \sim t$ if

$t \in \mathcal{F}(V)$ $\quad U \supset V$

and $s|_V = t$.

$s \in \mathcal{F}(U)$ $\quad t \in \mathcal{F}(V)$

$U \supset U \cap V \subset V$

$U \cap V = W$

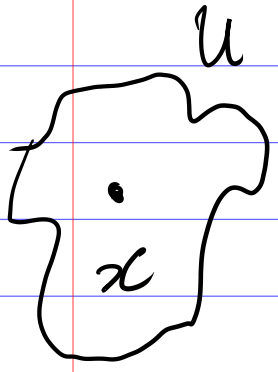
$s \sim s|_W = t|_W \sim t$

Suppose $X \subset \mathbb{R}^n$ (open) or: X is a metric space

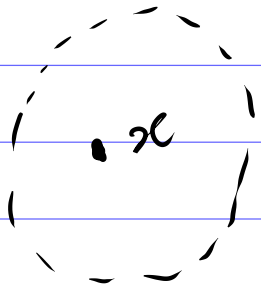
$$\mathcal{F}: \mathrm{Open}(X)^{op} \longrightarrow \mathrm{Set}$$

Stalks of $\mathcal{F}$ at $x \in X$

$$= \operatorname{colim}_{\substack{U \in \mathrm{Open}(X) \\ x \in U}} \mathcal{F}(U)$$



$$\cong \operatorname{colim}_{\varepsilon \to 0} \mathcal{F}(B_{x,\varepsilon})$$



the poset of open balls around x is <u>cofinal</u> in the poset of open neighborhoods of x

Def $(\mathcal{C}, W)$ relative category.

The homotopy (2,1)-category of $(\mathcal{C}, W)$ is a functor $\mathcal{C} \xrightarrow{L} \mathcal{C}[W^{-1}]$ such that for any category $\mathcal{D}$ the induced functor

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{L^*} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is a fully faithful functor whose essential image is given by functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ such that F maps elements of W to isomorphisms in $\mathcal{D}$.

$\mathcal{C} = \mathrm{Fun}(\mathrm{Open}(X)^{op}, \mathrm{Set}) = \mathrm{PreSh}(X)$

$W =$ stalkwise/local isomorphisms

Proposition There is an equivalence

$$\underbrace{\mathcal{C}[W^{-1}]}_{\text{1-category}} \rightleftarrows \underbrace{Sh(X)}_{\text{1-category}}$$

Proof (sketch)

a) $Sh(X) \hookrightarrow PreSh(X) = \mathcal{C} \to \mathcal{C}[W^{-1}]$

b) $\mathcal{C}[W^{-1}] \to Sh(X) = \mathcal{D}$

$PreSh(X) = \mathcal{C} \xrightarrow{a} Sh(X)$ associated sheaf

a sends local isos to isos.

c) $Sh(X) \to \mathcal{C}[W^{-1}] \xrightarrow{a} Sh(X)$

$F \overset{\psi}{\longmapsto} \iota F \longmapsto aF$

$F \to aF$ is an iso $\iff$ F is a sheaf

d) $\mathcal{C}[W^{-1}] \xrightarrow{a} Sh(X) \to \mathcal{C}[W^{-1}]$

$\iota \mathcal{F}' \longmapsto a\mathcal{F} \longmapsto \iota a \mathcal{F}$

$\iota(\mathcal{F} \xrightarrow[\text{local isomorphism}]{} a\mathcal{F})$

Example $\mathcal{C} = Fun(Open(X)^{op}, Grpd)$

Or $\mathcal{C} = Fun(Cart^{op}, Grpd)$

W = local equivalences /stalkwise

$\mathcal{C}[W^{-1}]$ is not equivalent to $Sh(X)$ $Sh(Cart)$

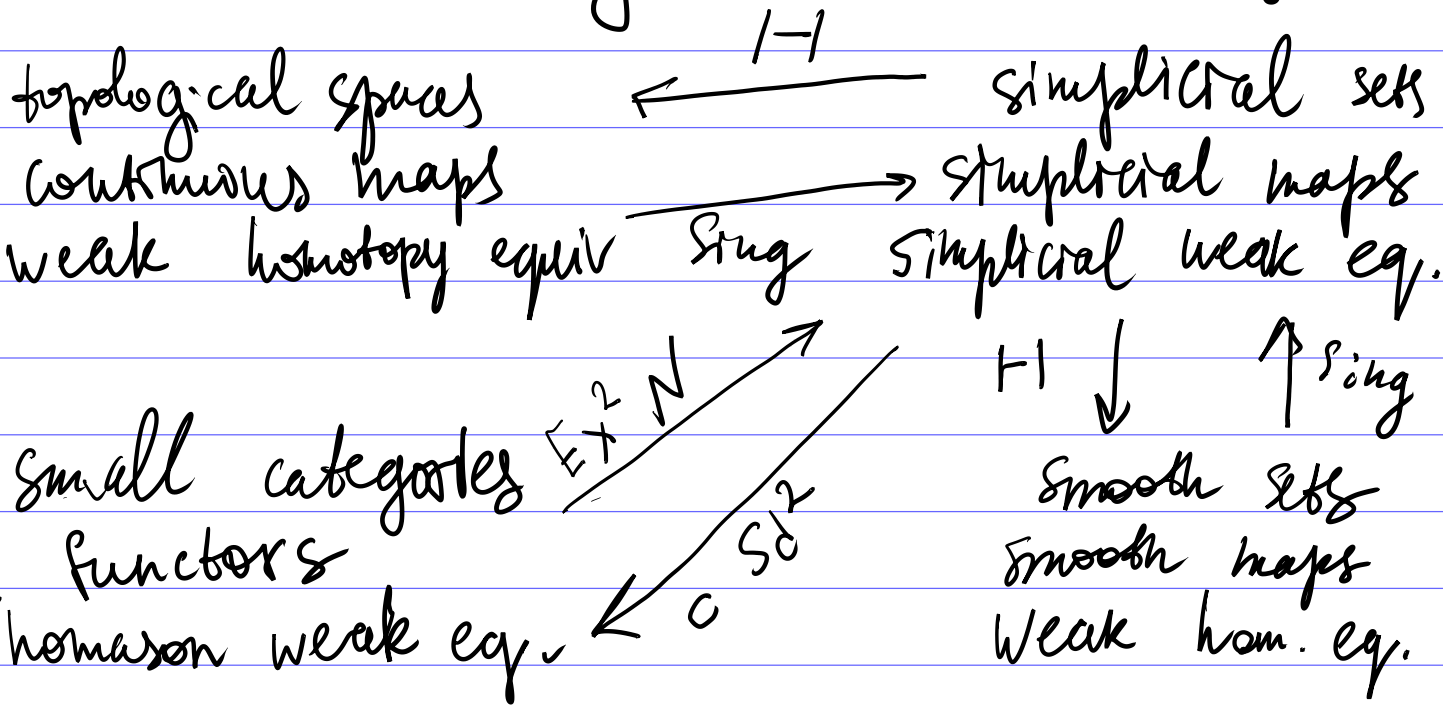
$\mathcal{C}[W^{-1}]$: 1-category, homotopy category of stacks

$Sh(X)$ $Sh(Cart)$: 2-category, $(2,1)$-category

$(\infty,1)$-category: like an ordinary category, but $\mathcal{C}(X,Y)$ is a space, not a set.

What is a space?

Practical answer: an object of any of the following relative categories:



What does it mean to enrich a cat in sSet?

equivalent models

	enriched: objects form a set	internal: objects form an object
strict / algebraic	enriched category: simplicial category	internal category: simplicial object in cats.
weak / geometric	Segal category	complete Segal space

Other models:

- Relative categories
- Quasicategories
- Riehl – Verity (a family of models)

Potentially model-independent: homotopy type theory (no working def of (∞,1)-cat.)

Lecture 9 : December 17

Next lecture: January 7

Homework 1: dmitripavlov.org/fft

Discord

What is a space (= ∞-groupoid) (in homotopy theory)?

Possible answers: (in practice)

topological spaces up to weak homotopy equivalences
↕
simplicial sets up to simplicial weak equiv.
smooth sets up to smooth weak equiv.
small categories up to Thomason weak equiv.

What does "up to" mean?

→ spaces form an $(\infty, 1)$-category

→ category enriched in spaces (again)

→ chicken-and-egg problem: need to talk about spaces first.

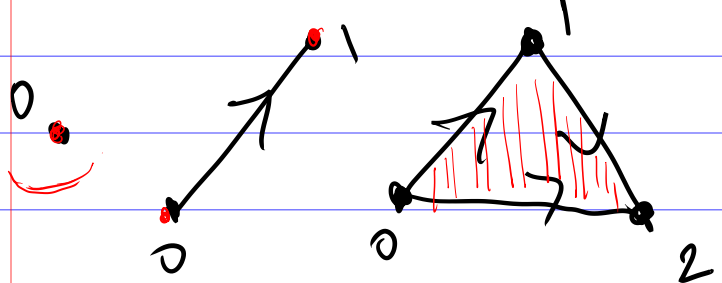
Simplicial sets: a practical crash-course

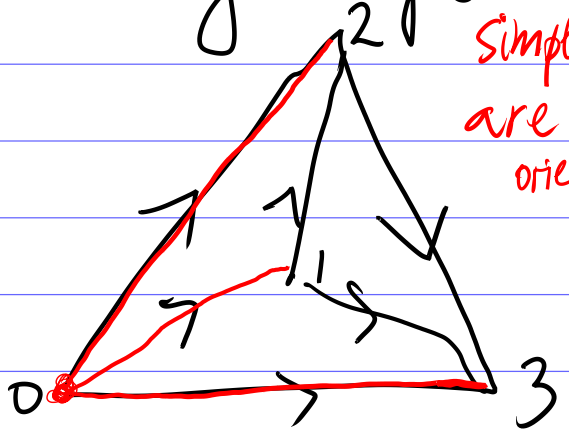
Def $\quad sSet = Fun(\Delta^{op}, Set)$

Δ = Ob: $\{0, 1, \ldots, n\}$ $\cong [n]$ $\quad n \geq 0$

vertices

Mor: nondecreasing maps





Simplices are oriented!

Typical simplicial set:



Why nondecreasing?
[can do 1) increasing 2) arbitrary]

Why simplices?
[can do: cubes, cells, ...]
answers on the next page!

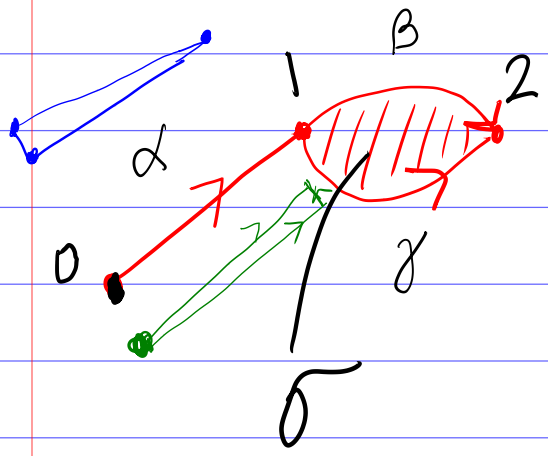
Limits and colimits in sSet: computed objectwise!

$(\lim_{i \in I} X_i)_n = \lim_{i \in I} ((X_i)_n)$

Def

$$\begin{array}{ccc} Cat & \xrightarrow{N} & sSet \\ \mathcal{C} & \longmapsto & N\mathcal{C} \end{array}$$

$$N\mathcal{C}_n = Fun(\{0 \to 1 \to 2 \to \cdots \to n\}, \mathcal{C}).$$



X

$X : \Delta^{op} \to Set$

$X([0]) = sSet(\Delta^0, X)$

$\| \qquad\qquad \|$

$\{0, 1, 2\} \qquad y_{[0]}$

$[0] \rightrightarrows [1]$

$\|$

$\{0\} \quad 0 \mapsto 0 \ (d^1), \ 0 \mapsto 1 \ (d^0) \quad \{0, 1\}$

$X([1]) = sSet(\Delta^1, X)$

$\|$

$\bullet \xrightarrow{d^1} \bullet \to \{\alpha, \beta, \gamma, s_0 0, s_0 1, s_0 2\}$

$\bullet \xrightarrow{d^0} \bullet \to$

$X([2]) = \{\sigma, s_0 s_0 0,$

$X(d^1) : X_1 \to X_0$ $\quad s_0 s_0 1, \ s_0 s_0 2,$

$\|$

d_1

$\alpha \mapsto 0$

$\beta \mapsto 1$

$\gamma \mapsto 1$

$s_1\alpha, \ s_0\alpha,$

$s_0\beta, \ s_1\beta, \ s_0\gamma, \ s_1\gamma\}.$

$X(d^0) : X_1 \to X_0$

$\|$

d_0

$\alpha \mapsto 1$

$\beta, \gamma \mapsto 2$

$X(s^0) = s_0$

$s_0(0) = s_0 0$

$\to \xrightarrow{s^0} \bullet \to X$

dmitripavlov.org/reading

Friedman

An elementary illustrated introduction to simplicial sets

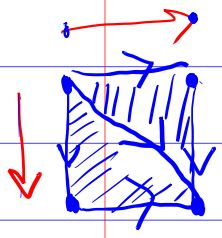
Why nondecreasing and not increasing (i.e., injective)?

⇝ semisimplicial sets

Defects: categorical products are not homotopy products

In fact: homotopy products require zigzags!

$$A \times B \to A \qquad A \times^h B \to B' \xleftarrow{\sim} B$$
$$\downarrow \qquad\qquad \downarrow$$
$$A \qquad\qquad A'$$
$$\qquad\qquad \uparrow \sim$$
$$\qquad\qquad A$$



Example: $A = B = S^1$

$\Rightarrow A \times B$ must have a 2-cell

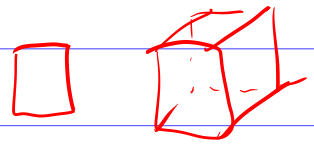
Why preserve ordering?

⇝ symmetric simplicial sets — $\Delta^h \to \Delta^n$

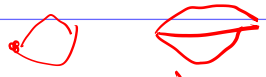
Simplicial complex ≡ concrete symmetric semisimplicial set.

Why simplices and not cubes, cells, etc.?

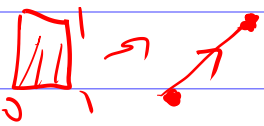
⇝ cubical set (Kan 1950s)



Same problem with products



⇝ cubical set with connections



Monoids, Categories $\longleftrightarrow$ simplicial objects $\Delta^{op} \to Set$

Homotopy coherent monoids and categories can be defined as certain simplicial objects!

The nerve construction for monoids (and categories)

= classifying space

Notation B: "bar construction" / delooping

shifts categorical degrees by 1

N: nerve

does not change categorical degrees.

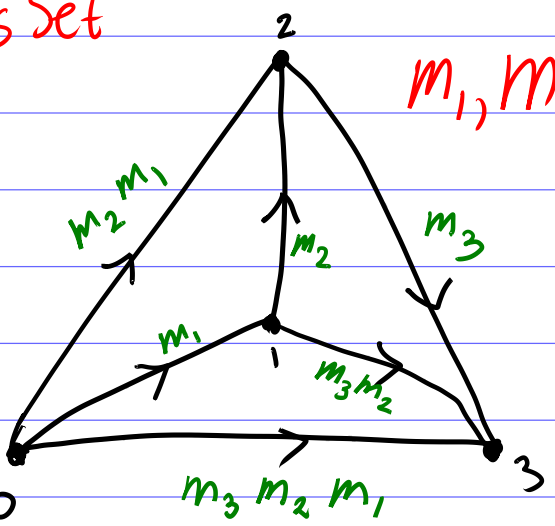
Example: $\pi_n(BG) \cong \pi_{n-1}(G)$, $\pi_n(NG) \cong \pi_n(G)$ } G: Picard groupoid $\equiv$ Sym. mon. groupoid w/ invertible obj.

$NG \simeq \Omega BG$

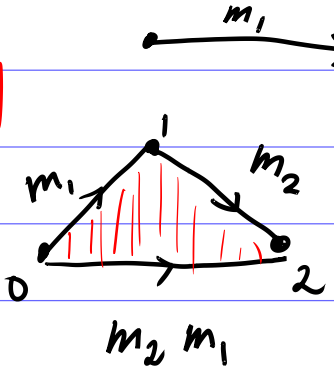
B: Monoid $\to$ sSet $= Fun(\Delta^{op}, Set)$

M: monoid $(BM)_n$:

$BM \in sSet$



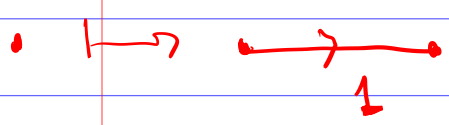
$m_1, m_2, m_3 \in M$



$m_1 \in M$

m_2

$m_1, m_2 \in M$



The edge $i \to j$ has $m_j \; m_{j-1} \; \cdots \; m_{i+1}$

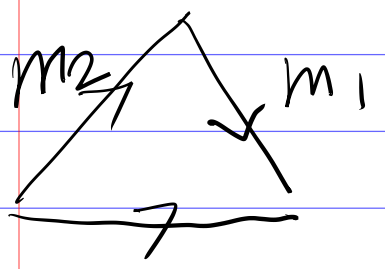
C: category $(NC)_n$: same pictures, but vertices are now objects in C and edges are morphisms

$M = \{1, e\}$ $1 = 1 \cdot 1$ $1 \cdot e = e \cdot 1 = e$

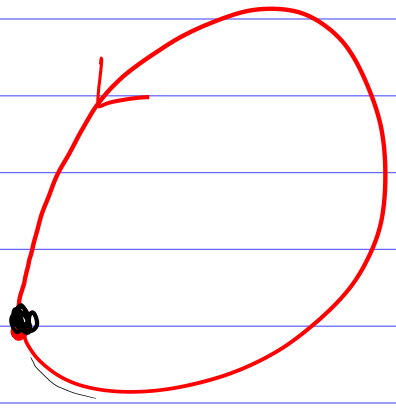
$e \cdot e = e$

M is a group $\Rightarrow M \simeq \Omega BM$

BM



$m_2 m_1$

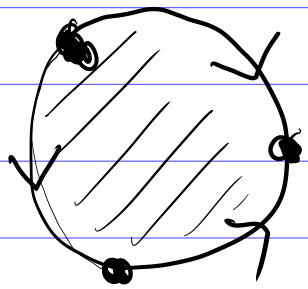


algebraic interpretation for BM:

$\underline{BM}$ = ∞-groupoid w/ a single object = ∞-group

$M[M_h^{-1}]$ = homotopy group completion of M = ΩBM

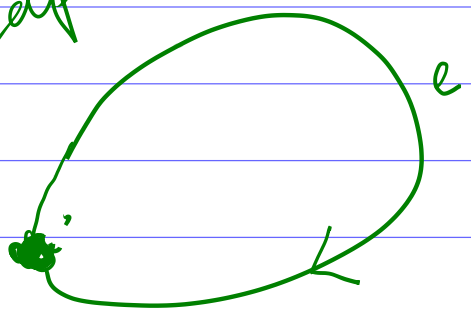


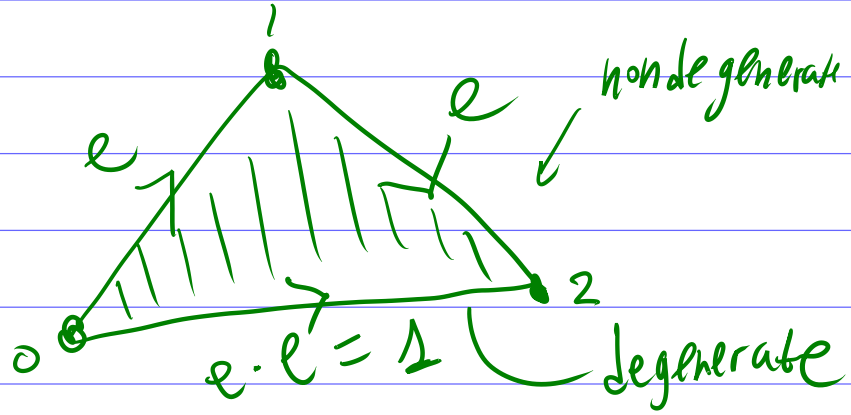


$M = \mathbb{Z}/2 = \{1, e\}$ $1 \cdot e = e \cdot 1 = e$ $1 \cdot 1 = 1$

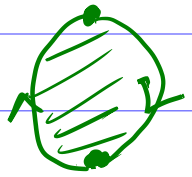
$e \cdot e = 1$

dunce cap





real projective plane



Example: $\mathcal{C} = \underline{B(\mathbb{Z}/2)}$

$N\mathcal{C}_n = 0 \xrightarrow{m_1} 1 \xrightarrow{m_2} \cdots \xrightarrow{m_n} n$

$\alpha_i \in \mathbb{Z}/2$. Nondegenerate: $m_i = [1]$.

Thus, $NB(\mathbb{Z}/2)$ is a model for $\underline{\mathbb{R}P^\infty}$ with a single n-cell for every $n \geq 0$.

$$\mathbb{R}P^\infty = S^\infty/(\mathbb{Z}/2)$$

How to characterize the images of $N: Cat \to sSet$ or $N: Grpd \to sSet$?

$$(N\mathcal{C})_n = \underset{Ob\mathcal{C}}{Mor\,\mathcal{C}} \times \underset{Ob\mathcal{C}}{Mor\,\mathcal{C}} \times \cdots \times \underset{Ob\mathcal{C}}{Mor\,\mathcal{C}}$$

Thus: $(N\mathcal{C})_{m+n} \xrightarrow{\cong} (N\mathcal{C})_m \underset{(N\mathcal{C})_0}{\times} (N\mathcal{C})_n$

Practical models for $(\infty,1)$-categories:

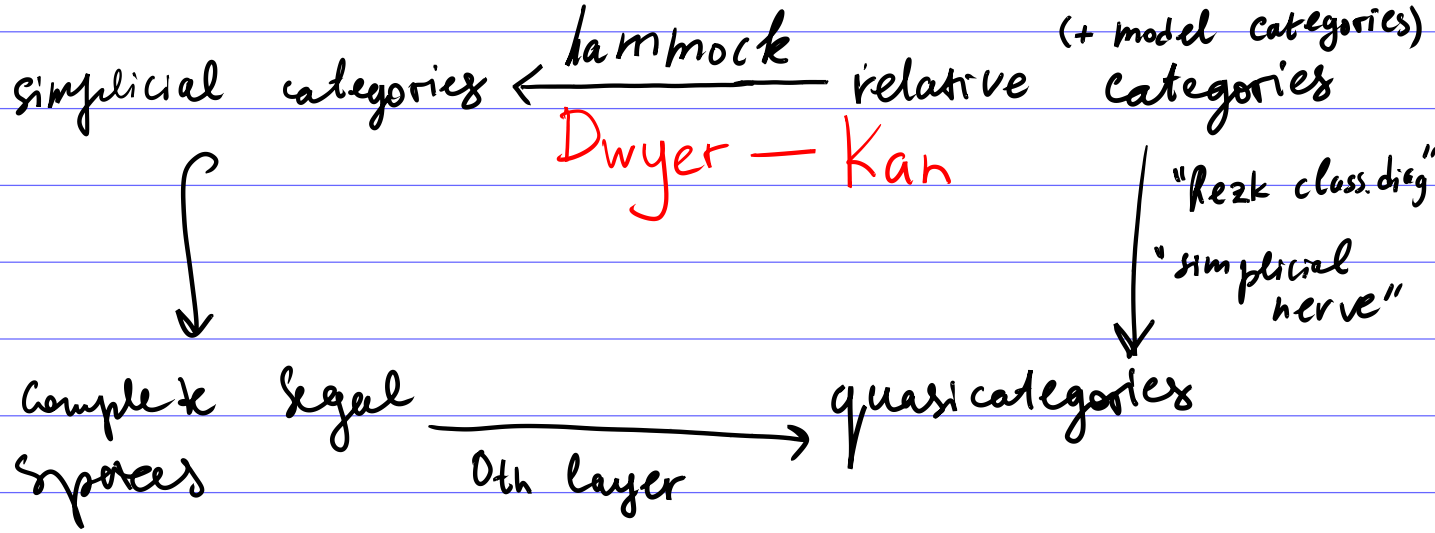
Quillen equivalent model categories:

Relative categories, model categories: examples, concrete computations

→ Quasi categories: theoretical constructions

→ Complete Segal spaces: geometric examples

Simplicial categories: quickly conveys the main idea; occasional theoretical constructions



Def A relative category is $(\mathcal{C}, W)$
$\mathcal{C}$: category; $W \subset \mathcal{C}$ subcategory with the same objects
↑ weak equivalences

Def A simplicial category is
a category enriched in simplicial sets
$\mathcal{C}$: $\forall X, Y \in \mathcal{C}$: $\mathcal{C}(X,Y) \in sSet$

Relative categories → simplicial categories

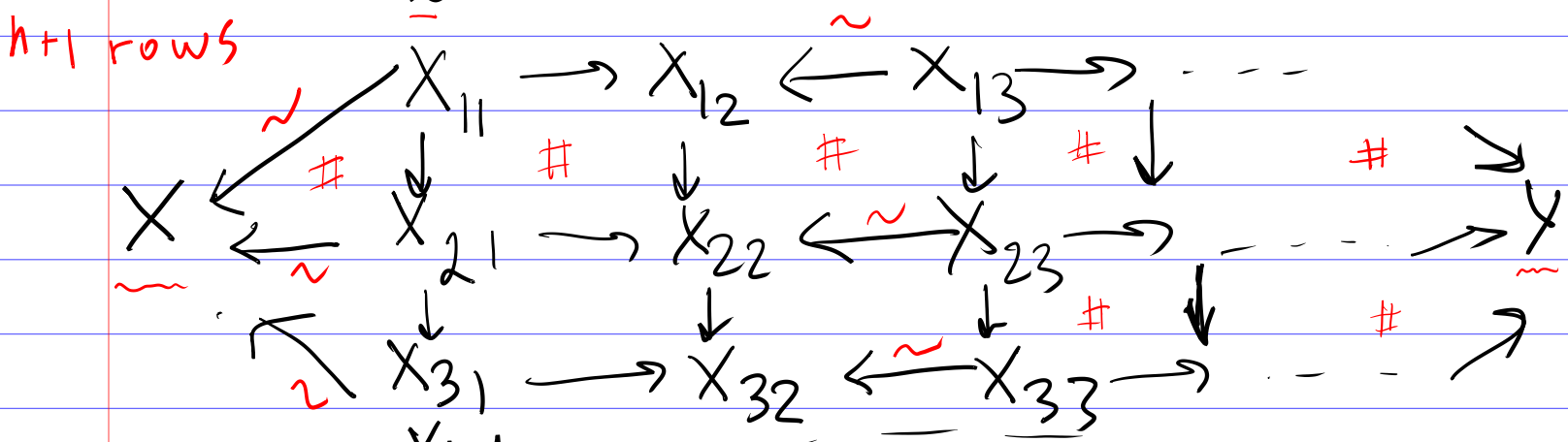
(Hammock localization)

relative cat $(\mathcal{C}, W) \longmapsto \mathcal{D} = L^H(\mathcal{C}, W)$ simplicial cat

$\mathrm{Ob}\,\mathcal{D} = \mathrm{Ob}\,\mathcal{C}$

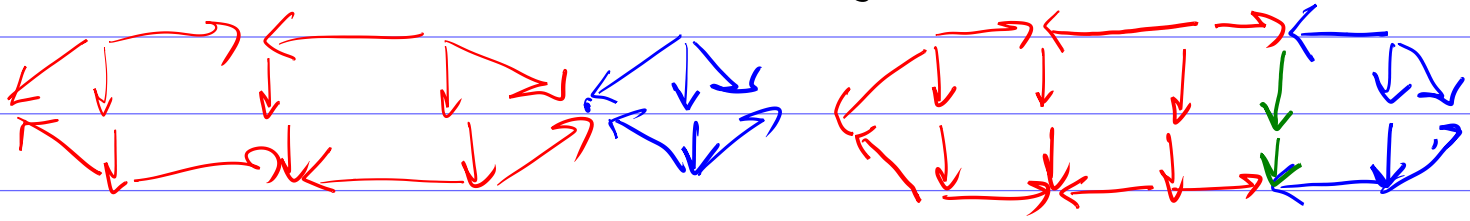
$X, Y \in \mathrm{Ob}\,\mathcal{D}$ $\quad \mathcal{D}(X, Y) \in s\mathrm{Set}$.

$\mathcal{D}(X, Y)_n$: equivalence classes of hammocks



Vertical & left-pointing maps $\in W$

- Can compose two composable columns
- Can insert identity column -



Thus: 0-simplices in $\mathcal{D}(X,Y)$ are

$$X \xleftarrow{\sim} X_1 \longrightarrow X_2 \xleftarrow{\sim} X_3 \to \cdots Y$$

Thus: $\pi_0 \mathcal{D}(X,Y) \cong \mathcal{C}[W^{-1}](X,Y)$.

category of fractions / homotopy cat.

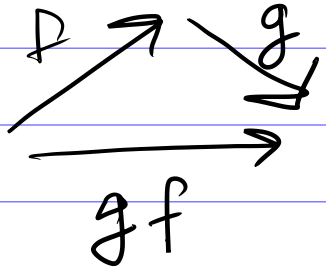
Relative categories & simplicial cats carry the same amount of info!

Model categories

A model structure on a relative category is auxiliary data that affects the specific presentation of the final answer and intermediate computations, but does not affect the final answer itself, up to a weak equivalence.

Def A model structure on a relative category $(\mathcal{C}, W)$ is $Cof, Fib \subset \mathcal{C}$ (cofibrations, fibrations) such that

- W satisfies 2-out-of-3:

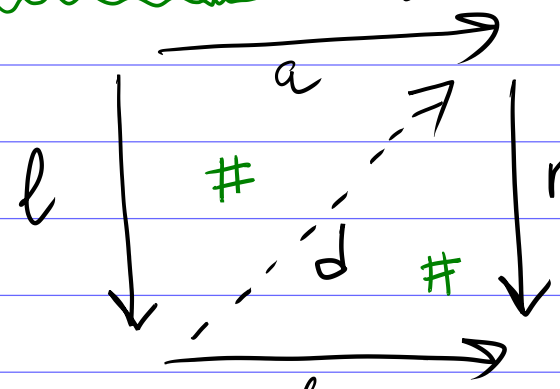


- $(Cof, Fib \cap W)$ and $(Cof \cap W, Fib)$ are weak factorization systems

Def $\mathcal{C}$ cat $L, R \subset \mathcal{C}$

(L, R) is a weak factorization system if

- $\forall f \in \mathcal{C}\ \exists l \in L\ \exists r \in R: f = rl$
- $L = {}^{\pitchfork}R$; $R = L^{\pitchfork}$ (closure)



$l \in L$

$r \in R \Leftrightarrow \forall l \in L$

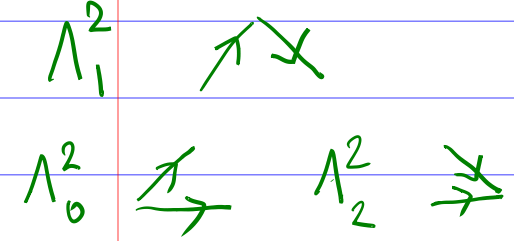
$\forall a, b: bl = ra$

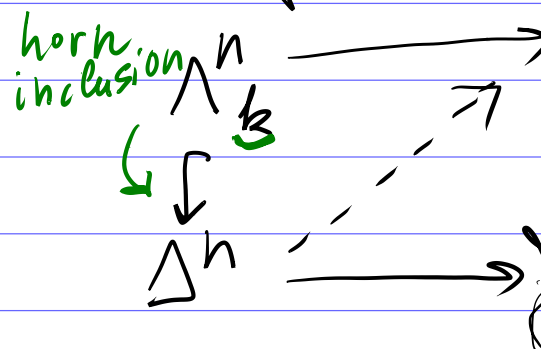
$\exists d: rd = b,\ dl = a$

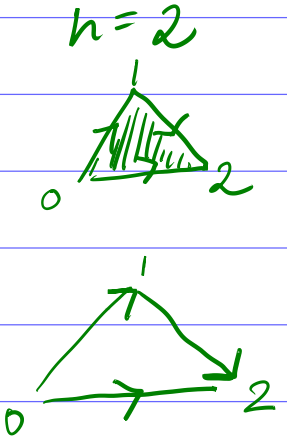
d is nonunique!

Example Simplicial sets (Kan-Quillen model structure)

- $\rightarrowtail$ Cof = monomorphisms (degreewise injection)
- $\twoheadrightarrow$ Fib = Kan fibrations

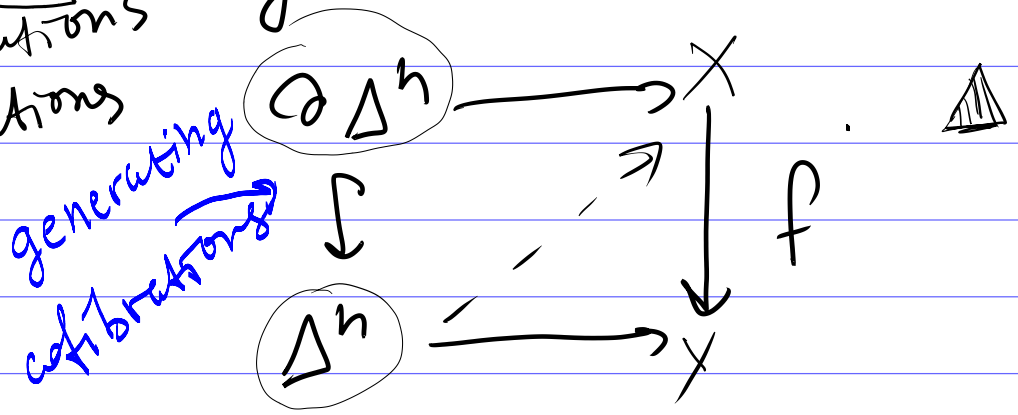






• $W \cap Fib$ = acyclic Kan fibrations

acyclic fibrations = trivial fibrations



• $W \cap Cof$ = anodyne maps

= acyclic cofibrations

= retracts of transfinite compositions of cobase changes of $\Lambda^n_k \hookrightarrow \partial\Delta^n$

• $W = (W \cap Fib) \circ (W \cap Cof)$

$= \{f : X \to Y \mid \forall K : K \twoheadrightarrow 1,$

$\mathrm{Hom}(Y, K) \xrightarrow{\sim} \mathrm{Hom}(X, K)\}$

simplicial homotopy eq

$= \{f \mid Ex^\infty f \text{ s.h.e.}\}$.

$Ex^\infty : sSet \to sSet$

Kan fibrant replacement functor

Geometric interpretation for Kan complexes

Kan complex $\equiv$ ∞-groupoid

$K \twoheadrightarrow 1$

k-morphisms for all $k \geq 1$ invertible!

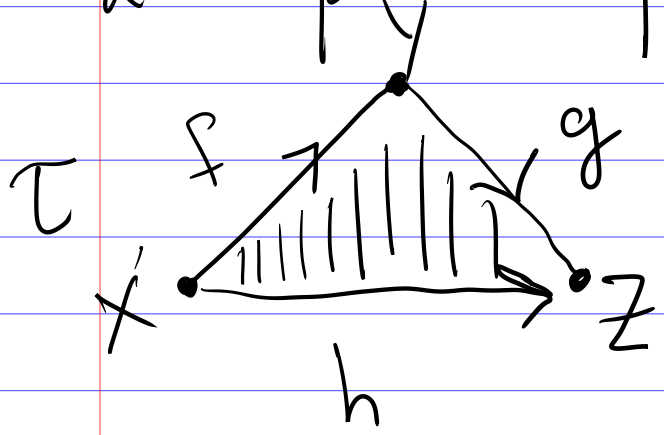
vertices of K = objects

1-simplices of K = 1-morphism



$\sigma : X \to Y$

2-simplices of K



$f : X \to Y$

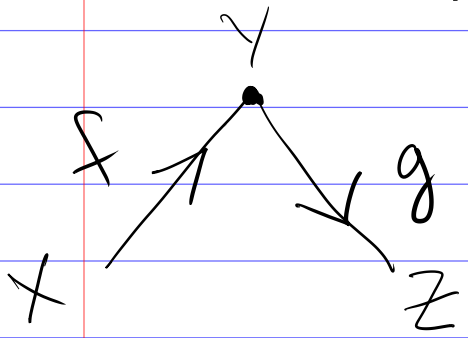
$g : Y \to Z$

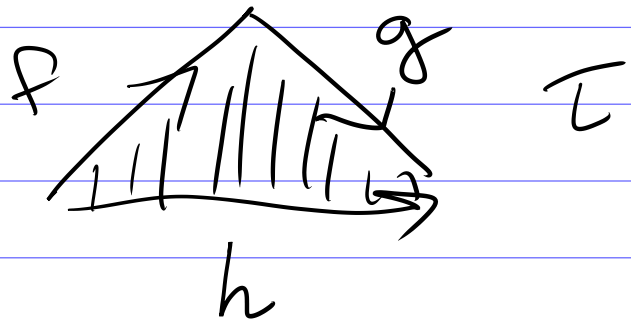
$h : X \to Z$

τ: a homotopy/2-morphism from $g \circ f$ to h

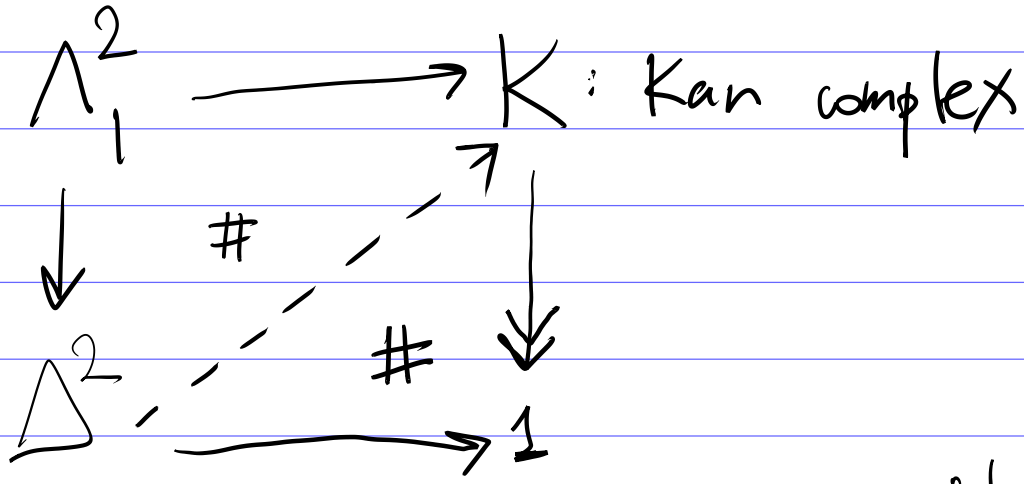
τ: isomorphism

To compose $f: X \to Y$ and $g: Y \to Z$,
find some $h: X \to Z$
isomorphic to $g \circ f: X \to Z$

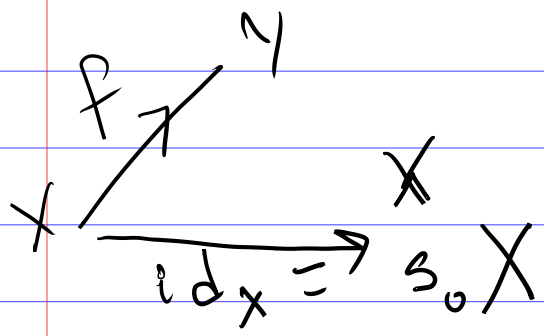


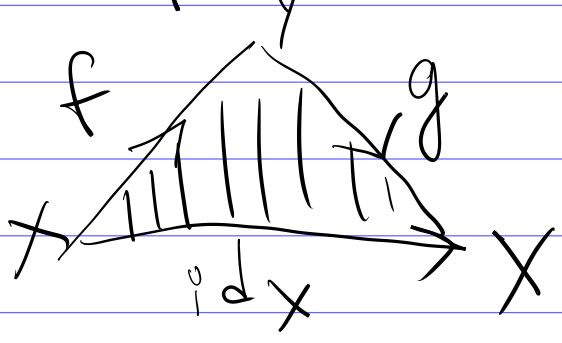


$\Lambda^2_1 \to K$ $\qquad$ $\Delta^2 \to K$



How to construct f^{-1}? $\quad g \circ f \sim id_X$





Toy examples to understand sSet:

natural / canonical model structures

Joyal – Tierney, 1992

on small categories / groupoids ("folk" model str)

Ob: small categories Mor: functors

$\mathcal{C} = Cat$ or $Grpd$

W = equivalences of cat.

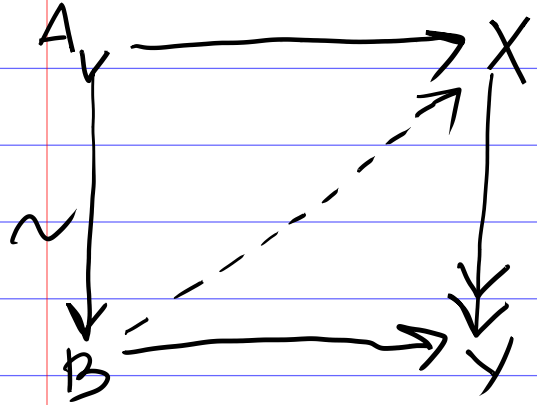
Cof = injective on objects

Fib = isofibrations $\mathcal{E} \xrightarrow{p} \mathcal{B}$

$$\forall e \in \mathcal{E} \quad \forall \varphi : p(e) \xrightarrow{\sim} b$$
$$\exists \psi : e \to e' : p(\psi) = \varphi$$

$Cof \cap W =$ inclusions of full subcategories that are equivalences

$Fib \cap W =$ surjective f.f. functors



Transferred model structures

Def $R: \mathcal{D} \to \mathcal{C}$ right adjoint functor

$\mathcal{C}$: model category $\quad (\mathrm{ModCat}/\mathcal{C})_R$ initial object

f: morphism in $\mathcal{D}$

$f \in W, \mathrm{Fib}, W \cap \mathrm{Fib}$ } transferred model structure on $\mathcal{D}$

$\Leftrightarrow R(f) \in W, \mathrm{Fib}, W \cap \mathrm{Fib}$ (in $\mathcal{C}$)

Remark Does not always exist.

Examples

a) $\mathrm{Top} \xrightarrow[\mathrm{Sing}]{} \mathrm{sSet}$ Serre model structure (Quillen)

b) $C^\infty\mathrm{Set} \xrightarrow[C^\infty\mathrm{Sing}]{} \mathrm{sSet}$ $\quad \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{Set})_{sh}$ (Cisinski) $\quad \mathrm{sSet}_{KQ}$

c) $\mathrm{Fun}(\Delta^{op}, \mathrm{Ab}) = s\mathrm{Ab} \xrightarrow{\text{forget}} \mathrm{sSet}_{KQ}$ (Quillen)

d) $\mathrm{Ch}_{\geq 0} \underset{N}{\overset{\Gamma}{\rightleftarrows}} s\mathrm{Ab}$ Dold-Kan

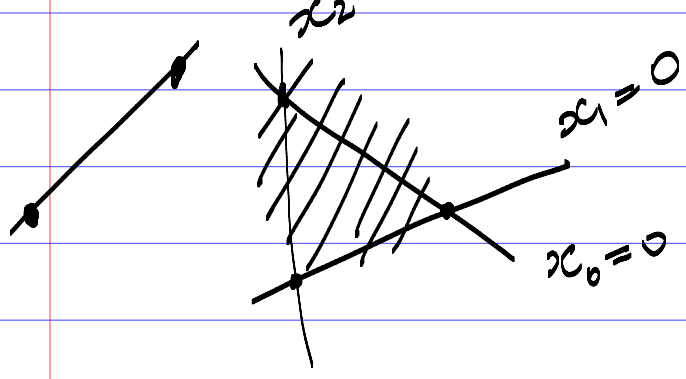
$d = \sum (-1)^i d_i$ $\quad (NA)_n = A_n / \text{deg. simpl.}$ $\quad N$ = normalized chains

e) $\mathrm{DGA}_{\geq 0} \to \mathrm{Ch}_{\geq 0}$ (Quillen)

f) $\mathrm{CDGA}^{\mathbb{Q}}_{\geq 0} \to \mathrm{Ch}_{\geq 0}$ (Quillen)

$$C^\infty \mathrm{Set} = \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{Set})_{sh}$$

$$\Delta^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \sum_i x_i = 1 \right\}$$



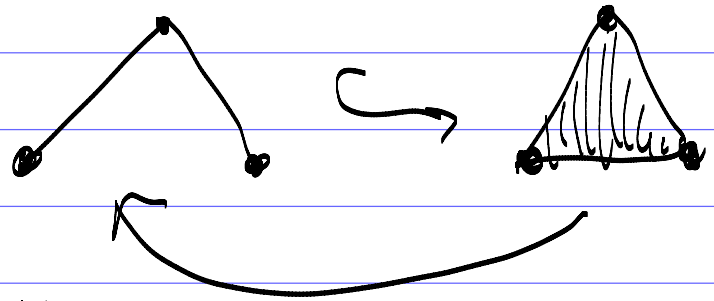
$\Delta^n \in \mathrm{Cart}$

$$\Delta^n \hookrightarrow \Delta^{n+1}$$

$$F \longmapsto ([n] \mapsto F(\Delta^n))$$

F ↑ C^∞ Set

smooth singular complex of F



smooth geometric realization

$$|-|_{C^\infty} : C^\infty \mathrm{Set} \rightleftarrows \mathrm{sSet}$$

M: manifold

$$\mathrm{Diff}(M)(S) = C^\infty(S, \mathrm{Diff}(M))$$

$$= C^\infty(S \times M, M)$$

$\forall s \in S$: $M \xrightarrow{\sim} M$

Prop $C^\infty\mathrm{Sing}\ \mathrm{Diff}(M) \simeq \mathrm{Sing}^{\mathrm{Diff}(M)}_{\text{Whitney}}$

The transferred (projective) model structure on functors / diagrams / presheaves

$\mathcal{C}$: category

M: model category

$$\mathrm{Fun}(\mathcal{C}, M) \xrightarrow{\text{evaluate on objects}} M^{\mathrm{Ob}\,\mathcal{C}}$$

(Kan) If M is cofibrantly generated, the transferred model structure exists

$W \cap \mathrm{Fib} = I^{\uparrow}$ (set; generating cofibrations)

$$F \xrightarrow{t} G$$

t is a fibration / weak eq $\iff \forall c \in \mathcal{C}$: $t(c)$ is fib / weak eq.

If $M = $ sSet (w/ Kan–Quillen m.s.), can also describe cofibrations explicitly

$$\mathrm{Fun}(\mathcal{C}, \mathrm{sSet}) = \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\Delta^{op}, \mathrm{Set})) = \mathrm{Fun}(\Delta^{op}, \mathrm{Fun}(\mathcal{C}, \mathrm{Set}))$$

t is a cofibrant object $\iff \forall n \in \Delta$:

$t_n = \hat{t}_n \sqcup$ degenerate n-simplices and $\hat{t}_n = \sqcup r$,

$\hat{t}_n \in \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ — r: retract of a representable

$Ch_{\geq 0}$: W = quasi-iso

fib = surjective in deg $\neq 0$

Cof = degree wise mon. w/ degreewise proj cokernel

A practical recipe for derived mapping spaces $L^H(C,W)(X,Y)$

$\mathcal{C}(X,Y) \in sSet$

Theorem ($\mathcal{C}$: (enriched) simplicial model cat

(Dwyer-Kan, 1980s) a) X: cofibrant $\emptyset \rightarrowtail X$

Y: fibrant : $Y \twoheadrightarrow 1$

Then $\mathcal{C}(X,Y) \xrightarrow{\sim} L^H(C,W)(X,Y)$

b) X, Y arbitrary:

cofibrant replacement: $\emptyset \rightarrowtail QX \xrightarrow{\sim} X$

fibrant replacement: $Y \xrightarrow{\sim} RY \twoheadrightarrow 1$

"derived mapping space"

$\mathcal{C}(QX,RY) \xrightarrow{\sim} L^H(C,W)(X,Y) =: \mathbb{R}Map(X,Y)$

$QX \xrightarrow{f} RY \qquad X \xleftarrow{\sim} QX \xrightarrow{f} RY \xleftarrow{\sim} Y$

(a hammock of length 3)

Example: homological algebra

$$\cdot \; Ch_{\geq 0} \underset{T}{\overset{N}{\rightleftarrows}} sAb \longrightarrow sSet$$

Transferred model structure $\equiv$ projective model structure

cofibrant replacement $\equiv$ projective resolution

all objects are fibrant

$X, Y \in Ch_{\geq 0}$

$RMap(X,Y) = Ch(QX, Y)$

$Ext^n(X,Y) = H_n(Ch(QX,Y))$

Slogan: Transferred model structures work well for algebraic structures with strict identities (uniquely defined ops)

Examples: a) $Ch_{\geq 0}$, DGA, CDGA$_{\mathbb{Q}}$

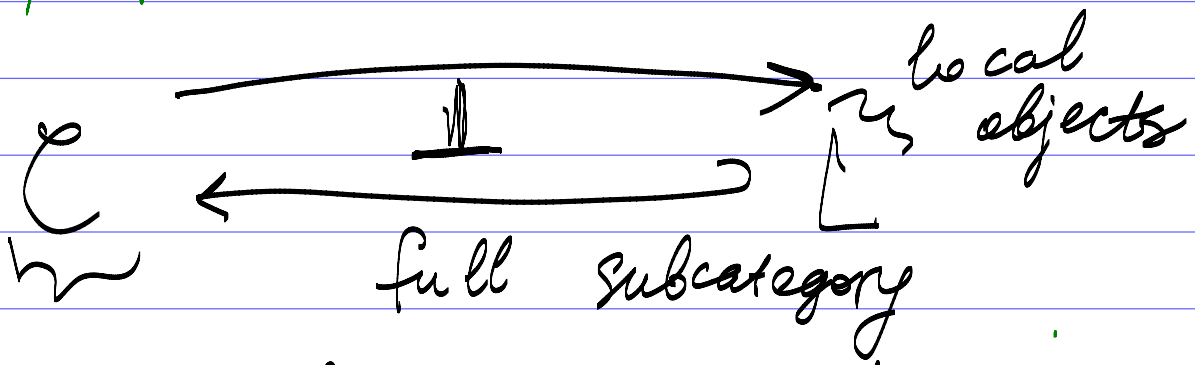
b) A_∞-DGA E_∞-DGA

c) sGrp, sMon, sRing, sCRing

d) A_∞-cat, sym mon. cat, simplicial cat

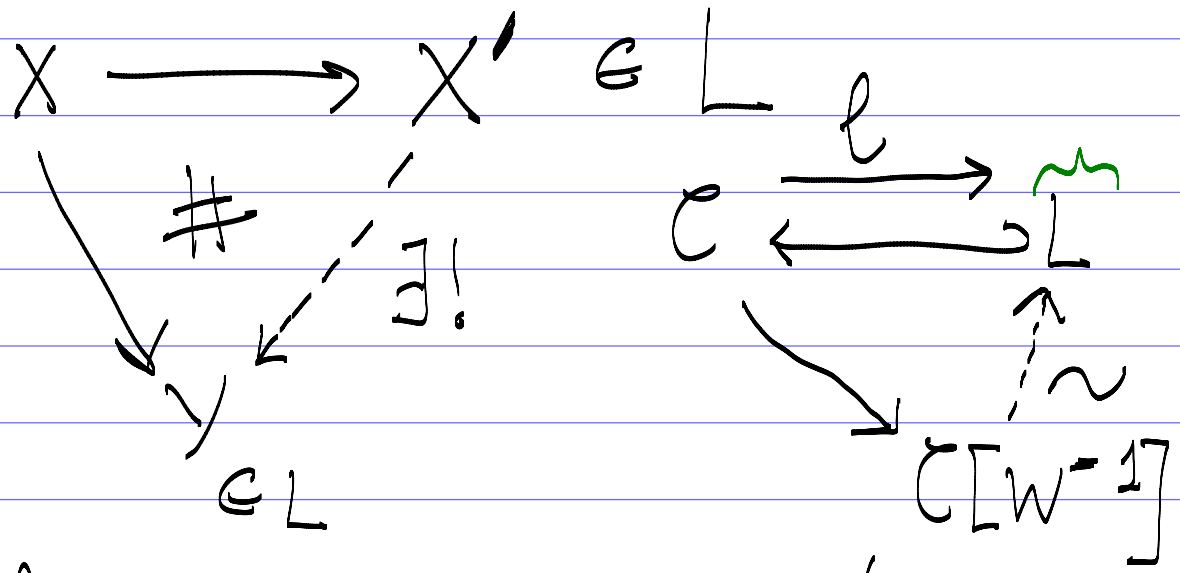
e) algebras over operads

Reflective localizations



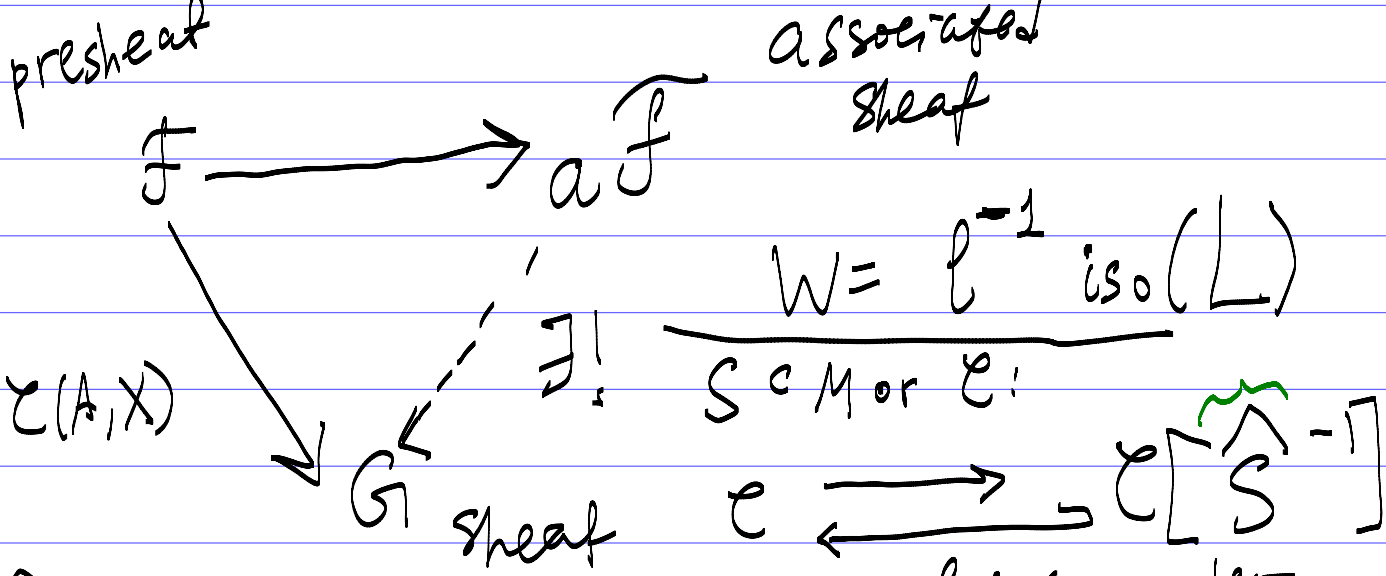
reflective localization

$X \in \mathcal{C}$ reflection of X into L



Sheaves:
S = Čech nerves
$\check{C}U \to X$

① $X \in L$:
$s: A \to B$
$\mathcal{C}(B,X) \xrightarrow{\simeq} \mathcal{C}(A,X)$



$\hat{S}$ = the saturation of S = closure under cobase changes and colim in the cat of morphisms

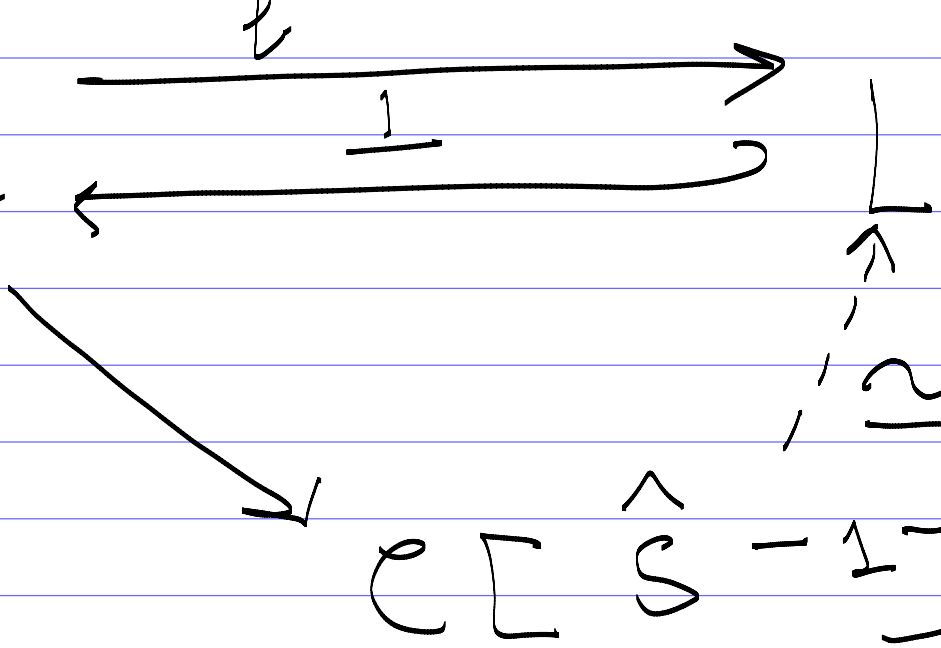
$\mathcal{C}$ category (locally presentable category)

S: a set of morphisms in $\mathcal{C}$

1) $L = \{ X \in \mathcal{C} \mid \forall\, s : A \to B \in S,\ \mathcal{C}(B, X) \xrightarrow{\cong} \mathcal{C}(A, X) \}$

2) $\hat{S} = \{ f : X \to Y \mid \forall A \in L :\ \mathcal{C}(X, A) \xrightarrow{\cong} \mathcal{C}(Y, A) \}$

Thm a)

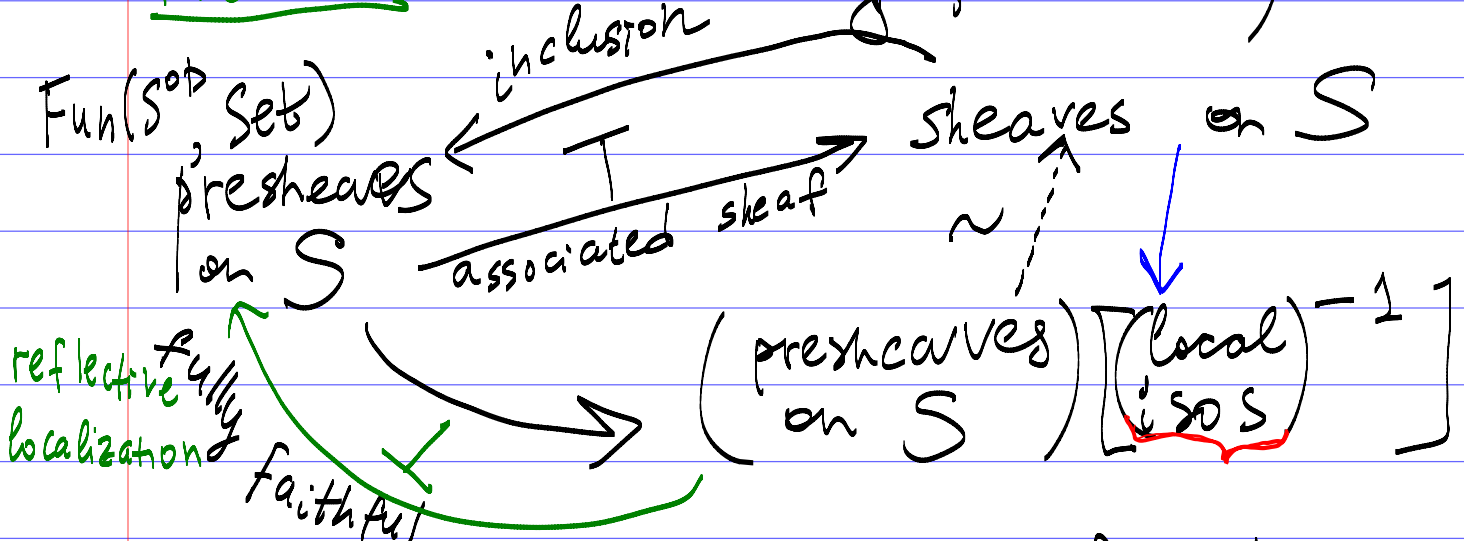


b) $\hat{S} = \ell^{-1}\, \mathrm{iso}(L)$

Left Bousfield localization

A homotopy coherent version of reflective localization

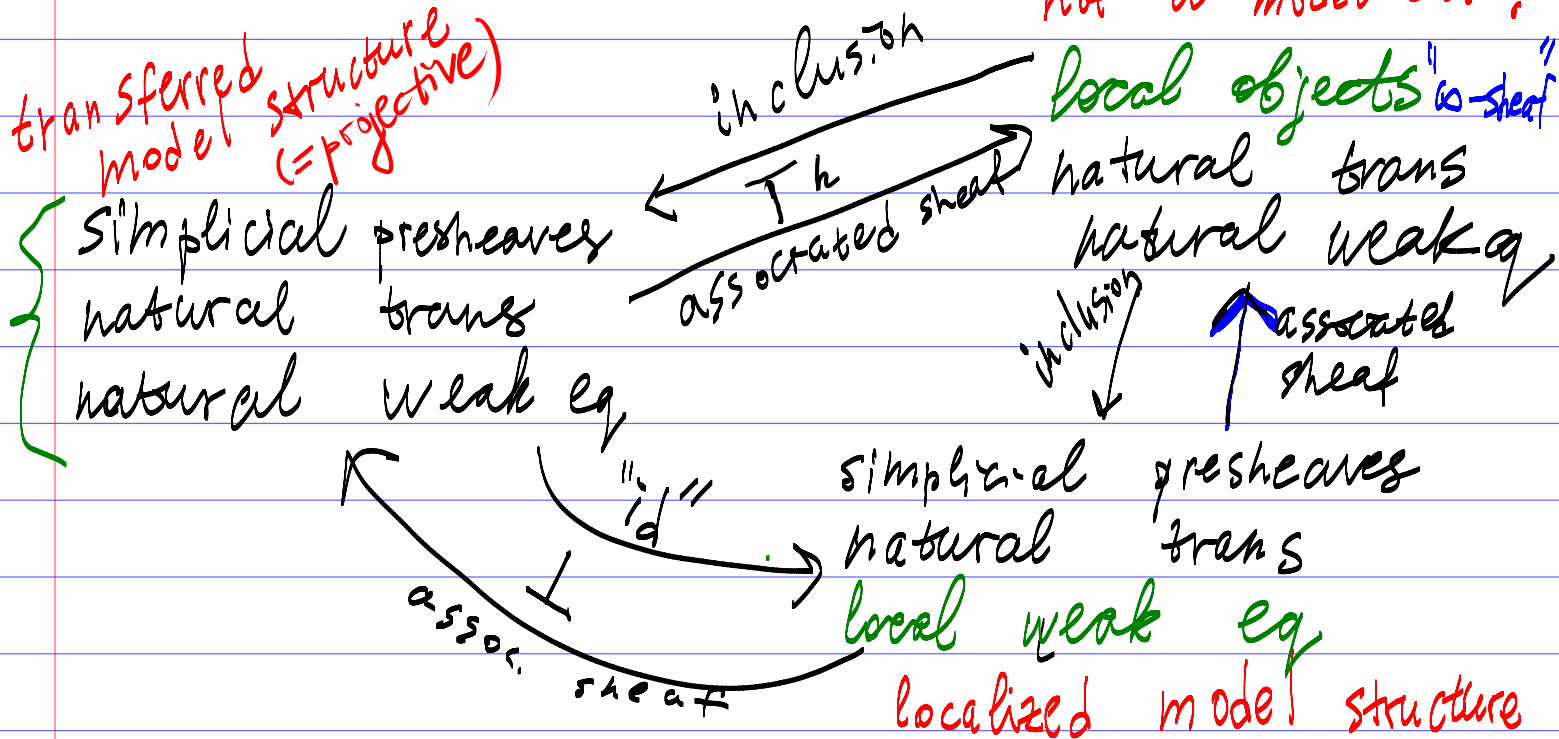
Recall S : site (e.g., $S = \mathrm{Cart}_k$)



Previously: (pre)sheaves of sets

Now (pre)sheaves of simplicial sets = simplicial presheaves

The homotopy coherent picture



Left Bousfield localizations

$\mathcal{C}$: model category — e.g., $\mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{sSet})$ w/ projective m.s.

S: a set of morphisms in $\mathcal{C}$ — e.g., Čech nerves of good open covers

Def: a) An S-local object is $X \in \mathcal{C}$ such that $\forall s \in S: s: A \to B$

$$L^H(\mathcal{C},W)(B,X) \xrightarrow{\sim} L^H(\mathcal{C},W)(A,X)$$

b) An S-local morphism ($\hat{S}$) is $f: X \to Y$ such that $\forall$ S-local object A:

$$\underbrace{\mathrm{RMap}}_{L^H(\mathcal{C},W)}(Y,A) \xrightarrow{\sim} \mathrm{RMap}(X,A).$$

Example $\mathcal{C}$ = simplicial presheaves

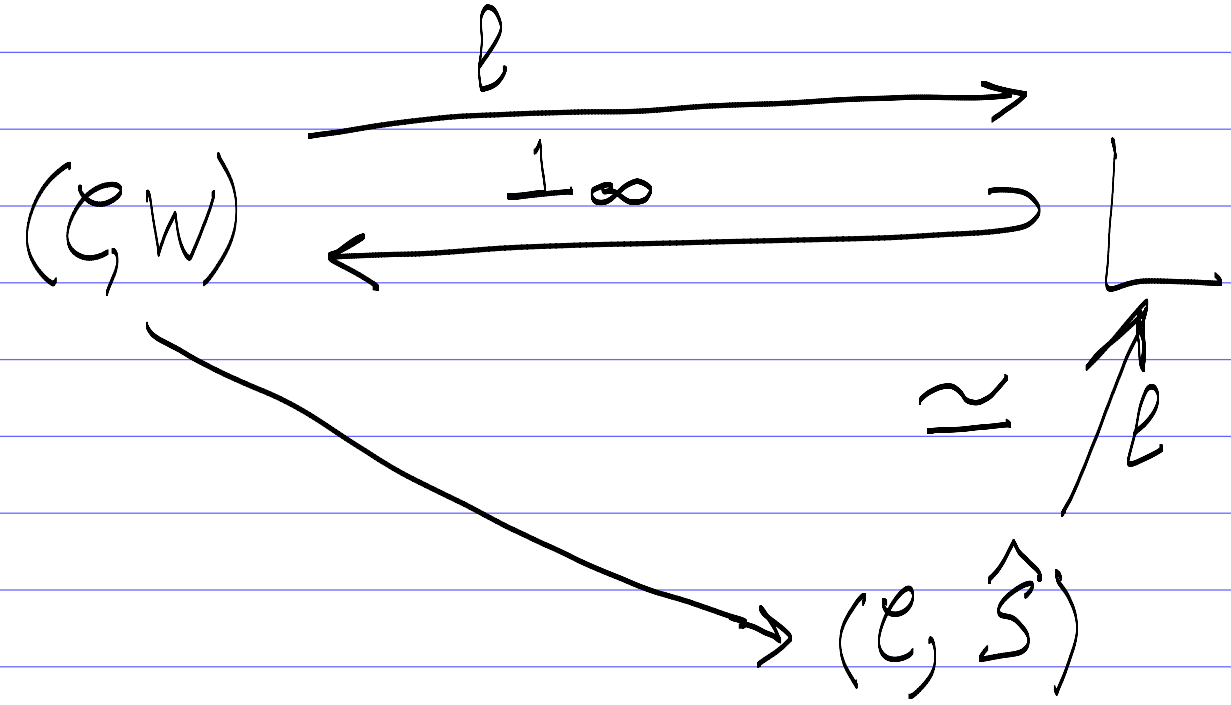
S = Čech nerves

a) S-local obj = presheaves satisfying ∞-descent

b) S-local morphisms = local weak eq

Thm $\mathcal{C}$ model cat (combinatorial)

a)



b) $\hat{S} = \ell^{-1} W_L$

A digression: Quillen adjunctions

Def $F: \mathcal{C} \longrightarrow \mathcal{D}$ $\mathcal{C}, \mathcal{D}$: mod. cat

is a **left Quillen functor** (Quillen pair / Quillen adjunction)

if F is a left adjoint functor

and F preserves cofibrations, acyclic cof.

Warning Typically, F does not preserve weak eq.

Prop $F: \mathcal{C} \longrightarrow \mathcal{D}$ ℓ Qf $QX \to X$, $Q \Rightarrow id$ ($\sim$)

Then $\mathcal{C} \xrightarrow{Q} \mathcal{C} \xrightarrow{F} \mathcal{D}$

(Q: cofibrant replacement)

preserves weak equivalences.

We write $\mathbb{L}F = F \circ Q$ (relative functor)

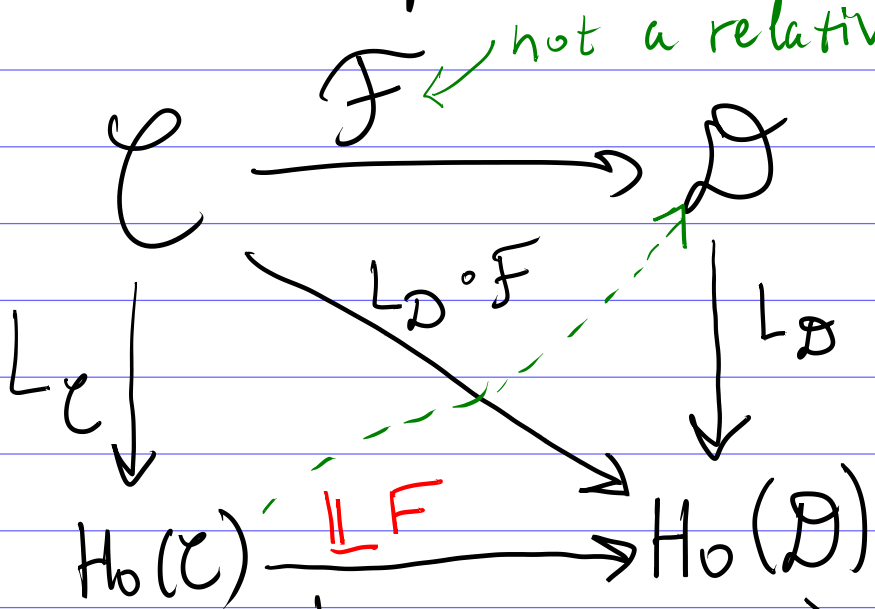
$\mathbb{L}F$: left derived functor of F

Thm (Blumberg–Richl, 2013) If $\mathcal{C}, \mathcal{D}$ are

cof. gen & simplicial, then $\mathbb{L}F$ lifts

to a homotopy coherent adjunction of simplicial categories

Remark The traditional definition of a left derived functor



$\mathrm{Lan}_{L_{\mathcal{C}}}(L_{\mathcal{D}} \circ F)$ total left derived functor

(Defective; $\mathbb{L}$ does not preserve compositions)

Higher Categories & Homotopical Alg, Ch 7

Modern definition (Hinich, Cisinski, ..., Maltsiniotis)

So, what is a left deriv. functor?

Replace $\mathrm{Ho}(-)$ w/ simplicial homotopy cat = homotopy $(\infty,1)$-category

e.g., hammock localizations

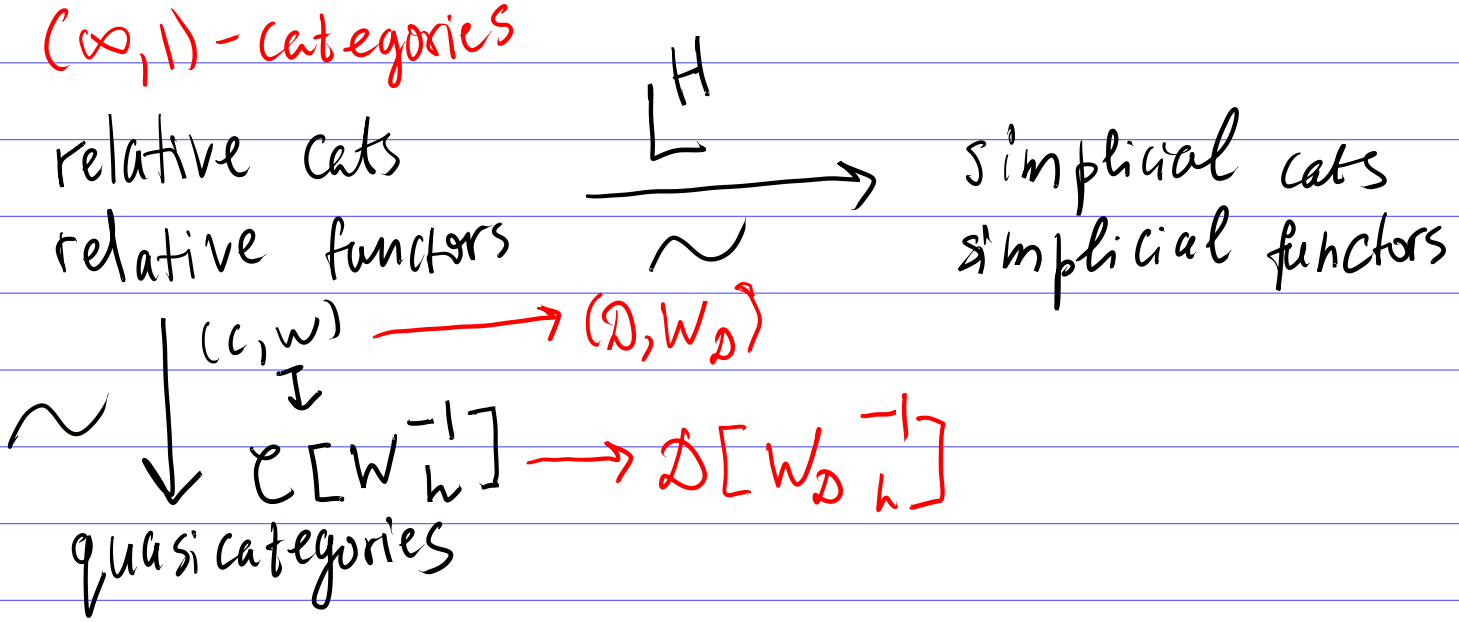
or: $(\mathcal{C}, W)$

Lan by its homotopy coherent analogue

For model categories: $\mathbb{L}F \simeq F \circ Q : \mathcal{C} \to \mathcal{D}$ $(\mathcal{C}, W_{\mathcal{C}}) \to (\mathcal{D}, W_{\mathcal{D}})$

a relative functor!

$(\infty,1)$-categories



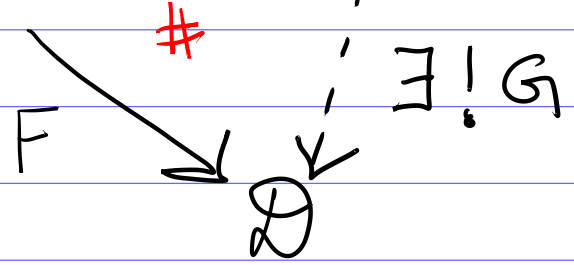
Def A **left Bousfield localization** of $\mathcal{C}$ (model cat) at S (a set of morphisms in $\mathcal{C}$) is

$\mathcal{C} \xrightarrow{\mathcal{L}} L_S\mathcal{C}$ left Quillen functor (left Quillen functor)

such that a) $(\mathbb{L}\mathcal{L})(S) \subset W$

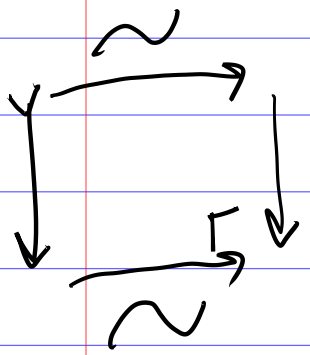
b) $\forall F$ (of left Quillen functor): $(\mathbb{L}F)(S) \subset W$

$\exists!\, G : G\mathcal{L} = F$ (left Quillen functor)



Observation a) The underlying cat of $\mathcal{C}$, $L_S\mathcal{C}$ are the same

b) Cofibrations are the same



c) Weak eq. of $L_S\mathcal{C}$ = S-local morphisms

Thm (Smith, ~1999) $L_S\mathcal{C}$ exists if

a) S is a set b) $\mathcal{C}$ is locally presentable c) $\mathcal{C}$ is cofibrantly generated d) $\mathcal{C}$ is left proper.

Proposition (Cisinski, HCHA)

Suppose $\mathcal{C} \xrightarrow{L} L_S\mathcal{C}$ (L preserves w.e.)

Then $\mathcal{C}[W_h^{-1}] \xrightarrow{L} L_S\mathcal{C}\left[\left(\begin{smallmatrix}S\text{-local}\\ \text{mor}\end{smallmatrix}\right)_h^{-1}\right]$

$\mathcal{C}[W_h^{-1}] \xleftarrow{I^h}$ fully faithful

reflective localization of $(\infty,1)$-cats.

Example: The Čech-local model structure $\simeq (\infty,1)$-sheaves

Cart: could be any site

$\mathcal{C} = sPSh(Cart) = Fun(Cart^{op}, sSet)$ projective

S = Čech nerves of good open covers.

$y: Cart \xrightarrow{\text{Yoneda}} \mathcal{C}$

$\mathcal{U} = \{U_i \hookrightarrow X\}$ good open cover if

$U_i = U_{i_0} \cap \cdots \cap U_{i_n} \in Cart \cup \{\emptyset\}$

$\check{C}\mathcal{U} \longrightarrow yX$

$(\check{C}\mathcal{U})_n = \coprod_{i:\, U_i \neq \emptyset} y(U_i)$ (a presheaf of sets)

kth face map $d_k: \check{C}\mathcal{U}_n \longrightarrow (\check{C}\mathcal{U})_{n-1}$

$\coprod_i yU_i \longrightarrow \coprod_j yU_j$

$yU_i = y(U_{i_0} \cap \cdots \cap U_{i_n}) \longrightarrow y(U_{i_0} \cap \cdots \widehat{U_{i_k}} \cdots \cap U_{i_n})$

k th degeneracy map: $s_i : (\check{C}\mathcal{U})_k \to (\check{C}\mathcal{U})_{n+1}$

$y_{U_{i_0} \cap \cdots \cap U_{i_n}} \to y_{U_{i_0} \cap \cdots \cap U_{i_k} \cap U_{i_k} \cap \cdots U_{i_n}}$

Example $\mathcal{U} = \{V, W\}$

$(\check{C}\mathcal{U})_0 = y_V \sqcup y_W$

$(\check{C}\mathcal{U})_1 = \underbrace{y_V \sqcup y_W}_{\text{degenerate}} \sqcup y_{(V \cap W)} \sqcup y_{(W \cap V)}$

Remark $y_V \sqcup^h_{y_{(V\cap W)}} y_W \xrightarrow{\sim} \check{C}\mathcal{U}$ (homotopy pushout)

Remark $\check{C}\mathcal{U}$ is cofibrant in the projective model structure on $sPSh(\mathrm{Cart})$

Corollary $\mathbb{R}\mathrm{Map}(\check{C}\mathcal{U}, \mathcal{F}) \simeq \mathrm{Map}(\check{C}\mathcal{U}, \mathcal{F})$

if $\mathcal{F}$ is a presheaf of Kan complexes

→ The simplicial set of compatible families over $\mathcal{U}$.

→ The map $\check{C}\mathcal{U} \longrightarrow yX$ S

induces the descent map

$$\begin{array}{ccc} \mathrm{RMap}(yX, \mathcal{F}) & \to & \mathrm{RMap}(\check{C}\mathcal{U}, \mathcal{F}) \\ \wr & & \wr \\ \mathrm{Map}(yX, \mathcal{F}) = \mathcal{F}(X) & \xrightarrow{\mathcal{D}} & \mathrm{Map}(\check{C}\mathcal{U}, \mathcal{F}) \end{array}$$

simp set of compatible families

$\mathcal{F}$ is an $(\infty,1)$-sheaf $\leftrightarrow$ $\mathcal{D}$ is a weak eq. for any $\mathcal{U}$.

Observation

$\mathcal{F}$ is an $(\infty,1)$-sheaf

$\Leftrightarrow$ $\mathcal{F}$ is S-local

$S = \{ \check{C}\mathcal{U} \to yX \mid \mathcal{U}$ good op cov of $X \}$.

$\mathrm{RMap}(yX, \mathcal{F}) \xrightarrow{\sim} \mathrm{RMap}(\check{C}\mathcal{U}, \mathcal{F})$

Def The Čech-local projective model structure on $sPSh(Cart)$ is $L_S\, sPSh(Cart)_{proj}$, where $S = \{\check{C}U \to yX\}$.

Remark Existence: Smith theorem

Weak equivalences = local w.e. (local Whitehead thm)
= stalkwise w.e (Cart)

$F \xrightarrow{\sim} G \iff \forall n \geq 0 : \operatorname{colim}_{\varepsilon>0} F(B^n_\varepsilon) \xrightarrow{\sim} \operatorname{colim}_{\varepsilon>0} G(B^n_\varepsilon)$

Amabel-Debray-Haine Differential cohomology

Cofibrations = projective cofibrations

Fibrant objects = prestacks of Kan complexes that are $(\infty,1)$-sheaves

Example $\mathbf{B}G$, G Lie group.

$$(\mathbf{B}G)(U) = B\,C^\infty(U, G)$$

$\mathbf{B}G : \mathrm{Cart}^{op} \to sSet$ — B: delooping

trivial principal G-bundle over U

$$B\,H = N\begin{pmatrix} Ob: * \\ End(*) = H \end{pmatrix}$$

Observation $\mathbf{B}G$ is fibrant

a) nerves of groupoids are Kan complexes.

b) descent: $\mathcal{U} = \{V, W\}$, $V \cup W = X$

$\mathbb{R}\mathrm{Map}(X, \mathbf{B}G)$

$$(\mathbf{B}G)(X) \xrightarrow{\sim} (\mathbf{B}G)(V) \times^h_{(\mathbf{B}G)(V\cap W)} (\mathbf{B}G)(W)$$

homotopy pullback

(trivial) principal G-bundles over X

cocycle data for principal G-bundles trivialized over $\{V, W\}$

Ob: $t : V \cap W \to G$ — transition function

Mor $\alpha : V \to G$, $\beta : W \to G$

$t_1 \to t_2$

$t_2 \alpha = \beta t_1$

Essential surjectivity: $(V \times G) \sqcup (W \times G)$

$\hookrightarrow (V \cap W) \times G$, t

Fullness: α and β glue together

Faithfulness: $h \mapsto (h|_V, h|_W)$

Segal Γ-spaces, Δ-spaces, etc.

Goal: Encode homotopy coherent

monoids, commutative monoids,
(Δ-spaces) (Γ-spaces)

groups, categories, n-categories, etc.
(group-like Γ-spaces) (Segal cat / complete Segal space) (complete n-fold Segal space)

$(M \times M) \times M \to M \times (M \times M)$

$M \times M \xrightarrow{\#^h} M \times M \to M$

Two approaches:

a) Stasheff et al.: A_∞-spaces (algebraic approach)
1960s
(algebras over operads)

b) Segal et al.: Γ-objects / Δ-objects
1970
(presheaves; in a left Bousfield loc.)
= functor of points

Pros and cons

a) A_∞ - spaces, E_∞ - spaces, etc.

algebraic approach → colimits are hard

transferred model structure → cofibrancy is hard ($\simeq$ free)
(retracts of free objects)

→ intuitive

A_∞ - spaces → spaces
ΩX

→ sometimes difficult to encode examples

b) Δ - spaces, Γ - spaces, etc.

geometric approach → colimits are objectwise
(followed by the localization functor)

left Bousfield localization → cofibrancy is easy (objectwise)

→ nonintuitive

→ easy to encode examples
(e.g., bordisms)

Recall monoids in Sets

$$\text{Monoid} \xrightarrow[\text{deloop}]{B} \text{Cat} \xrightarrow[\text{nerve}]{N} {}_s\text{Set}$$

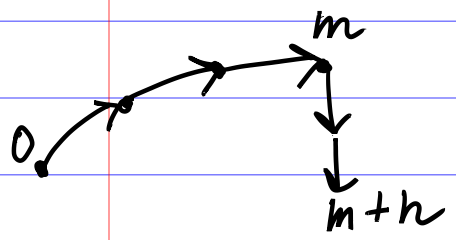
Grothendieck:

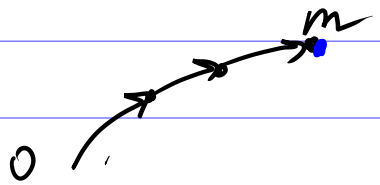
The essential image of N .

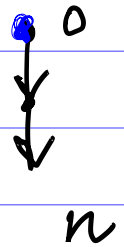
= simplicial sets X that satisfy the Segal conditions

$$X_{m+n} \xrightarrow{(*)}_{\simeq} X_m \times_{X_0} X_n$$

$$\begin{array}{ccc} X_{m+n} & \to & X_m \\ \downarrow & & \downarrow \\ X_n & \to & X_0 \end{array}$$







N.B.: add $X_0 \xrightarrow{\simeq} *$

Suppose $X \in sSet$

X satisfies the Segal conditions (*)

Then $\exists\, \mathcal{C} \in Cat$: $N\mathcal{C} \cong X$.

$Ob\,\mathcal{C} = X_0$

$Mor\,\mathcal{C} = X_1$

$1 : Ob\,\mathcal{C} \xrightarrow{s_0} Mor\,\mathcal{C}$

$\bullet \longmapsto \bullet$

$s, t : Mor\,\mathcal{C} \to Ob\,\mathcal{C}$

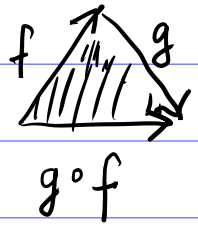
$X_1 \;\underset{d_0 = t}{\overset{d_1 = s}{\rightrightarrows}}\; X_0$

$(Mor\,\mathcal{C}) \times (Mor\,\mathcal{C}) \to Mor\,\mathcal{C}$

$t, Ob\,\mathcal{C}, s$

$X_1 \times_{X_0} X_1 \cong X_2$

d_1



Now: (homotopy coherent) monoids in spaces

$$\infty\text{-Monoid} \xrightarrow[\text{deloop}]{B} \infty\text{-Cat} \xrightarrow[\text{nerve}]{N} s\mathrm{Space} = \mathrm{Fun}(\Delta^{op}, \underbrace{s\mathrm{Set}}_{\text{Kan-Quillen}})$$

The essential image of N

= simplicial spaces that satisfy the Segal conditions

$$(*)\quad X_{m+n} \xrightarrow{\simeq} X_m \underset{X_0}{\times^h} X_n$$

(= Segal spaces).

For $N \circ B$: $X_0 \xrightarrow{\simeq} *$

How to encode (*) in a model category?

(*) is formulated as a homotopy limit property

$\Rightarrow$ objects satisfying (*) are stable under holims

$\Rightarrow$ probably have a reflective subcategory

Def Model structure on Segal spaces:

"global model structure"

a) Start with $Fun(\Delta^{op}, sSet)_{proj/inj}$

b) Left Bousfield localize at Segal maps: $(**)$

$$\coprod^h_{y_{[0]}} y_{[n]} \to y_{[m+n]}$$

Prop Fibrant objects are fibrant presheaves that satisfy $(*)$.

Proof $(*) \iff (**)$-local objects.

Suppose $X : \Delta^{op} \to sSet$.

$$RMap((**), X)$$

$$X_{m+n} \xrightarrow{\simeq} X_m \times_{X_0}^h X_n.$$

Derived Yoneda lemma

$X : \mathcal{C}^{op} \longrightarrow sSet$

$A \in \mathcal{C}$

$\underbrace{Fun(\mathcal{C}^{op}, sSet)}_{proj}$

$RMap(\underbrace{y_A}_{\text{cofibrant}}, X)$

$\simeq$

$Map(\underbrace{Q\, y_A}_{\simeq}, RX)$

$\simeq$

$Map(y_A, RX)$

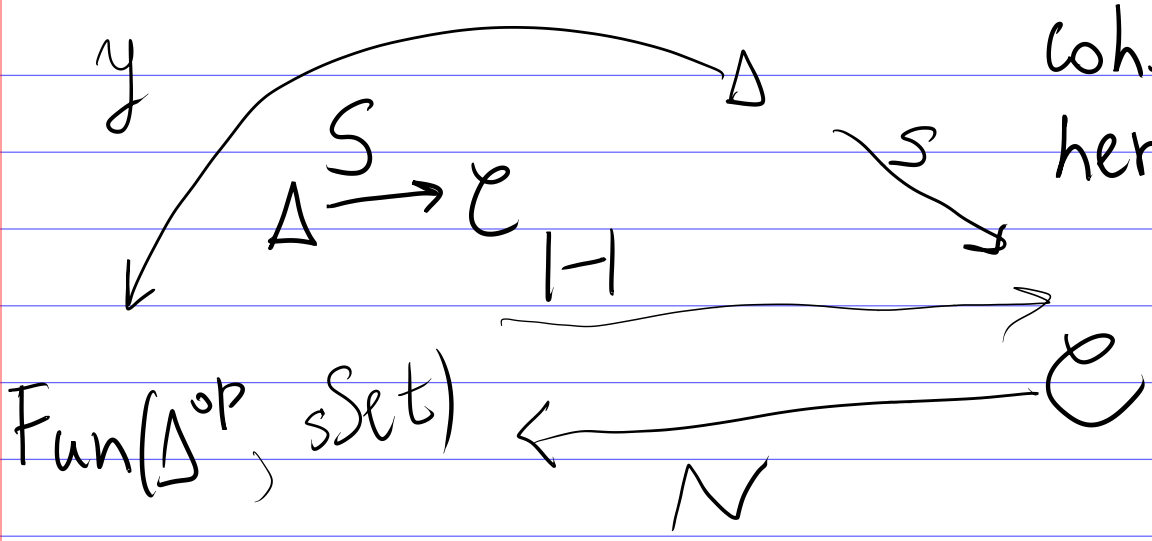
$\simeq$

$(RX)(A) \xleftarrow{\sim} \widetilde{X(A)}$

$\mathbb{L}\mathcal{L} \simeq \mathbb{L}\mathrm{Lan}_y(-)$ $\quad \Delta^{[n]}$

$$\mathrm{Fun}(\Delta^{op}, sSet) \underset{\text{proj}}{\overset{\mathcal{L}}{\underset{T}{\rightleftarrows}}} \mathrm{Fun}(\Delta^{op}, sSet)_{\text{Segal}}$$

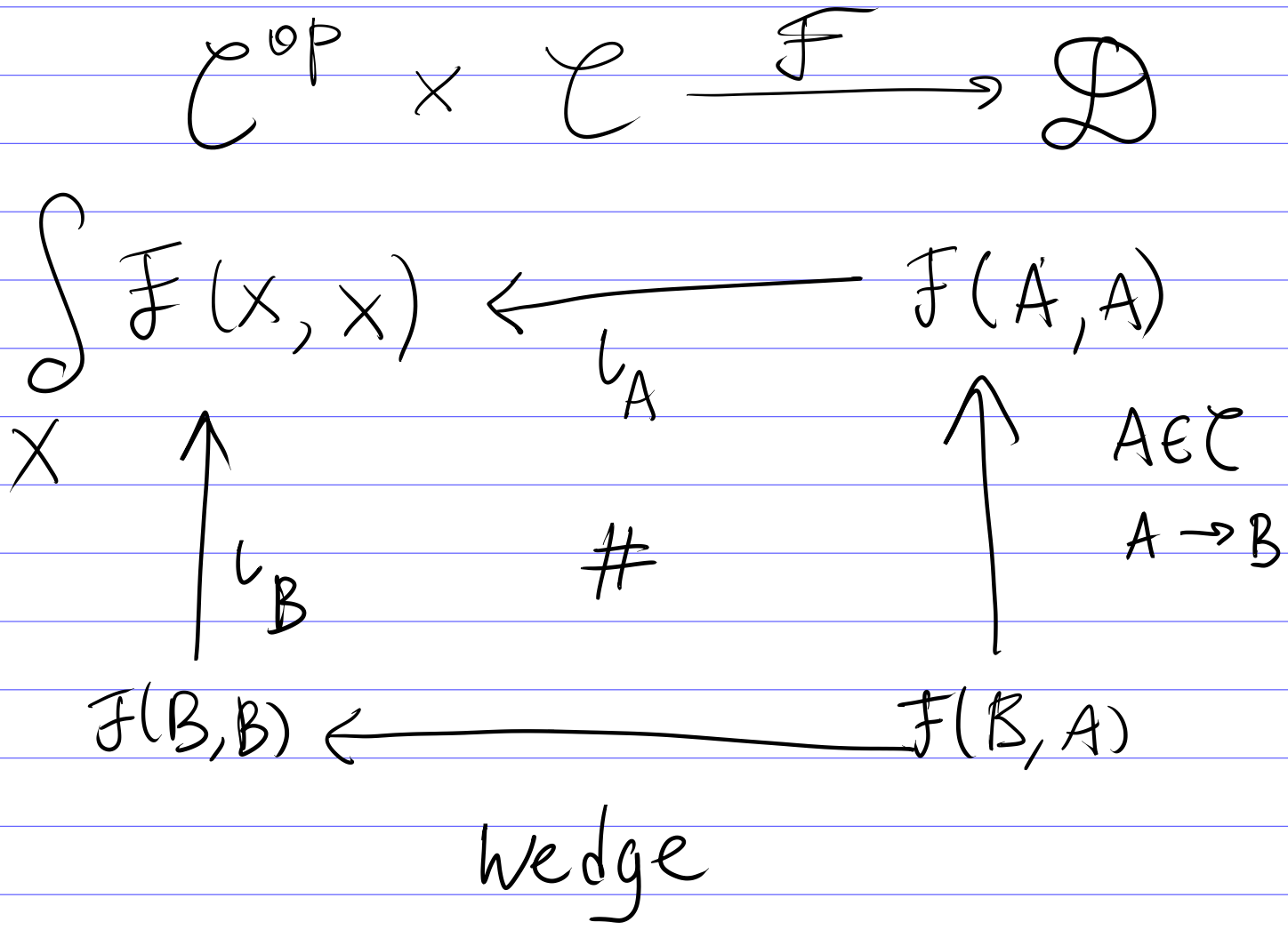
$sSpace$ $\quad N = RT = T \circ \tilde{R}$ $\quad \infty\text{-}Cat$ (encoded by their homotopy coherent nerve)



$$(NX)_n = \mathcal{C}(S_n, X)$$

$$|F| = \Delta \otimes_{\Delta^{op}} F = \int^{\Delta_n} \Delta_n \times F([n])$$

coend



Fosco Loregian

Coend calculus Ex 1.4.5 (Stokes)

The resulting model structure encodes flagged ∞-categories

(= some invertible morphisms are singled out)

a) For ∞-monoids ($= \Delta$-spaces)

$\rightsquigarrow$ require $X_0 \xrightarrow{\sim} *$ encoded by $\emptyset \to y_{[0]}$

b) For Segal categories:

$\rightsquigarrow$ require X_0 to be discrete

c) For complete Segal spaces (Rezk)

$\rightsquigarrow$ require $X_0 \xrightarrow{\sim} X_1^X$

(encoded by $N(\circlearrowright) \to y_{[0]}$.

d) For ∞-groupoids / ∞-groups

$\rightsquigarrow$ require $X_1^X \xrightarrow{\sim} X_1$

encoded by $y_{[1]} \to N(\circlearrowright)$

$\widetilde{L_H \mathcal{C}}(-,-)$
hammock localization

$(***)$ $\emptyset \longrightarrow y_{[0]}$

$\mathbb{R}\mathrm{Map}\left(\underline{\quad\quad}, X\right)$

$\mathbb{R}\mathrm{Map}(\emptyset, X) \longleftarrow \mathbb{R}\mathrm{Map}(y_{[0]}, X)$

$\|\ \mathrm{Map}(\emptyset, RX) = \Delta^0 = 1 \quad \|$

$1 = * \xleftarrow{\sim} X_0$

Def A Segal monoid ($= \infty$-monoid) is a local object in

$L_S\, \mathrm{Fun}(\Delta^{op}, s\mathrm{Set})_{proj/inj}$

$S = (**) \cup (***)$

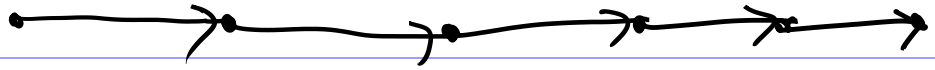
$(**)$: $X_{m+n} \to X_m \underset{X_0}{\times} X_n \simeq X_m \times X_n$

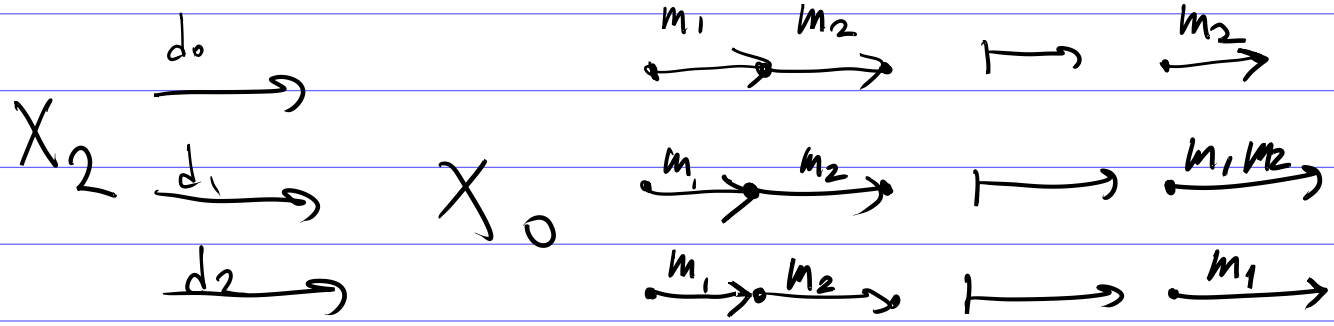
$(***)$: $X_0 \xrightarrow{\sim} *$

Examples

a) M: ordinary monoid

$X_n = \underbrace{M \times M \times \cdots \times M}_{n \text{ times}}$





inner face maps: multiply; outer: discard; degeneracy: $\xrightarrow{1}$

b) M: simplicial monoid.

$M \in sSet$.

c) (Segal) $\mathcal{C}$: small category w/ finite products

$N\mathcal{C}$: want a monoid structure encoding $(-)\times(-)$ (categorical product)

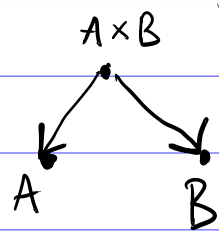
Problem: $A \times B$ is not uniquely defined, only up to a unique iso.

Instead: $X : \Delta^{op} \longrightarrow sSet$; $X_1 = N\mathcal{C}$

X_n = the nerve of ~~cat~~ groupoid; objects:

X_0

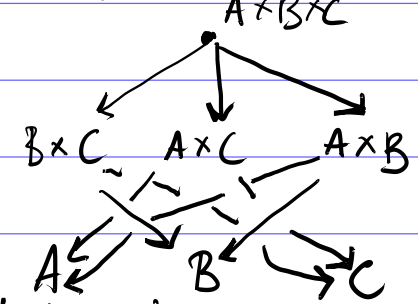
X_1: A • X_2



$A \in \mathcal{C}$

$A, B \in \mathcal{C}$

X_3: 1970: Segal Categories and cohomology theories.

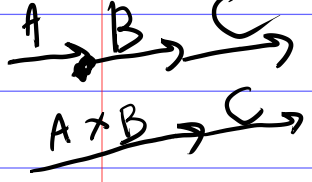


outer face maps: discard all objects containing the given vertex

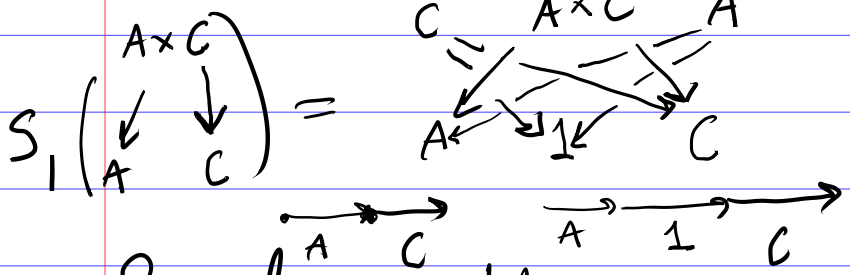
$$d_0(\dots) = B \times C \quad (\text{with maps to } B \text{ and } C)$$

degeneracy maps: add a vertex $\underline{1}$ (terminal) object in $\mathcal{C}$

inner face maps: discard all objects where only one out of 2 vertices is present



$$d_1(\dots) = A \times B \times C \quad (\text{with maps to } A \times B \text{ and } C)$$

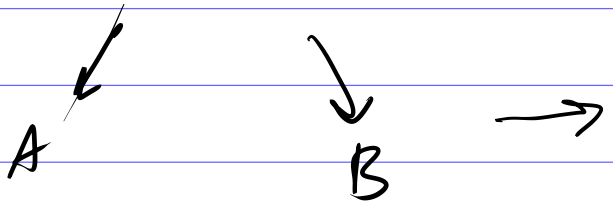


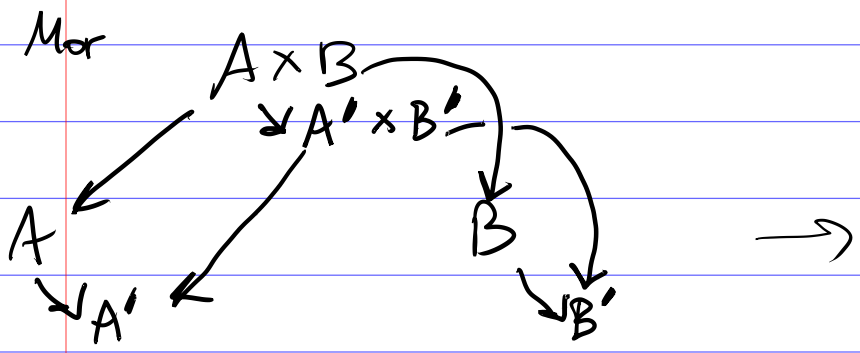
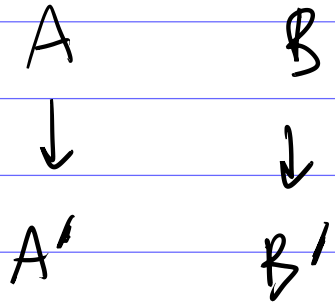
Segal conditions: products are unique up to a unique iso

$$X_2 \to X_1 \times_{X_0} X_1$$

$$\sim \downarrow \text{Reedy fibrant } X \quad \downarrow \quad \mapsto \quad A, B$$

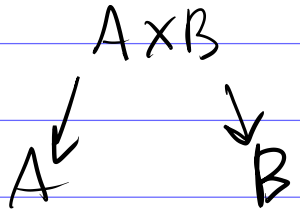
$$X_1 \times_{X_0} X_1 \qquad A \times B \to (A) \quad (B)$$

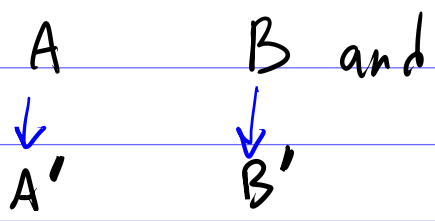
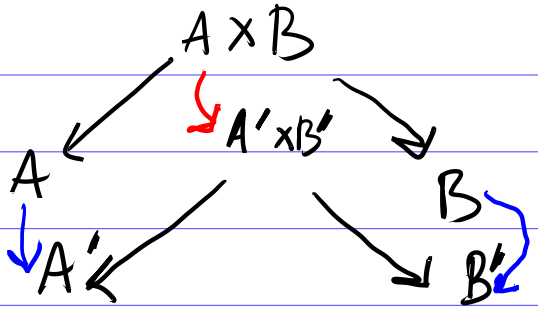
Ob: $A \times B$  Ob A, B

Mor  Mor 

Equivalence $\equiv$ essentially surjective and fully faithful functor

Surjective: Given A, B

pick 

Full: Given  and 

Faithful: uniqueness part of the univ property of categorical products.

d) $\mathcal{C}$: small category w/ monoidal str. $\otimes$

Want: $N\mathcal{C}$: a monoid str. enc. $(-)\otimes(-)$.

Problem: $\otimes$ is not associative or unital (only up to iso)

Solution X_n = nerve of cat

$X_1 = N\mathcal{C}$ $\quad X_0 \cong *$

Objects: $n=0$: (ignore units for now)

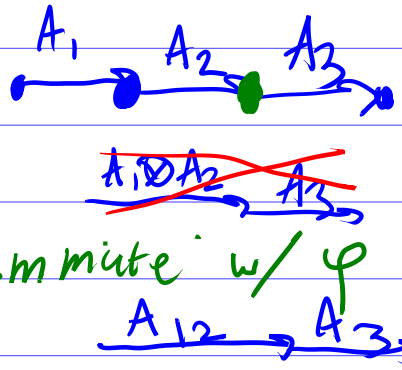
$n=1$: A_1

$n=2$: $(A_1, A_2, A_{12}, A_1 \otimes A_2 \xrightarrow{\cong} A_{12})$

$n=3$: $A_1, A_2, A_3, A_{12}, A_{23}, A_{13},$

$A_1 \otimes A_2 \xrightarrow{\cong} A_{12}, \dots)$

$A_{12} \otimes A_3 \xrightarrow[\varphi]{\cong} A_{123}.$



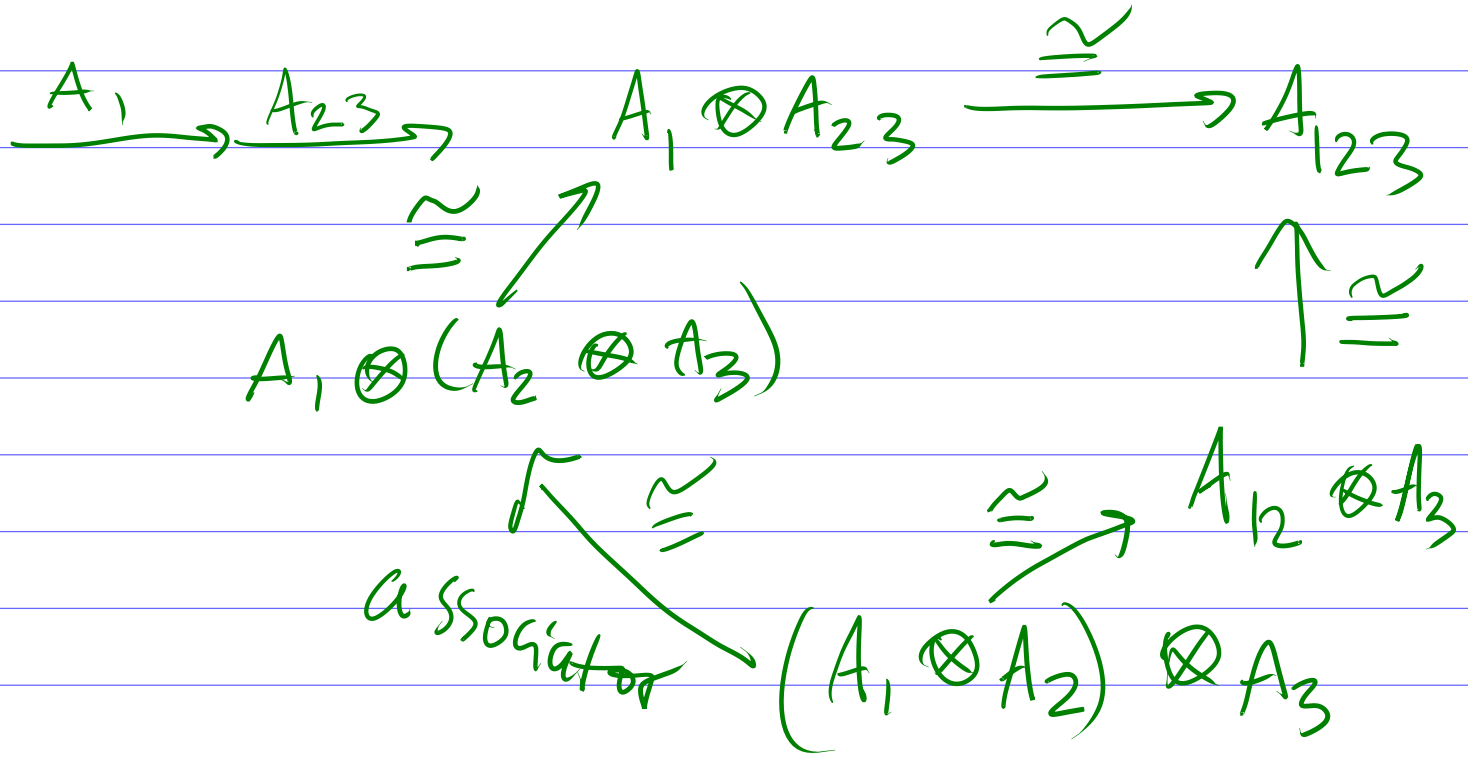
Morphisms: natural isos that commute w/ φ

Face maps: outer: discard A_1 or A_n

$(A_{12}, A_3, A_{12} \otimes A_3 \xrightarrow{\simeq} A_{123})$

inner: keep data where 1, 2 are both present or not present.

Degeneracy maps: insert $\mathbb{1}$



$\Delta^{op} \longrightarrow Grpd$

$\left(A_1 \otimes (A_2 \otimes A_3)\right) \otimes A_4$

$\downarrow \cong$ Mac Lane's coherence theorem.

$\left((A_1 \otimes A_2) \otimes (A_3 \otimes A_4)\right)$

Segal conditions $\Leftrightarrow$ Mac Lane's coherence theorem for monoidal categories.

Lecture 15: February 11

e) The category of spans

(2,1) - category (invertible 2-morphisms)

$\mathcal{C}$: category with pullbacks

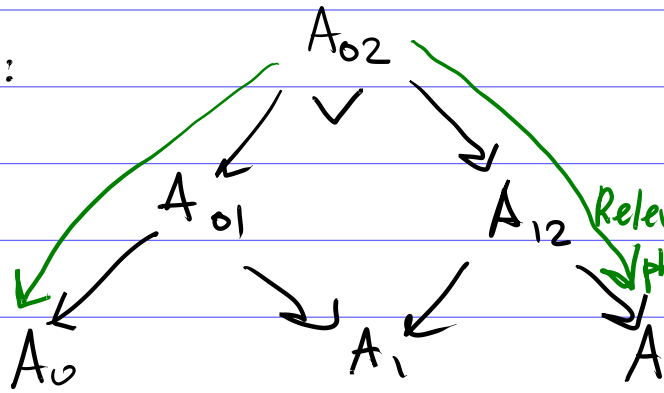
$A_{01} \to A_0 \times A_1$

Span $\mathcal{C}$: $\mathrm{Ob} = \mathrm{Ob}\,\mathcal{C}$ $A_0, A_1, \ldots$

$\mathrm{Mor}(A_0, A_1) =$ $A_0 \leftarrow A_{01} \rightarrow A_1$ (span)

Geometric interpretation: correspondences;

Composition of morphisms:



Relevance for physics: Spaces of fields (cobordisms)

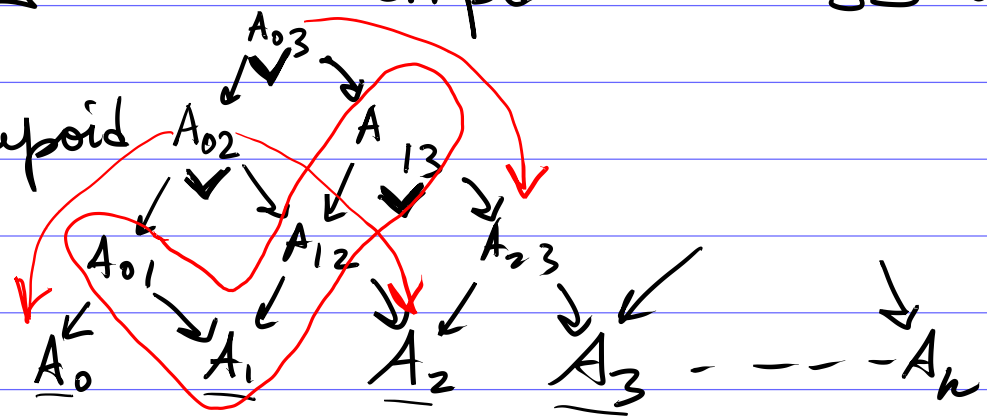
Composition is not uniquely defined; only up to unique iso.

Solution: encode Span $\mathcal{C}$ as a Segal object (in groupoids)

$\mathrm{Span}\ \mathcal{C}: \quad \Delta^{op} \longrightarrow \mathrm{Grpd} \xrightarrow{N} \mathrm{sSet}$

$(\mathrm{Span}\ \mathcal{C})_n$: groupoid

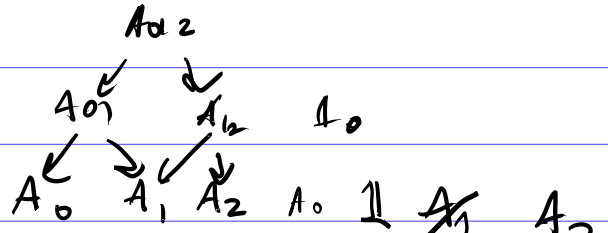
Objects:



Isomorphisms: natural transformations; Id on the bottom row

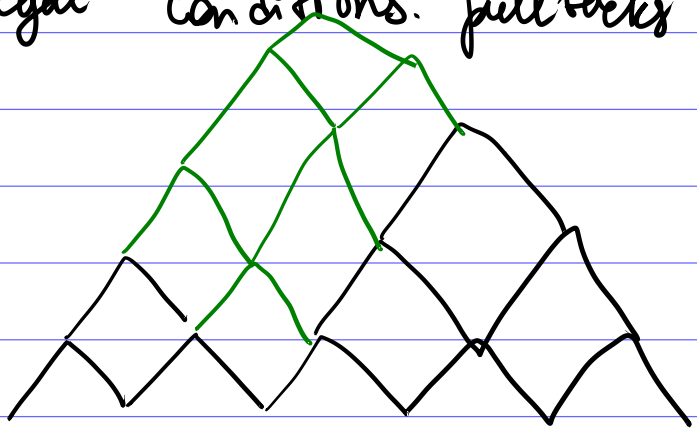
Simplicial structure maps:

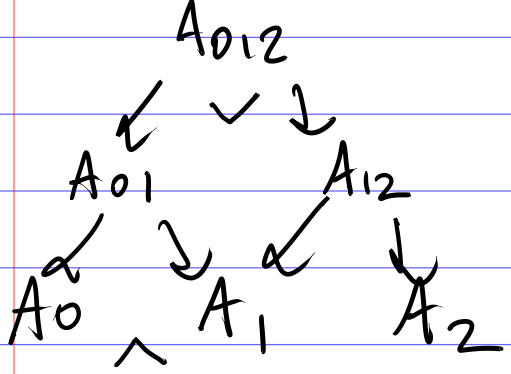
a) Face maps: delete ✓;

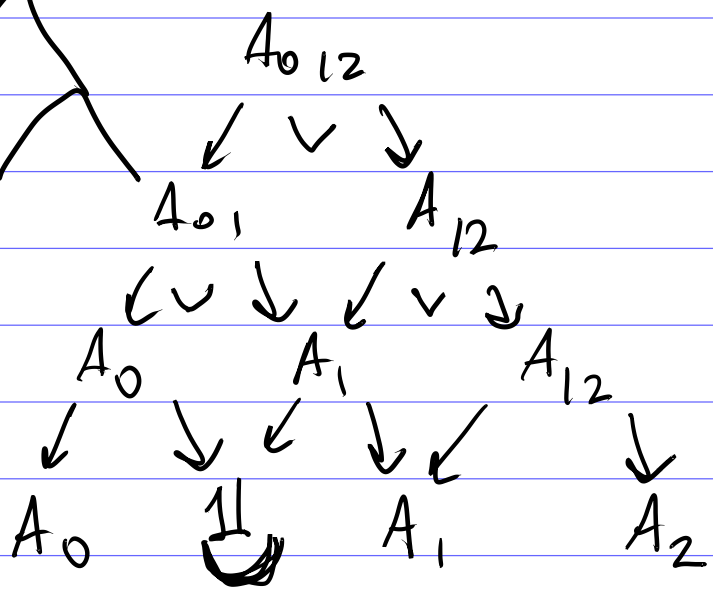


b) Degeneracy maps: insert $\mathbb{1}$; duplicate ✓

Segal conditions: pullbacks unique up to unique iso

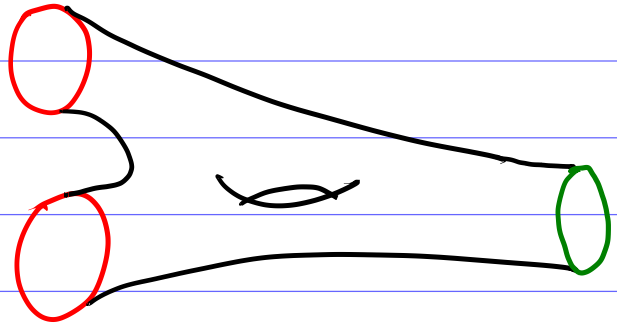


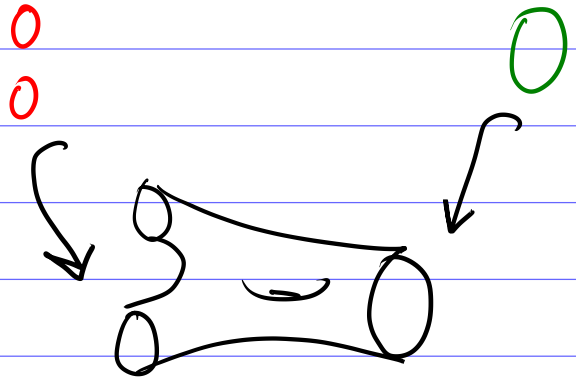




Spans in the opposite category = cospans

e.g., bordisms:

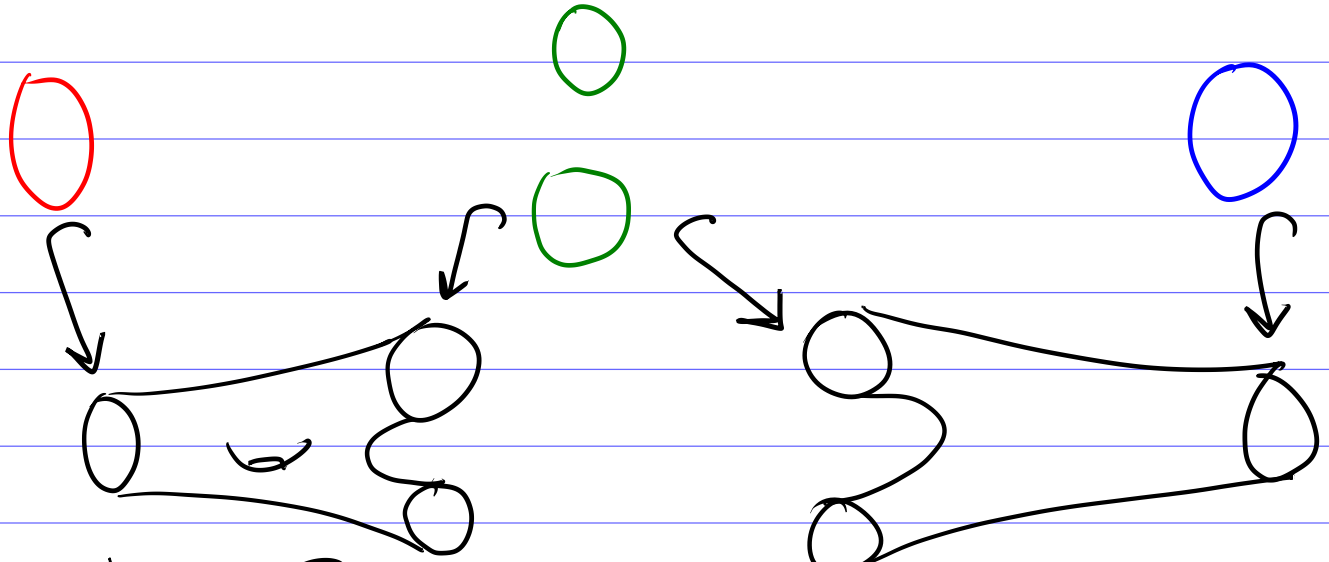


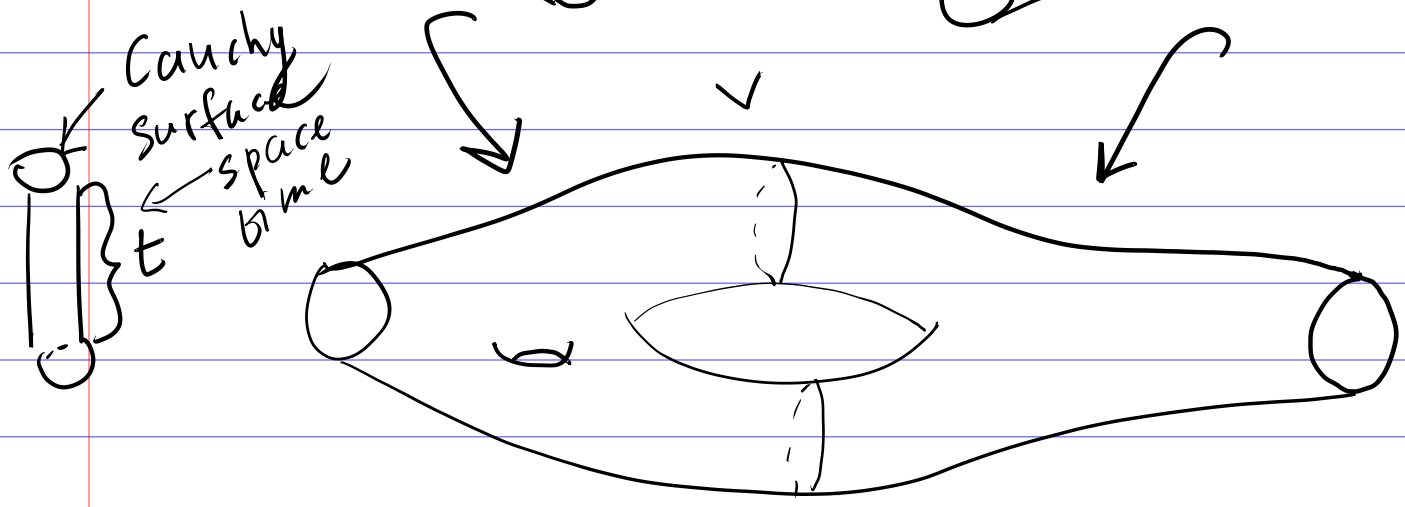


gluing of bordisms

= composition of cospans

Defect: only works for topological bordisms





The completeness condition for Segal categories

$\Delta^{op} \to Grpd$

$X: \Delta^{op} \to sSet$ Segal category

Redundancy: two notions of invertible 1-morphism

a) $f \in (X_1)_0^X$

b) $f \in (X_0)_{/1}$ $\quad A \mapsto id_A$

We have a canonical map $X_0 \xrightarrow{s_0} X_1^X$ (the X_1^X circled, s_0 and $\star$ marked)

Def X is complete if $\star$ is a weak eq.

Example a) Not complete: ordinary nerve:

$\mathcal{C}$: category

$X_n =$ the discrete set of functors $\{0 \to 1 \to \cdots \to n\} \to \mathcal{C}$

$X_0 = Ob\,\mathcal{C}$ $\quad X_1 = Mor\,\mathcal{C}$ (set) $\quad X_1^X = Iso\,\mathcal{C}$

$X_0 \xrightarrow{s_0} X_1^X$: $Ob\,\mathcal{C} \xrightarrow[X \mapsto id_X]{\nsim \text{ if } \mathcal{C} \text{ has nonidentity isos}} Iso\,\mathcal{C}$

b) Complete: Rezk nerve

$X_n = Fun(\{0 \to 1 \to \cdots \to n\}, \mathcal{C})^{\simeq}$ groupoid

$X_0 = \mathcal{C}^{\simeq}$, $X_1 = Mor(\mathcal{C})^{\simeq}$ $\quad X_1^X = Iso\,\mathcal{C}$

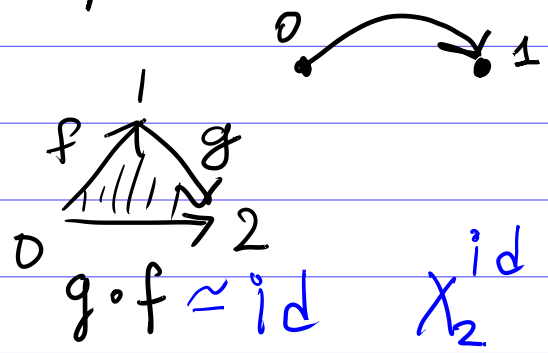
$\mathcal{C}^{\simeq} \xrightarrow{s_0} Iso\,\mathcal{C}$ $\quad A \mapsto A \xrightarrow{id} A$ $\quad \begin{matrix} A \\ \downarrow f \\ B \end{matrix} \mapsto \begin{matrix} A & \xrightarrow{id} & A \\ \downarrow f & & \downarrow f \\ B & \xrightarrow{id} & B \end{matrix}$

$X : \Delta^{op} \to sSet \quad (\Delta^{op} \to Grpd)$

$X_0 =$ ∞-groupoid of objects

$X_1 =$ ∞-groupoid of morphisms

$$X_1 \underset{d_0 = t}{\overset{d_1 = s}{\rightrightarrows}} X_0$$



$$X_1 \times_{s, X_0, t} X_1 \xleftarrow{\sim} X_2 \xrightarrow{d_1} X_1$$

(g, f) $g \circ f \simeq id$

(d_0, d_2)

d_1

id

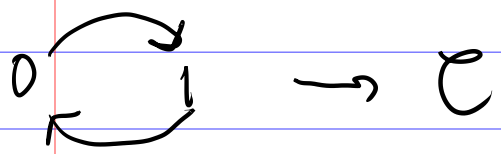
s_0

(g,f) $\downarrow$ f

$X_1 \leftarrow im = \{f \mid \exists g.\ g \circ f \simeq id\}$

X_0

Remark $X_1^X = \mathbb{R}Map\left(N(\leftrightarrows), X\right)$



Left Bousfield localization for complete Segal spaces

$\mathrm{Fun}(\Delta^{op}, \mathcal{S}^{Set})$ proj/inj = Reedy

(*) $y_{[m]} \sqcup_{y_{[0]}} y_{[n]} \longrightarrow y_{[m+n]}$

X: fibrant object.

$X_{m+n} \xrightarrow{\sim} X_m \times^h_{X_0} X_n$

(**) $N\left(0 \rightleftarrows 1\right) \longrightarrow y_{[0]}$

$\mathrm{RMap}(-, X)$ X: fibrant obj

$X_0 \xrightarrow{\sim} \mathrm{RMap}\left(N\left(0 \rightleftarrows 1\right), X\right)$

$s_0 \searrow \quad \| \quad X_1^X$

$N\mathcal{C} \xrightarrow{\sim} N_{Rezk}\mathcal{C}$, $N_{Rezk}\mathcal{C}$ is complete.

a) $N_{Rezk}\mathcal{C}$ is complete (Segal object in groupoids)

$[0] \mapsto \mathcal{C}^X$

$[1] \mapsto (\mathcal{C}^{\to})^X$

$\cup$

$I_{s_0}\mathcal{C}$

$\mathcal{C}^X \xrightarrow[s_0]{f \mapsto id_f} I_{s_0}\mathcal{C}$ $\xrightarrow{s}$ $\mathcal{C}$, $x \mapsto id_x$

ess surj: $X \mapsto \begin{array}{ccc} X & \xrightarrow{id_X} & X \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\sim} & Y \end{array}$ (f)

full: $\begin{array}{ccc} X & & X \xrightarrow{id_X} X \\ {\scriptstyle f}\downarrow & \mapsto & \downarrow{\scriptstyle f} \\ Y & & Y \xrightarrow[id_Y]{} Y \end{array}$

faithful ✓

b) $N\mathcal{C} \xrightarrow{\sim} N_{Rezk}\mathcal{C}$ (not an objectwise weak eq.)

$\uparrow$

local weak eq in left Bousfield loc w.r.t. Segal maps & completion maps

Notation: If $X : \Delta^{op} \to sSet = Fun(\Delta^{op}, Set)$

$[n] \in \Delta$ $\qquad \downarrow ev_{[n]}$

Then: $X_n : \Delta^n \to Set$ $\qquad Set$

(Not the same as $X([n])$.)

Observations

1) $(N_{Rezk}\mathcal{C})_n$ = nerve of $\mathrm{Fun}(0 \xrightarrow{\sim} 1 \xrightarrow{\sim} 2 \to \cdots \xrightarrow{\sim} n, \mathcal{C})$

2) $(N_{Rezk}\mathcal{C})_0 \xrightarrow[\simeq]{s_0^n} (N_{Rezk}\mathcal{C})_n$ simplicial homotopy equivalence!

induced by $\{0 \xrightarrow{\sim} 1 \xrightarrow{\sim} 1 \xrightarrow{\sim} 2 \to \cdots n\} \xrightarrow{\sim} \{0\}$

3) $N_{Rezk}\mathcal{C} \simeq \underset{n \in \Delta^{op}}{\mathrm{hocolim}} (N_{Rezk}\mathcal{C})_n$

$\simeq (N_{Rezk}\mathcal{C})_0 = N\mathcal{C}.$

4) $\Rightarrow$ Theorem: $N\mathcal{C} \xrightarrow{\sim} N_{Rezk}\mathcal{C}$

$$\left(N_{\text{Rezk}}\,\mathcal{C}\right)_{\underbrace{n}_{\text{fixed}}}\left(\underbrace{[m]}_{\text{varies}}\right)$$

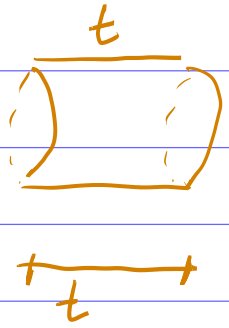
$$= N\left(\text{Fun}\left(\{0 \to 1 \to \cdots \to m\}, \mathcal{C}\right)^{\simeq}\right)_n$$

$$\begin{array}{ccccccc}
A_{00} & \to & A_{01} & \to & \cdots & \to & A_{0m} \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
A_{10} & \to & A_{11} & \to & \cdots & \to & A_{1m} \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
\vdots & & \vdots & & & & \vdots \\
\sim & & \sim & & & & \sim \\
A_{n0} & \to & A_{n1} & \to & \cdots & \to & A_{nm}
\end{array}$$

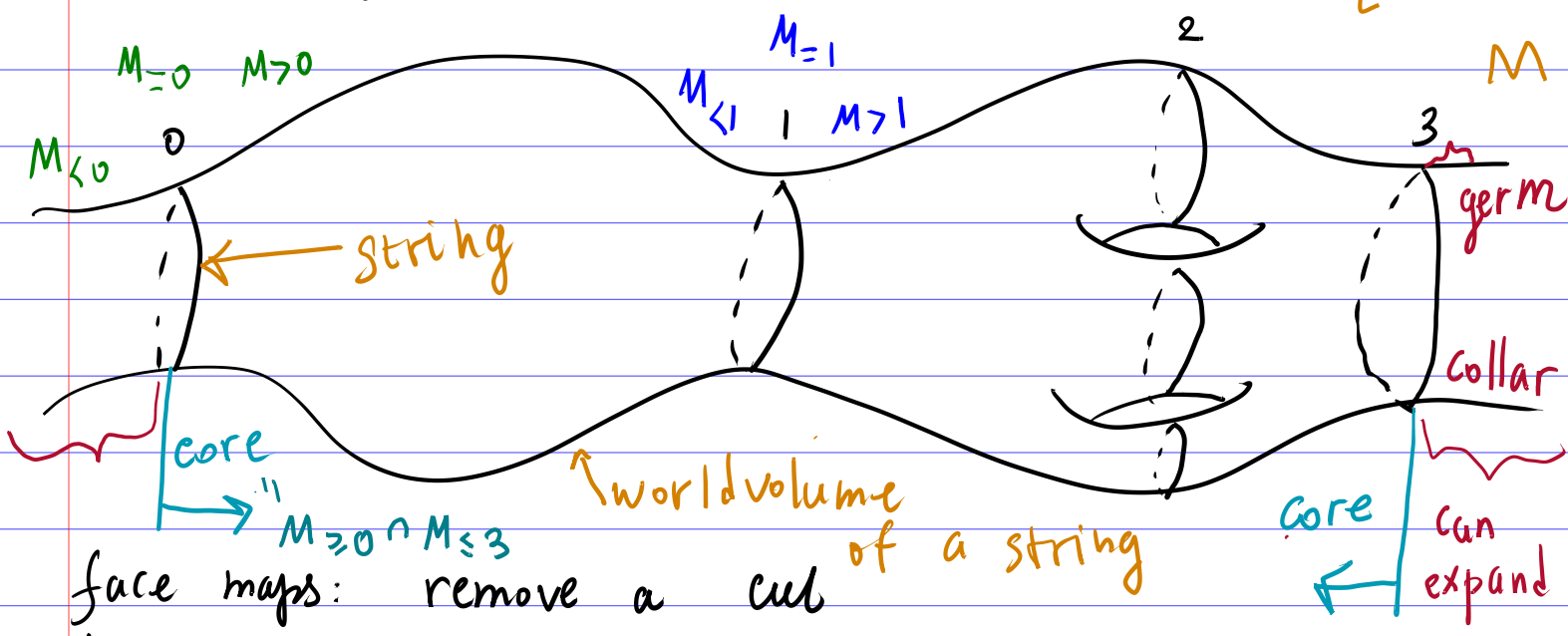
$$\left(N_{\text{Rezk}}\,\mathcal{C}\right)_n = N\,\text{Fun}\left(\{0 \tilde{\to} 1 \tilde{\to} 2 \tilde{\to} \cdots n\}, \mathcal{C}\right)$$

Lecture 16: February 18, 2024

Categories of bordisms



$[n] = [3]$: groupoid of bordisms



face maps: remove a cut

degeneracy maps: duplicate a cut

morphisms in the groupoid: diffeomorphisms that preserve cuts

Def A cut on a manifold M is a triple $(M_{<0}, M_{=0}, M_{>0})$ of subsets of M such that $\exists\, h: M \to \mathbb{R}$:

- 0 is a regular value of h
- $(M_{<0}, M_{=0}, M_{>0}) = (h^{-1}(-\infty, 0), h^{-1}\{0\}, h^{-1}(0, \infty))$

A cut "knows" about incoming/outgoing bordism

An ordering on cuts: $M_{<0} \subset N_{<0}$

$(M_{<0}, M_{=0}, M_{>0}) \prec (N_{<0}, N_{=0}, N_{>0})$

Def A cut tuple on $\mathcal{M}$ is cuts

$$C_0 \leq C_1 \leq \cdots \leq C_m$$

We get a functor $\Delta^{op} \longrightarrow$ Set

face map: remove a cut

degeneracy map: duplicate a cut

$$[m] \mapsto \{\text{set of cut } [m]\text{-tuples on } \mathcal{M}\}$$

Def Given $d \geq 0$, the Segal object of d-dim bordisms is $\Delta^{op} \longrightarrow$ sSet (KQ) that sends $[m] \in \Delta$ to N of a cat

Ob: bordisms:

- $\mathcal{M}$ d-dim smooth manifold
- C a cut $[m]$-tuple on $\mathcal{M}$

$C = (C_0, \ldots, C_m)$ $C_k = (C_{<k}, C_{=k}, C_{>k})$

$C_{\leq k} = C_{<k} \cup C_{=k}$, $C_{\geq k} = C_{=k} \cup C_{>k}$

such that

the core of $(\mathcal{M}, C) = C_{\geq 0} \cap C_{\leq m}$

is compact

Mor: cut-respecting embeddings of bordisms

$\varphi : \mathcal{M} \hookrightarrow \mathcal{M}'$ open embedding

such that $\varphi^* C' = C$

and $\operatorname{im} \varphi \supset \operatorname{core}(\mathcal{M}', C')$ } expand collars



Remark φ restricts to a bijection on cores.



Remark N(cat of bordisms and emb. (diffeomorphisms))

KQ $\simeq$ N(groupoid of bordisms and germs of diffeos)

geom str.



Remark Can also require

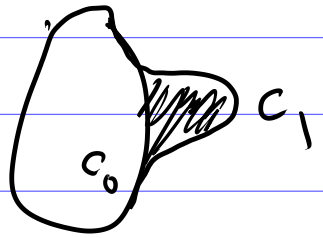
$$C_{\leq 0} \subset C_{<1}$$

$$C_{=0} \subset C_{<1}$$

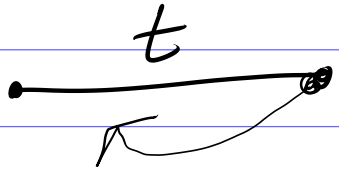


$$C_{=0} \cap C_{=1} = \emptyset$$

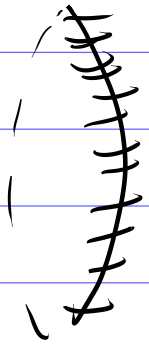
Disallow:



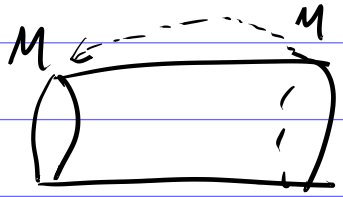
identity bordisms: are problematic



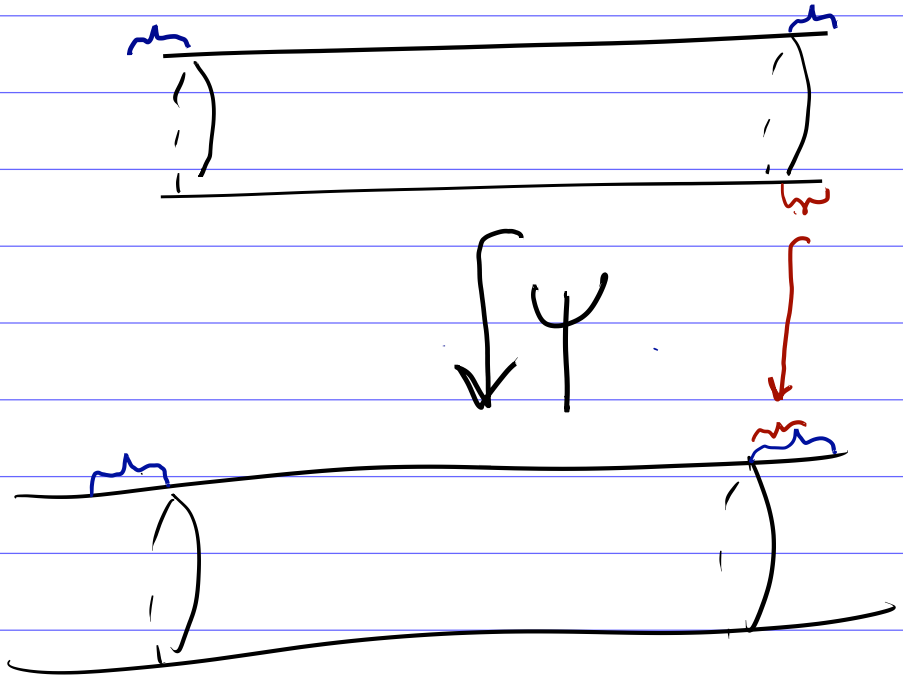
$t \to 0$ $t = 0$



(Adding isotopies partially resolves the problem.)

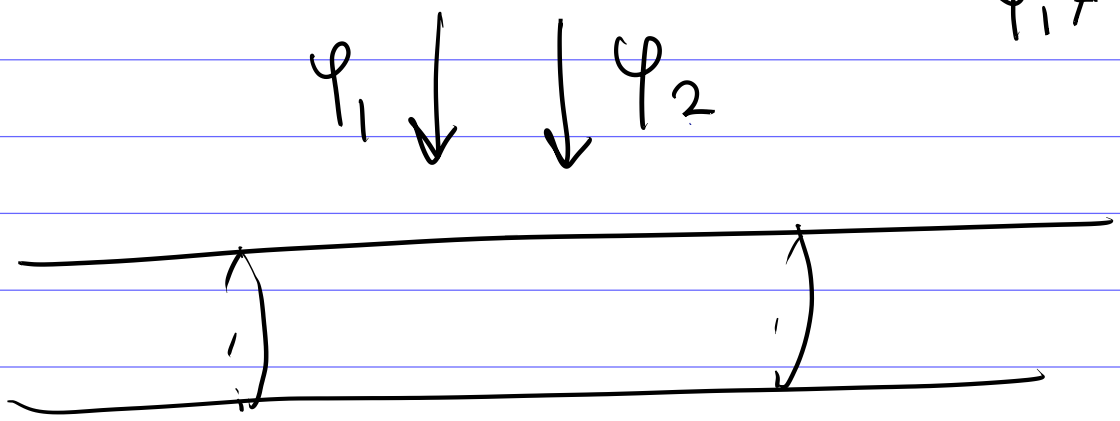


With isotopies: have identity bordisms up to isotopy



$\varphi_1 \circ \psi = \varphi_2 \circ \psi$

$\varphi_1 \neq \varphi_2$



In $\mathcal{N}(\text{emb})_{kQ}$, ψ is invertible,

hence $\varphi_1 \circ \psi = \varphi_2 \circ \psi$ implies

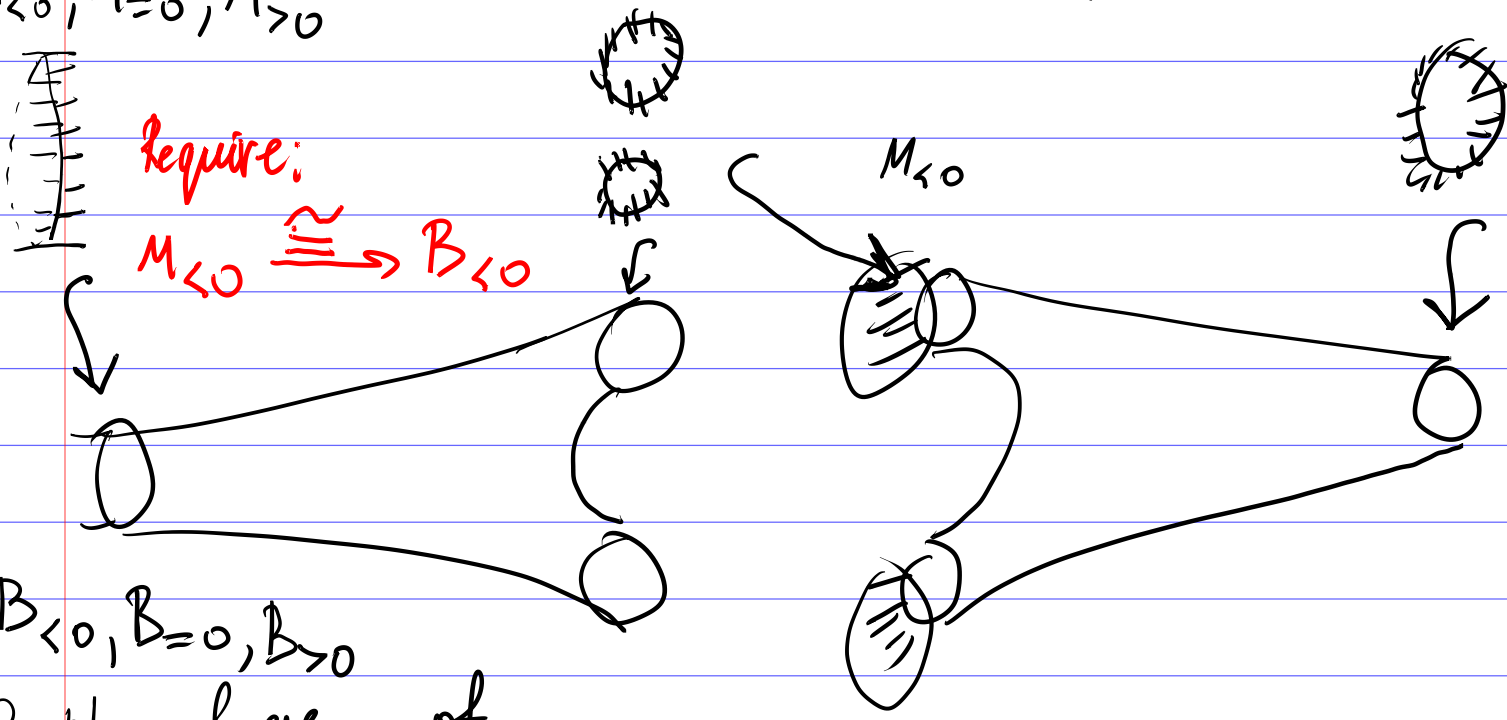
$$\varphi_1 \circ \underbrace{\psi \circ \psi^{-1}}_{1_2} \simeq \varphi_2 \circ \underbrace{\psi \circ \psi^{-1}}_{1_2}$$

$$\varphi_1 \qquad\qquad \varphi_2$$

Bordisms as cospans

$M_{<0}, M_{=0}, M_{>0}$

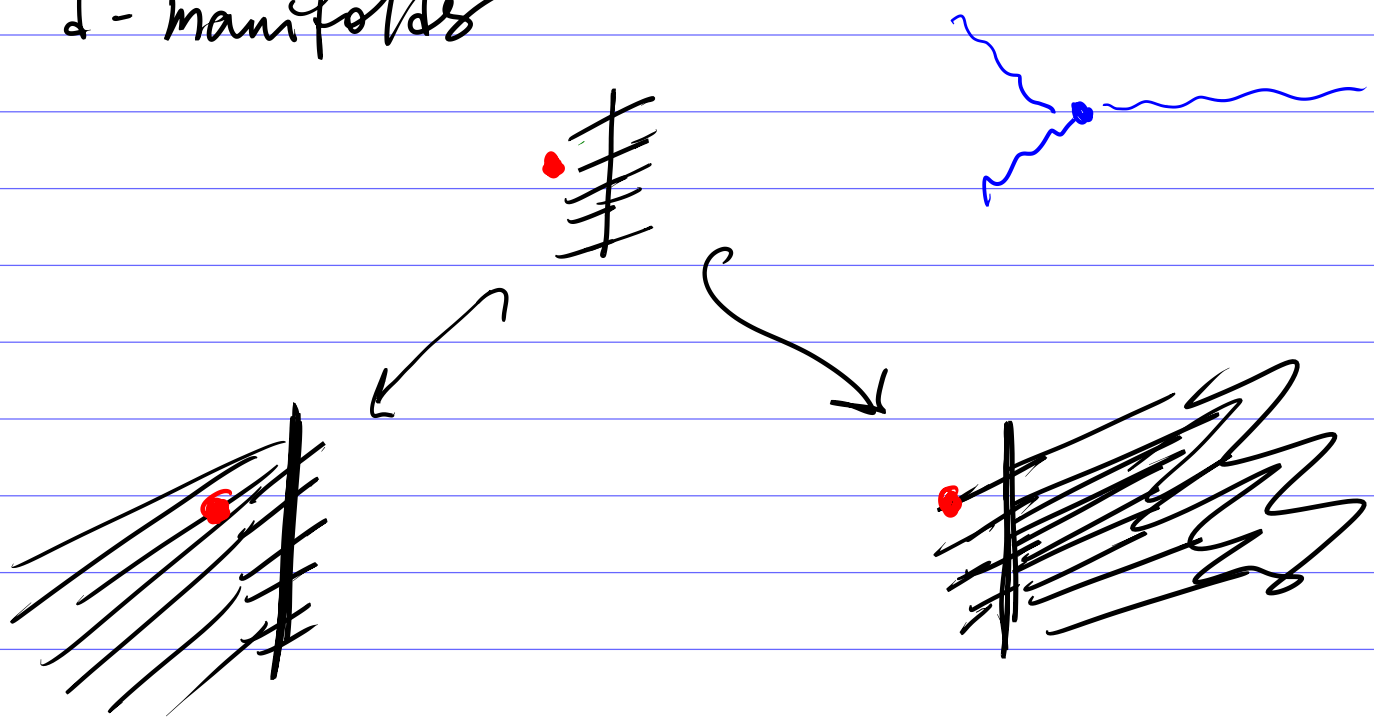
require: $M_{<0} \xrightarrow{\simeq} B_{<0}$



$B_{<0}, B_{=0}, B_{>0}$

Both legs of a cospan are embeddings of d-manifolds

Stolz – Teichner 2012



More things to take care of:

1) Want a d-category of bordisms not a 1-category

$\Delta \rightsquigarrow \Delta^{\times d}$ n-fold Segal space

2) Want a sym. mon. str $\otimes = \sqcup$ Γ-objects

3) Want geometric structures on bordisms

4) Want smooth families of bordisms.

5) Want isotopies = (∞,d)-cat of bordisms
(make bordisms dualizable)

February 25, 2024

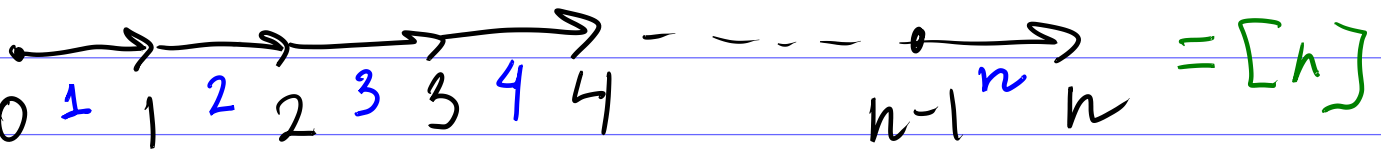
Symmetric monoidal structures and Γ-objects.

(Segal 1970: Cats and cohom. theo)

Like Δ-spaces, but using Γ.

$\Gamma = \mathrm{Fin\,Set}_*^{op} = \mathrm{Fin\,Set\,Partial}^{op}$

Recall $[n] \in \Delta$ is interpreted as



The corresponding object on Γ is

$\{1, 2, 3, \ldots, n\} = \langle n \rangle$

(Instead of composing n arrows, take the product of n elements.)

Def A Γ-space is $\Gamma^{op} \xrightarrow{X} sSet$

$\mathrm{Fin\,Set}_* \to sSet$

such that $X_0 \xrightarrow{\sim} *$

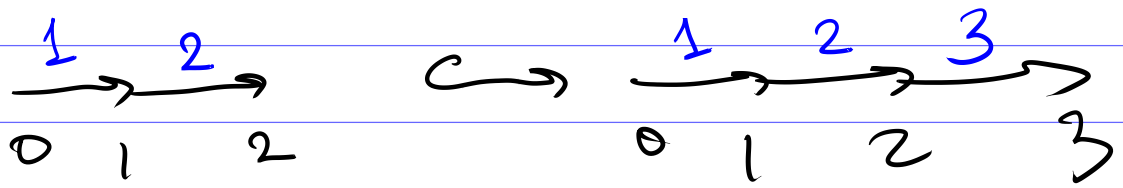
$X_{m+n} \xrightarrow{\sim} X_m \times X_n$

Also works $sSet \rightsquigarrow \mathcal{V}$

Which maps of finite sets to allow?

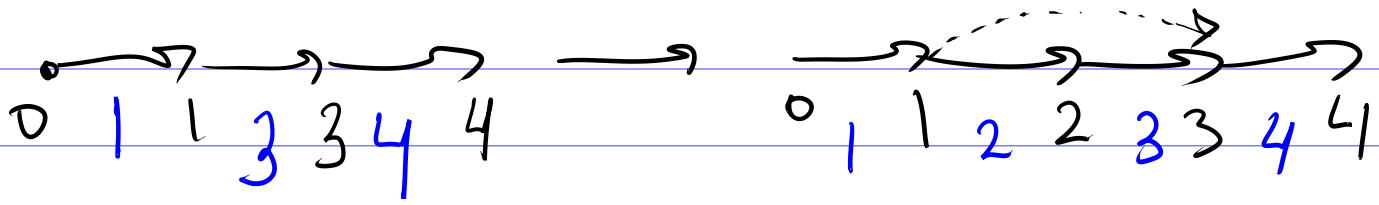
Recall In Δ, three types of maps

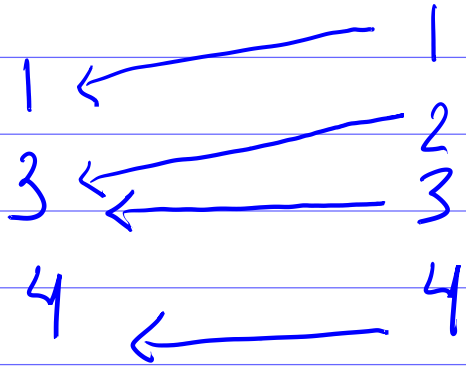
- outer face maps $X_3 \to X_2$



$\{1,2\}$ ← discard the third element in the tuple $\{1,2,3\}$ $X_3 \to X_2$

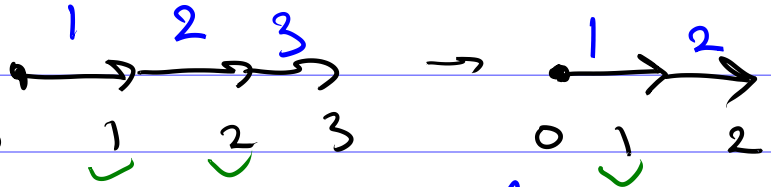
- inner face maps

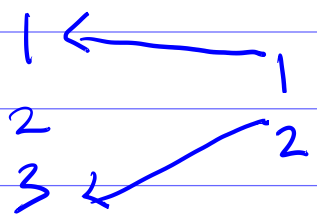




multiply the 2nd and 3rd elements

- degeneracy maps





insert the neutral element in the 2nd position

Remark In both Δ and Γ, every map decomposes as a composition of outer face maps, inner face maps, and degeneracy maps.

(Γ has nonidentity automorphisms, which must be allowed in the decomposition)

Comparison of Δ and Γ

Def

$$\begin{array}{ccc} \Delta & \xrightarrow{\beta} & \Gamma \\ \downarrow & & \downarrow \\ [n] & \longmapsto & \langle n \rangle \end{array}$$

$f: [m] \to [n]$ $\qquad$ $\beta(f): \beta[m] \to \beta[n]$

$$\begin{array}{ccc} \langle m \rangle & \xleftarrow{g} & \langle n \rangle \\ \| & & \\ \{1, \cdots, m\} & & \end{array}$$

$g^{-1}\{i\} = \{f(i-1)+1, \cdots, f(i)\}$

$$\begin{array}{ccc} i-1 & \xrightarrow{i} & i \\ \updownarrow & & \updownarrow \\ f(i-1) & \longrightarrow & f(i) \end{array}$$

$f(i-1) \to f(i-1)+1 \to \cdots \to f(i)-1 \to f(i)$

Remark $\quad \operatorname{dom} g = (f(0), f(m)]$.

Example

$CMon \longrightarrow \Gamma\text{-}Set$

$(X, \cdot) \longmapsto FinSetP \xrightarrow{X} Set$

A finite: $X^A \longrightarrow X$

$X_{\langle n \rangle} = X^n = \underbrace{X \times \cdots \times X}_{n \text{ times.}}$

$f: \langle m \rangle \longrightarrow \langle n \rangle$ FinSetP

$f^{-1}\{j\}$ $\quad$ j

$X_f : X_{\langle m \rangle} \longrightarrow X_{\langle n \rangle}$

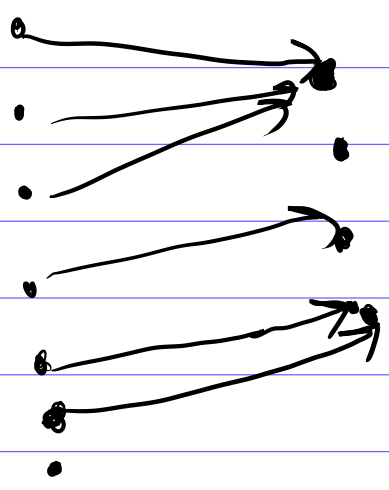
$\| \qquad \|$

$X^m \xrightarrow{X_f} X^n$

$j \in \langle n \rangle$:

$(X_f)_j : X^m \longrightarrow X$

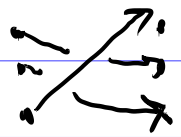
$X^{f^{-1}\{j\}} \xrightarrow{\pi \text{ projection}} X \xrightarrow{\text{multiplication}} X$



$f^{-1}\{j\} \subset \langle m \rangle$

Remark $\Sigma_n \curvearrowright X_{\langle n \rangle}$

$Aut(\langle n \rangle) \xrightarrow{\sim} \langle n \rangle \longrightarrow \langle n \rangle$



$\Gamma^{op} \xrightarrow{\mathbb{Z}} CMon$ $\qquad$ $\mathbb{B}\mathrm{ord}^{\mathrm{framed}}_{\infty,d}\left[k\text{-mor}^{-1}_{0\le k\le d}\right]$

$\| \qquad\qquad\qquad\qquad \simeq \mathbb{S}$

FinSet^{P}

$A \in \qquad \mapsto \mathbb{Z}[A] = \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{A}$

$f: A \to B \quad \mapsto \quad \mathbb{Z}[A] \to \mathbb{Z}[B]$

$a \in \quad \mapsto \begin{cases} f(a) \in \\ 0, \text{ if } a \notin \text{ pointf} \end{cases}$

Example $\boxed{X \in CMon:}$

$\Gamma^{op} = \mathrm{FinSet}_{P} \to \mathrm{Set}$

$\mathbb{Z}$ $\downarrow$ $\nearrow CMon(-, X)$

FGFree CMon

(opposite) algebraic theory of commutative monoids.

$\mathbb{Z} \in$ Ring initial CRing

$\mathbb{S} \in$ EwRing ∞-initial

$\mathbb{J}_\infty \mathbb{S} \xrightarrow{\cong} \mathbb{Z}$

Remark

Γ itself can be thought of as follows:

$A \leftrightarrow \mathbb{S}[A]$ (sphere spectrum)

$\Gamma \xrightarrow{\sim} (CMon^{\infty}_{\mathbb{S}})^{op}$

Example $\mathcal{C}$ small cat w/ finite prod.

$$\Gamma^{op} \xrightarrow{X} Cat \xrightarrow{N} sSet$$

$X_n =$ cat; obj: ...

$\Sigma_n \curvearrowright X_n$ by permuting objects in the corresponding diagram

$\Sigma_2 \curvearrowright$ $A \leftarrow A\times B \rightarrow B$ $\quad B \leftarrow A\times B \rightarrow A$

Example $\mathcal{C}$ small cat w/ sym. mon. str $\otimes$

$$\Gamma^{op} \xrightarrow{X} Cat \xrightarrow{N} sSet$$

$X_{\langle n\rangle}$: cat; obj:

$\Phi \subset \langle n\rangle$, $\Phi \neq \emptyset$: $A_\Phi \in \mathcal{C}$

$\left.\begin{array}{l} \Phi \subset \langle n\rangle,\ i \in \langle n\rangle \\ \Phi < i \end{array}\right\}$ $A_\Phi \otimes A_{\{i\}} \xrightarrow{\varphi} A_{\Phi\cup\{i\}}$

$\Sigma_n \curvearrowright X_n$: use Maclane's coherence thm for s.m.c.

Model structure: $Fun(\Gamma^{op}, sSet)_{proj/inj}$ / Reedy

$\rightsquigarrow$ left Bousfield localization at Γ

$$\begin{cases} y_{\langle m\rangle} \sqcup y_{\langle n\rangle} \to y_{\langle m+n\rangle} \\ \emptyset \to y_{\langle 0\rangle} \end{cases}$$

$\langle m\rangle \to \langle m+n\rangle$

$\langle m\rangle \leftarrow \langle m+n\rangle$ (via $[id, \downarrow]$)

$\langle m+n\rangle = \langle m\rangle \sqcup \langle n\rangle$

fibrant objects = special Γ-objects: $\Gamma^{op} \xrightarrow{X} Kan$ (proj)

$X_0 \xrightarrow{\sim} *$, $X_{m+n} \xrightarrow{\sim} X_m \times X_n$ (homotopy coherent commutative mon.)

Connections to other approaches:

Lecture 18
2024-3-3

Thm $\exists$ Quillen equivalences

$$\Gamma\text{-spc} = Fun(\Gamma^{op}, sSet) \rightleftarrows Alg_{E_\infty}$$

$\langle m\rangle \mapsto \mathbb{F}_1 \sqcup \cdots \sqcup \mathbb{F}_1$

(transferred along the forgetful functor)

where $E_\infty \xrightarrow{\sim} Comm$, $\Sigma_n \curvearrowright (E_\infty)_n$ free.

Thm $\exists$ equivalence of quasicategories

$$\mathcal{N}(\Gamma\text{-spc})[W^{-1}] \rightleftarrows QAlg_{Comm}$$

quasicategor.

Remark Group-like Γ-spaces also require (very special Γ-spaces)

(☆) $y_{\langle 1\rangle} \sqcup y_{\langle 1\rangle} \xrightarrow{[L_0, M]} y_{\langle 2\rangle}$

$L_0: \langle 2\rangle \to \langle 1\rangle$: $1 \mapsto 1$

$M: \langle 2\rangle \to \langle 1\rangle$: $1, 2 \mapsto 1$

$X_2 \xrightarrow{\sim} X_1 \times X_1$ (shear map)

$(x, y) \longmapsto (x, x \cdot y)$.

The resulting Quillen adjunction

$$\Gamma_{spc} \rightleftarrows \Gamma_{very\ sp}$$

$Sp_{\geq 0}$ connective spectra

Sp infinite loop spaces

implements homotopy group compl & algebraic/topological K-theory

Symmetric monoidal $(\infty,1)$-categories:

$$\Gamma^{op} \longrightarrow Fun(\Delta^{op}, sSet)$$

special Γ-object complete Segal space

$$\Leftrightarrow \Gamma^{op} \times \Delta^{op} \longrightarrow sSet \quad \Leftrightarrow \Delta^{op} \longrightarrow Fun(\Gamma^{op}, sSet)$$

Bordism category as a sym mon $(\infty,1)$-cat.

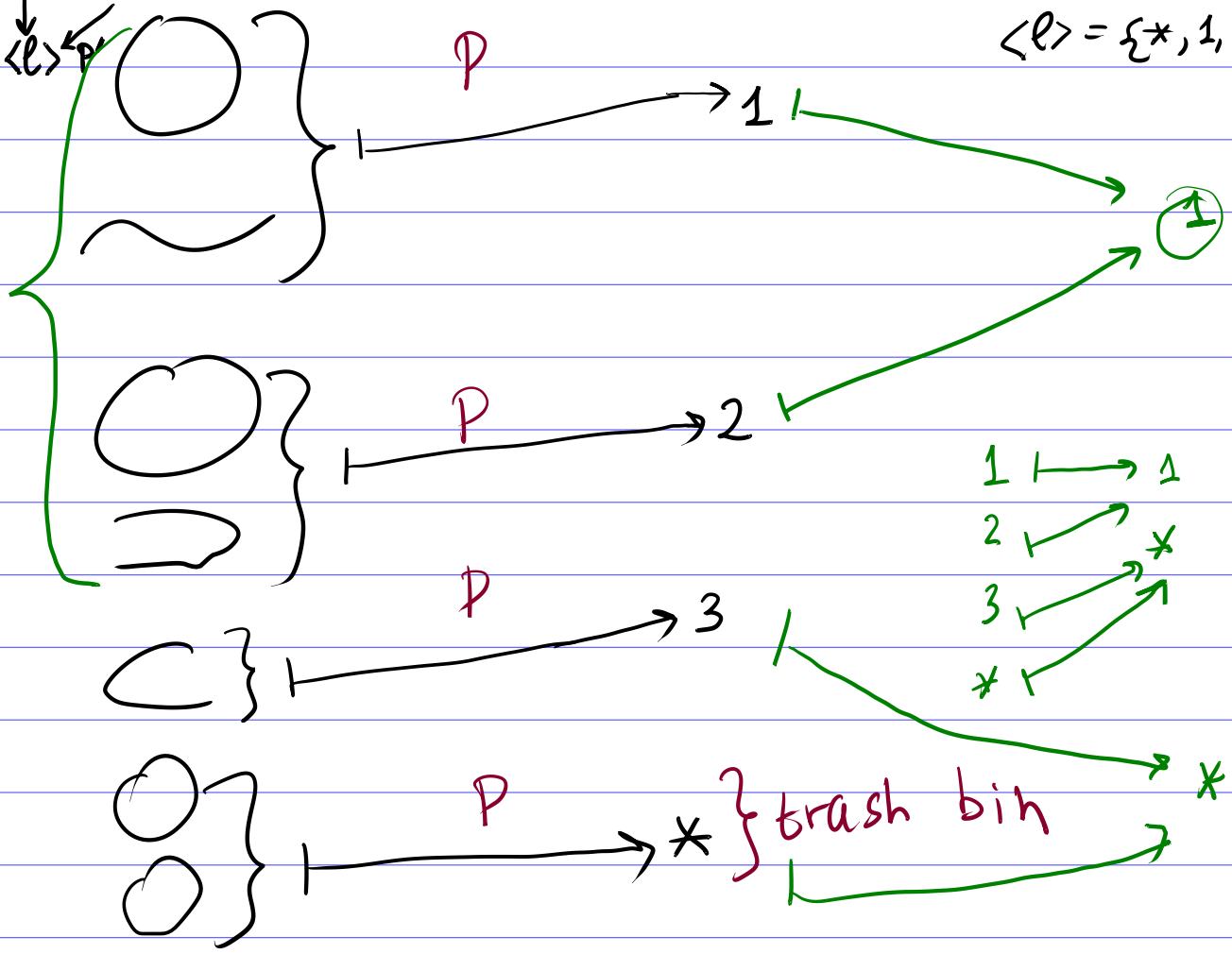
Def $Bord_d : \Gamma^{op} \times \Delta^{op} \xrightarrow{Cat\ N} sSet$

$\langle \ell \rangle, [m] \longmapsto N(\cdots)$

Ob: (M, C, P), $P: M \longrightarrow \langle \ell \rangle$ (finite set)

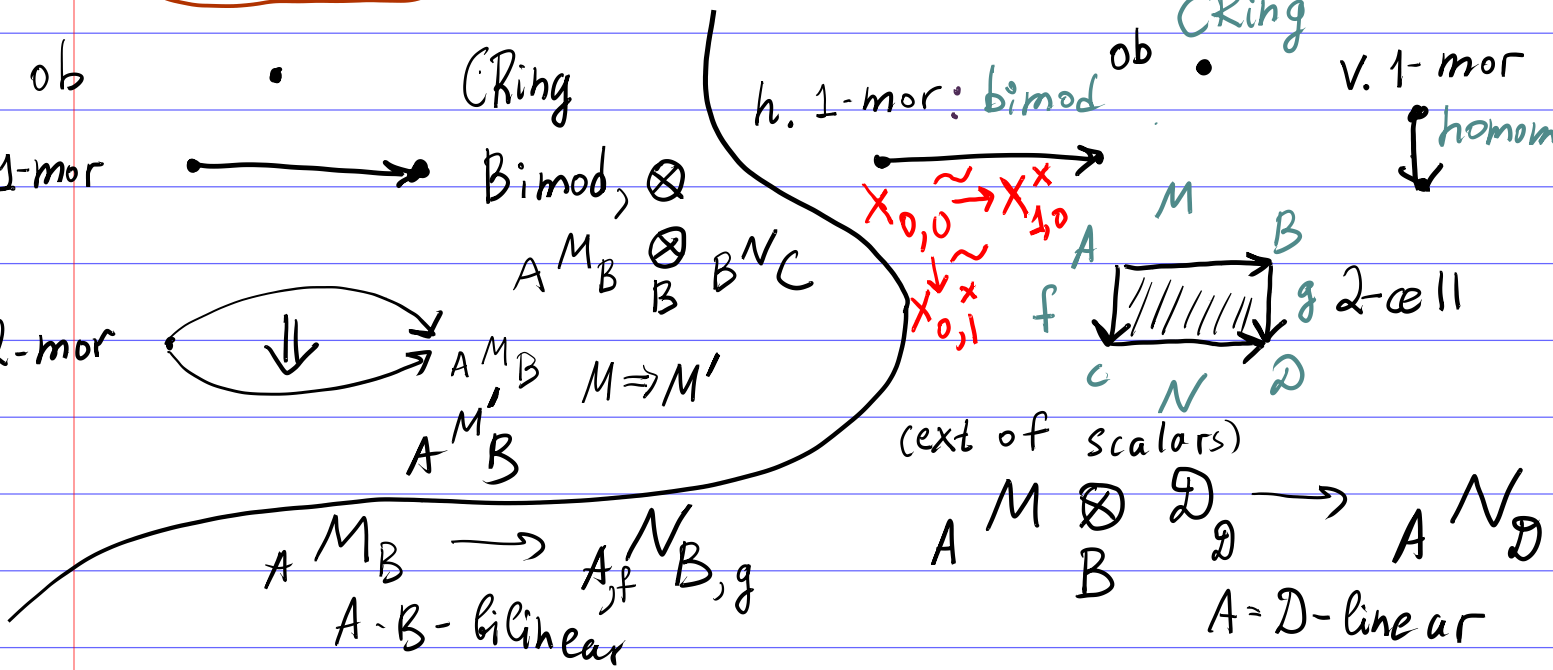
Mor: $M \hookrightarrow M'$, $P \downarrow$, $\langle \ell \rangle \xleftarrow{P'}$

$\ell = 3$, $\langle \ell \rangle = \{*, 1, 2, 3\}$



(∞,d)-categories and symmetric monoidal (∞,d)-cat

bicategories vs double categories



To collapse the vertical direction in order to get bicategories from double categories, require vertical morphisms to be invertible.

$\Rightarrow$ globularity condition

model category

Def A simplicial object $\Delta^{op} \xrightarrow{X} V$ is globular if

$$X_0 \simeq \underbrace{S}_{\text{simplicial set}} \otimes \underbrace{\mathbb{1}}_{\text{terminal object (in } V)} \simeq \operatorname{hocolim}_I \mathbb{1}$$

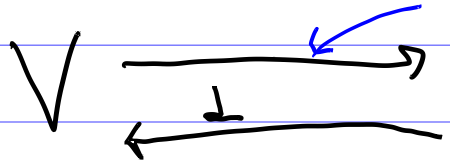
e.g., $(\infty,1)$-cat

$V = \mathrm{Fun}(\Delta^{op}, sSet)$ CSS

Model structures for globularity

- Assume V fits into a Quillen adjunction

(★)



invert everything up to homotopy (∞-groupoids)

- Assume the derived counit is a weak equivalence

$(\infty, 2)$-category $X : \Delta^{op} \times \Delta^{op} \to sSet$ [Grpd]

1) Globularity condition

forces $X^{\times}_{0,1} \xrightarrow{\simeq} X_{0,1}$

$X^{\times}_{0,1}$: the space of invertible vertical 1-morphisms

$X_{0,1}$: the space of vertical 1-morphisms

[all vertical 1-morphisms are invertible]

2) Vertical completeness condition

forces $X_{0,0} \xrightarrow{\simeq} X^{\times}_{0,1}$

$X_{0,0}$: the space of objects

[invertible vertical 1-morphisms are induced by homotopies between objects]

3) Horizontal completeness condition

forces $X_{0,0} \xrightarrow{\simeq} X^{\times}_{1,0}$

$X^{\times}_{1,0}$: the space of invertible horizontal 1-morphisms.

(Counter)example: The double category of algebras, bimodules, etc.

$X: \Delta^{op} \times \Delta^{op} \longrightarrow Grpd \xrightarrow{N} sSet$

k: base ring $\quad Alg_k$ algebras over k

$X_{0,0} = \{Ob: Alg_k ;\ Mor:$ isos of algebras$\}$

$X_{0,1} = \{Ob:$ homs of algebras $A_0 \xrightarrow{f} A_1$

$\downarrow$

Mor $\quad A_0 \xrightarrow{\cong} A_0'$, $f \# f'$, $A_1 \xrightarrow{\cong} A_1'$

$X_{1,0} = \{Ob:\ {}_A M_B$ $\quad Mor:$ isos of bimodules

$p(a \cdot m \cdot b) = f(a)\, p(m)\, g(b)$

$A_0 \xrightarrow{f \cong} A_1$, $M_0 \xrightarrow{p} M_1$, $B_0 \xrightarrow{\cong g} B_1$

$X_{1,1} = \{$ Ob: intertwiners

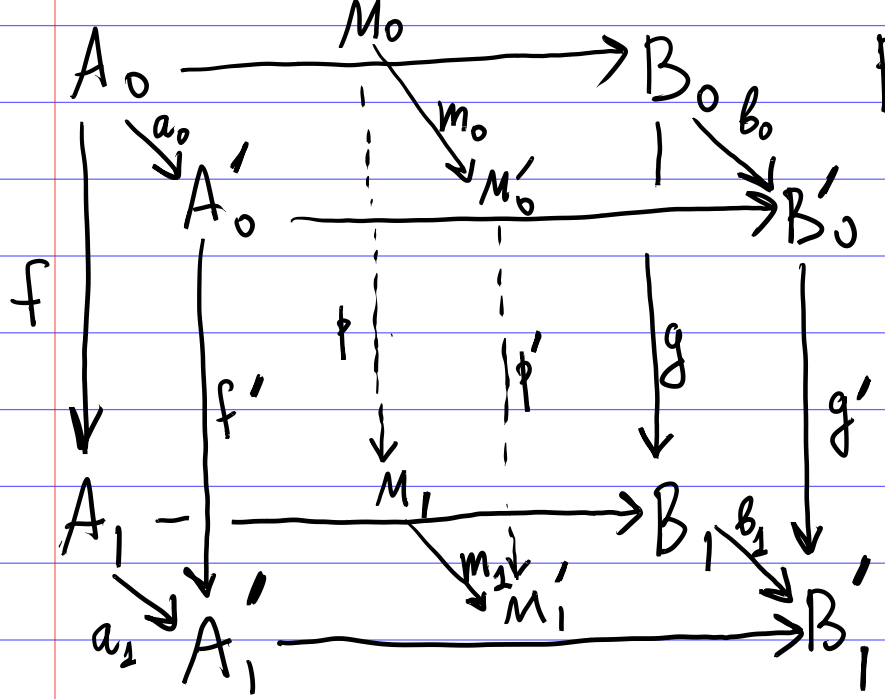
$$\begin{array}{ccc} A_0 & \xrightarrow{M_0} & B_0 \\ {\scriptstyle f}\downarrow & \swarrow p & \downarrow{\scriptstyle g} \\ A_1 & \xrightarrow[M_1]{} & B_1 \end{array}$$

$p : M_0 \to M_1$ k-linear map

$\phi(a \otimes m \otimes b)$

$= f(a)\, p(m)\, g(b)$

$$\Longleftrightarrow \quad \underset{A_0}{M_0} \otimes \underset{B_0}{B_1} \xrightarrow{P} \underset{A_0}{A_1} \otimes \underset{A_1}{M_1} \otimes \underset{B_1}{B_1}$$



$$P \xrightarrow[r]{\cong} P'$$

$p' \circ m_0 = m_1 \circ p$

$X_{2,0}$: Ob: $A \overset{M}{\to} B \overset{N}{\to} C$

Mor

$$\begin{array}{ccccc} A_0 & \xrightarrow{M_0} & B_0 & \xrightarrow{N_0} & C_0 \\ f\downarrow\cong & & \cong\downarrow g & & \cong\downarrow h \\ A_1 & & M_1 & & N_1 \quad B_1 \quad C_1 \end{array}$$

horizontal Segal condition:

$$X_{2,0} \xrightarrow{\cong} X_{2,0} \times_{X_{0,0}} X_{1,0}$$

$$\simeq \downarrow \qquad\qquad \uparrow \simeq$$

$$X_{1,0} \qquad X_{1,0} \times_{X_{0,0}} X_{1,0}$$

$$\simeq \Leftarrow X_{1,0} \twoheadrightarrow X_{0,0}$$

fibration of groupoids "isofibration"

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ A' & & B' \end{array} \qquad \begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ A' & \cong & B' \end{array} \qquad \begin{array}{ccc} M & \to & B \\ \downarrow & & \downarrow \\ M' & & B' \end{array}$$

$M' = M$

$B' = B$

1) globularity:

$$X^{\times}_{0,1} \xrightarrow{\sim} X_{0,1} = \mathbb{N}$$

Ob: invertible homs of algebras $\begin{smallmatrix} A_0 \\ \downarrow f \\ A_1 \end{smallmatrix}$ $\not\simeq$ Ob: homs of algs $\begin{smallmatrix} A_0 \\ \downarrow f \\ A_1 \end{smallmatrix}$

Mor: $\begin{array}{ccc} A_0 & \xrightarrow{\cong} & A_0' \\ {\scriptstyle f}\downarrow & \# & \downarrow{\scriptstyle f'} \\ A_1 & \xrightarrow[\cong]{} & A_1' \end{array}$

X is not globular!

X can be made globular by requiring f to be invertible

Remark In a 2-cell (f, g, ϕ), f and g are required to be invertible, but ϕ can be non invertible.

Vertical completeness condition:

True!

$X_{0,0} \xrightarrow{\sim} X_{0,1}^{\mathbb{X}} = \mathrm{Fun}(\overset{\cong}{\mathbb{I}}, X_{0,0})$

Ob: $A \longmapsto$ Ob: A_0, $A \xrightarrow{\cong} A_1$

Mor: $A \xrightarrow{\cong} A' \longmapsto$ Mor: $A_0 \xrightarrow{\cong} A_0'$, $f \downarrow \cong$, $A_1 \xrightarrow[\cong]{} A_1'$, $f \downarrow f'$

$A \longmapsto$ $A \downarrow id$, A

$A \xrightarrow{s} A' \longmapsto$ $A \xrightarrow{s} A'$, $id \downarrow$, $\downarrow id$, $A \xrightarrow[s]{} A'$

ess surj

$A_0 \xrightarrow{id} A_0$, $id \downarrow$, $\downarrow f$, $A_0 \xrightarrow[f]{} A_1$

Horizontal completeness condition

$X_{0,0} \longrightarrow X^{X}_{1,0}$

invertible $\Leftrightarrow$ Morita equivalence

Ob: A $\quad A \mapsto {}_A A_A$ $\quad$ Ob: ${}_A M_B$

Mor: A_0 $\quad A_0 \mapsto A_0 \; A_0 \; A_0$

$f: A_0 \to A_1$ $\quad f \mapsto f: A_1 \; A_1 \; A_1$

Mor ${}_{A_0} M_0 \; B_0$

$f: A_0 \to A_1$, $M_0 \to M_1$, $g: B_0 \to B_1$

Faithful ✓

Full:

$$\begin{array}{ccc} A & & A \\ & \overset{p}{\downarrow} & \\ f\downarrow & B & \downarrow g \\ B & & B \end{array}$$

$p(a_1 a_2 a_3) = f(a_1)\, p(a_2)\, g(a_3)$

$a_1 = a_2 = 1$: $p(a_3) = p(1) \cdot g(a_3)$

$a_2 = a_3 = 1$: $p(a_1) = f(a_1) \cdot p(1)$

$f(1) = g(1) = 1$

Essential surjectivity:

$$\begin{array}{ccc} A & A & A \\ \cong\downarrow & \cong\downarrow & \downarrow\cong \\ A & M & B \end{array}$$

False!

${}_A A^n \quad {}_A M_n(A)$

A comm: $A \not\cong M_n A$

Remark We can force X to be horizontally complete by adding Morita equivalences to $X_{0,0}$

Then if $\mathcal{V}$ is combinatorial, there is a set of morphisms $\tilde{S}$ in $\mathcal{V}$ such that

$$L_S \mathcal{V} \underset{I}{\overset{\sim}{\rightleftarrows}} sSet \text{ is a Q. equiv.}$$

Remark Suppose $\mathcal{V}$ and S are as above.

Then $$Fun(\Delta^{op}, \mathcal{V}) \underset{const}{\overset{colim}{\rightleftarrows}} \mathcal{V} \underset{I}{\overset{*}{\rightleftarrows}} sSet$$

and: $S' = \{ y_{[n]} \otimes v \to y_{[0]} \otimes v \} \cup \{ y_{[0]} \otimes S \}$

Thus, $Fun(\Delta^{op}, \mathcal{V})$ and S' satisfy the assumption (*)

Def a) The globular model structure on $Fun(\Delta^{op}, \mathcal{V})$ is $L_{S'} Fun(\Delta^{op}, \mathcal{V})_{proj/inj/Reedy}$

$S' = y_{[0]} \otimes S$, where $\mathcal{V}, S$ are as above

X is S'-local $\Leftrightarrow$ X_0 is S-local $\Leftrightarrow$ X_0 is an ∞-groupoid

b) complete globular Segal model structure: on $Fun(\Delta^{op}, \mathcal{V})$:

add: $y_{[m]} \sqcup_{y_{[0]}} y_{[n]} \longrightarrow y_{[m+n]}$

and: $\mathcal{N}(\circlearrowright) \longrightarrow y_{[0]}$

Local obj: $X_{m+n} \xrightarrow{\sim} X_m \times^h_{X_0} X_n$, $X_0 \xrightarrow{\sim} X_1^X$, X_0 ∞-groupoid

$$S' = y_{[0]} \otimes S$$

$$= \{ y_{[0]} \otimes g \mid g \in S \}.$$

X is S'-local $\Leftrightarrow$ $\mathbb{R}\mathrm{Map}(\widetilde{y_{[0]} \otimes g}, X)$ is a weak eq.

$\Leftrightarrow$ $\mathbb{R}\mathrm{Map}(g, X_0)$ is a weak eq.

$\Leftrightarrow$ X_0 is S-local

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \times V \xrightarrow{\otimes} \mathrm{Fun}(\mathcal{C}^{op}, V)$$

$$(F^{\Downarrow}, A^{\Downarrow}) \mapsto (c \mapsto \underbrace{F(c)}_{\in \mathrm{Set}} \otimes A) \in V$$

$$\mathrm{Set} \times V \xrightarrow{\otimes} V$$

$$(G^{\Downarrow}, A^{\Downarrow}) \mapsto \coprod_{G} A$$

$n \geq 0$: category number; V, S: as above

Def The model category of globular complete n-fold Segal objects in V is a Weak n-category object in V

- V if $n = 0$

$n > 0$: • the model category of globular complete Segal objects with values in the model category of globular complete $(n-1)$-fold Segal objects in V.

An (∞, n)-cat is a cat h.c. enriched in $(\infty, n-1)$-cat

$V = sSet$: (∞, n)-categories

$V = Set$: n-categories

$V = Fun(\Gamma^{op}, sSet)$: Sym mon (∞, n)-cat

$V = Ch$: enriched in chain complex: differential graded n-categories

$n = 1$ cats enriched in Ch; Fukaya; Floer

$V = sPSh(Cart)_{\check{C}ech}$: Smooth (∞, n)-categories.

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$V = sPSh(\underbrace{Cart}_{\text{encodes smooth families of bordisms}} \times \underbrace{Cart}_{\text{encodes isotopies}})_{\check{C}ech, \underbrace{shape}_{\simeq sSet_{KQ}}}$: our setup.

Generalizes to SCart, etc.

smooth symmetric monoidal (∞, n)-category

$V = sPSh(Cart * \Gamma \times Cart)$: our setup

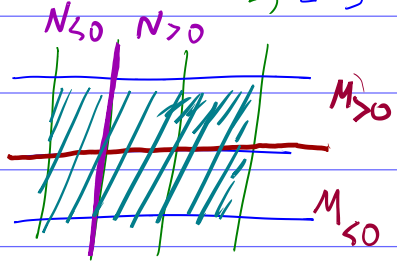
Lecture 20: March 24, 2024

Bordisms as a higher category

sym. mon. str. cat. $\quad$ sym mon $\quad$ (∞, d)-cat

$\Gamma \times \Delta^{\times d} \longrightarrow \mathrm{Cat} \xrightarrow{N} \mathrm{sSet}$ $\quad m = [3], [2]$

$\langle \ell \rangle, [m_1], \ldots, [m_d] \mapsto$



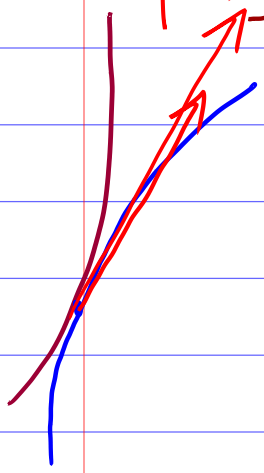
Ob: M : d-dimensional smooth manifold

$p: M \to \langle \ell \rangle$

C: cut m-grid on M:

$1 \le i \le d$: cut $[m_i]$-tuple C^i on M

nonexample:

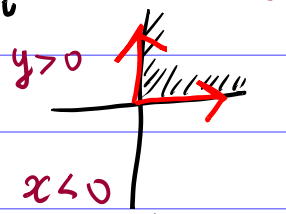


transversality: $\forall S \subset \{1, \ldots, d\}$

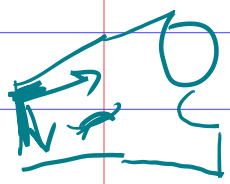
$\forall j_i : S \to \mathbb{Z} : \quad 0 \le j_i \le m_i$

$\exists h_j : M \to \mathbb{R}^S : \forall i \in S : \pi_i \circ h_j : M \to \mathbb{R}$ induces $C^i_{j_i}$

Also: regular values of h form an open neighborhood of $0 \in \mathbb{R}^S$.



→ Compactness: $C_{[j, j']} = \bigcap_{i \in S} C_{[j_i, j'_i]}$ is compact



Core $= C_{[0, m]}$

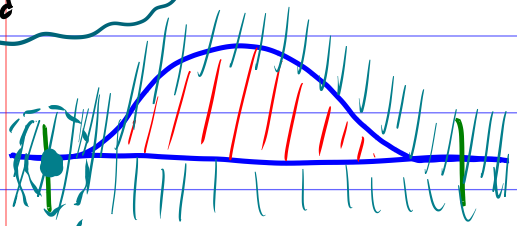


Globularity

Extracting a vertex in the ith direction $(1 \le i \le d)$ yields a simplicially degenerate bordism in directions $> i$

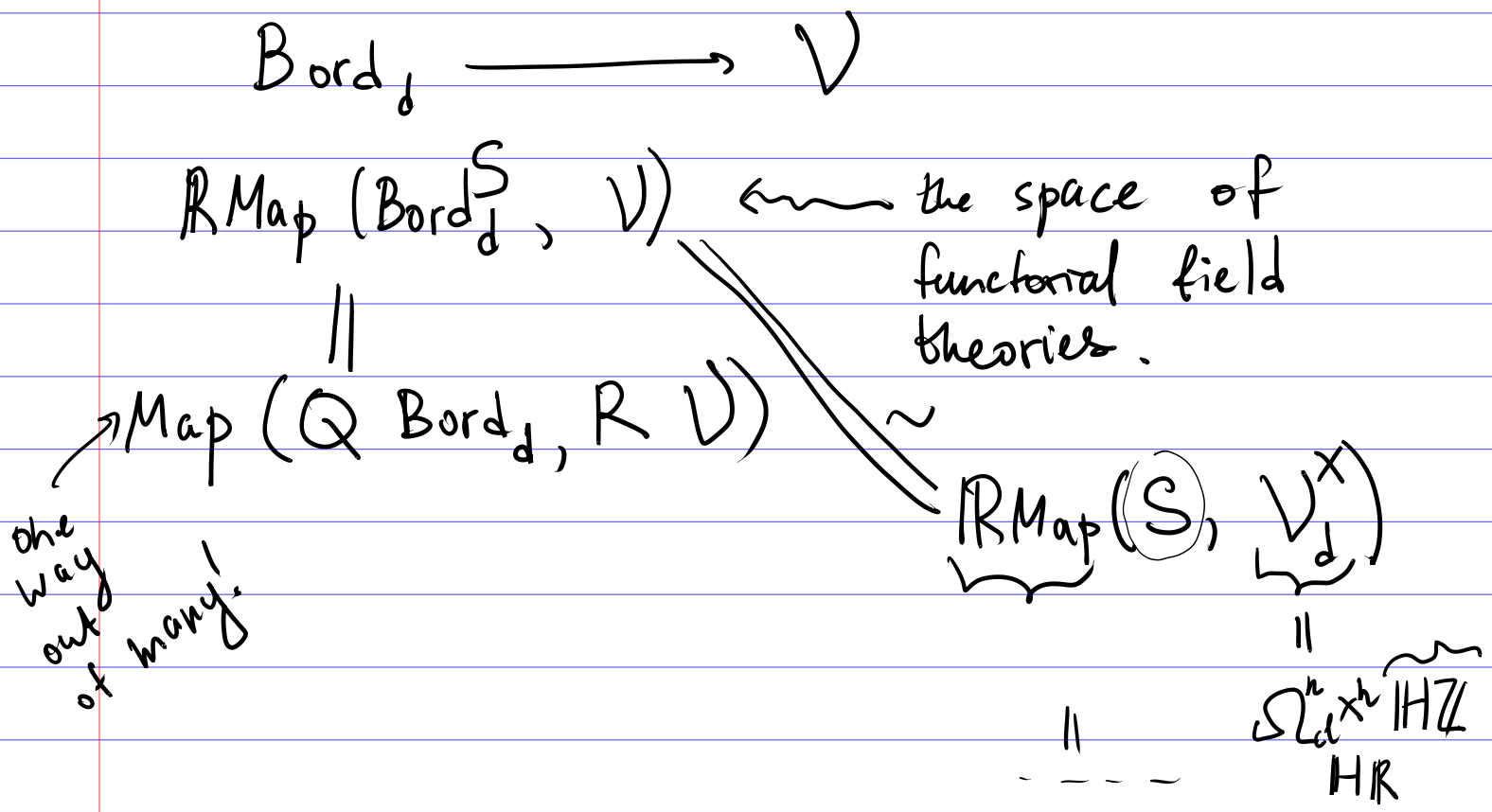
(after passing to a small open neighborhood of the core.)

Def An FFT is a (derived) map $\mathrm{Bord}_d \to \mathcal{V}$



$m = [1], [1]$

What's left: smoothness, isotopies, geometric structure


$\mathrm{Bord}_d \longrightarrow V$
$\mathbb{R}\mathrm{Map}(\mathrm{Bord}_d^{fr}, V)$
the space of functorial field theories.
$\simeq$
$\mathrm{Map}(Q\,\mathrm{Bord}_d, R\,V)$
one way out of many!
$\mathbb{R}\mathrm{Map}(S, V_d^{fr})$
$\Omega^n_{fr} X^h H\mathbb{Z}$
HR

Smooth category theory

Recall: The Čech model structure on simplicial presheaves.

start with $sPSh(Cart) = Fun(Cart^{op}, sSet)_{proj/inj}$

left Bousfield localization at $\check{C}\mathcal{U} \to yX$, $\{U_i \to X\}_{i \in I}$

More generally: The Čech model structure on left proper combinatorial model cat. $\mathcal{V}$-valued presheaves, $\mathcal{V}$:

$L_S Fun(Cart^{op}, \mathcal{V})$

$\mathcal{V} \times Set \to \mathcal{V}$

$A, T \mapsto \coprod_T A$

$S = A \otimes (\check{C}\mathcal{U} \to yX)$, $A \in \mathcal{V}$

Local objects: presheaves $\mathcal{F}$ that satisfy ∞-descent

$$\mathcal{F}(X) \xrightarrow{\sim} \operatorname{holim}_{j \in \Delta/I} \mathcal{F}(U_j)$$

Smooth category theory $\mathcal{V}$ = model cat of higher cat

~~SSS~~

Example $\mathcal{V} = Cat$ (canonical/natural/categorical model structure)

Subexamples 1) Smooth cat of vector spaces: $C^\infty Vect$ (target category)

$C^\infty Vect: Cart^{op} \to Cat$

$Cart \to S \mapsto C^\infty Vect$ category of $C^\infty(S)$-modules (Serre-Swan)

$(S \to T) \mapsto (M \mapsto M \otimes_{C^\infty T} C^\infty S)$ poorly behaved in ∞-dim cases

Alternative: vector bundles over $S \simeq$ f.g. proj. mod over $C^\infty S$

2) Smooth vector spaces

Concretely

$V(S)$: module over $C^\infty S$

Smooth abelian groups

$\mathcal{A} = Sh(Cart, Ab) \simeq Ab(Sh(Cart, Set))$ "$C^\infty(S, V)$"

$\mathbb{R}_{C^\infty} \in CMon(\mathcal{A})$ smooth commutative rings

$V \in Mod_{\mathbb{R}_{C^\infty}}$ smooth vector space

$\mathbb{R}_{C^\infty}(S) = C^\infty(S, \mathbb{R})$ $\quad V^* = Hom(V, \mathbb{R}_{C^\infty})$

V is complete and separated: $V \xrightarrow{\cong} V^{**}$ ($\Leftrightarrow$ reflexive)

Smooth category of smooth vector spaces

$S \mapsto C^\infty Vect(S)$: category

Cart

Ob: $Mod_{C^\infty(S,\mathbb{R})}$

Concretely:

s.v.s. V

$M \in \mathcal{A}$

$M \otimes M \xrightarrow{\mu} M$

$\mathbb{Z} \to M$

$(M \otimes M) \otimes M \cong M \otimes (M \otimes M)$

$\neq$

$M \otimes M \to M \leftarrow M \otimes M$

Mod_M

$X \in \mathcal{A}$

$M \otimes X \to X$

$\forall S \in Cart$

$V(S) \in Mod_{C^\infty S}$

$\forall f: S \to T$

$C^\infty T \to C^\infty S$

$V(T) \to V(S)$

Morally: $V(S) = C^\infty(S, V)$

Example $Z \in \mathrm{Man}$

$\underline{C^\infty}(Z, \mathbb{R}) \in \mathrm{SVS}$

$S \to T$

$S \in \mathrm{Cart}$

$\downarrow$

$C^\infty(S \times Z, \mathbb{R})$ $\qquad C^\infty(T \times Z, \mathbb{R}) \to C^\infty(S \times Z, \mathbb{R})$

Example $\mathrm{Hom}\left(\underline{C^\infty}(Z, \mathbb{R}), \mathbb{R}_{C^\infty}\right)$

$= \left(\underline{C^\infty}(Z, \mathbb{R})\right)^*$

$= \mathcal{D}^\infty_{cs}(Z, \mathbb{R})$

Compactly supported distributions on Z

$\forall S: C^\infty(S \times Z, \mathbb{R}) \longrightarrow \underline{C^\infty S}$

$S = pt: C^\infty(Z, \mathbb{R}) \xrightarrow{\psi} \mathbb{R}$ determines

ψ maps smooth families of functions to smooth families of numbers

Thm (? L. Schwartz ?)

$\psi \leftrightarrow$ distribution on Z,
i.e., a continuous
linear map
$C^\infty(Z, \mathbb{R}) \to \mathbb{R}$
(Whitney topology)

Thm $(\mathcal{D}^\infty_{cs}(Z, \mathbb{R}))^* \cong C^\infty(Z, \mathbb{R})$

Cor $C^\infty(Z, \mathbb{R})$ and $\mathcal{D}^\infty_{cs}(Z, \mathbb{R})$
are complete separated s.v.s.

2) Smooth limits and smooth colimits

Recall

$I \in \mathrm{Set}$

$\mathrm{Vect}^I \underset{(i \mapsto V) \leftarrow\!\shortmid V}{\overset{\oplus \text{ direct sum}}{\longleftarrow}} \mathrm{Vect}$

$\mathrm{Vect}^I \xrightarrow{\prod \text{ direct product}} \mathrm{Vect}$

unit: $\{V_i\}_i \to \{\bigoplus_j V_j\}_i$

$V_i \xrightarrow{\text{canonical inclusion } v_i} \bigoplus_j V_j$

counit: $\bigoplus_j W \to W$ (codiagonal)

unit: $W \xrightarrow{\Delta \text{ diagonal}} \prod_j W$

counit: $\{\prod_j V_j\}_i \to \{V_i\}_i$ (canonical projections)

Now

$I \in C^\infty\mathrm{Set}$

"

$\mathrm{Fun}(\mathrm{Cart}^{\mathrm{op}}, \mathrm{Set})$

e.g., $I \in \mathrm{Cart}$

$\mathcal{D}_{cs}$

$C^\infty\mathrm{Vect}_{cs}^I \underset{f^*V \leftarrow\!\shortmid V}{\overset{\mathcal{D}_{cs}}{\longleftarrow}} C^\infty\mathrm{Vect}_{cs}$

$V \mapsto \mathcal{D}_{cs}(I, V)$

$f: I \to pt$

$\Gamma_{C^\infty} = C^\infty(I, V)$

unit: $V \to \mathcal{D}_{cs}(I,V) \times I$

$v \mapsto \delta_{i,v}$ delta-function

counit: $\Psi \mapsto \Psi(1)$

$\mathcal{D}_{cs}(I, W) \to W$

$\Omega^{\text{top}}(I, W) \otimes \mathcal{O}_I$

$C^\infty\mathrm{Vect}_{cs}$

unit $W \xrightarrow{\text{const}} C^\infty(I, W)$

$I \in \mathrm{Cart}$

$C^\infty\mathrm{Vect}_{cs}(\widehat{S})$

$S \in \mathrm{Cart}$

counit $\Gamma_{C^\infty}\left(\begin{smallmatrix} V \\ \downarrow \\ I \end{smallmatrix}\right) \times I \xrightarrow{\text{eval}} V$ (over I)

classical action + topological sector $= C^\infty\mathrm{Vect}_{cs}(I \times S)$

$\Gamma_{C^\infty}(V) = \left(\exp\left(\frac{i}{\hbar}\tilde{S}(\dots)\right)\right)$

$\mathrm{Hom}(M, B\nabla U(1)) \in U(1)$

$\mathrm{Bord}_d^M \to$ prequantum vectors $\leftarrow$ d-mir $U(1)$

bra $\to$ vectors $\leftarrow$ ket

Lan $\to$ vectors $\leftarrow$ Feynman path $\int$

Bord_d

The bordism category as a smooth higher category

April 7: no lecture (eclipse)
April 14: next lecture

$d \geq 0$: $\mathrm{Bord}_d : \mathrm{Cart}^{op} \times \Gamma^{op} \times (\Delta^{\times d})^{op} \to sSet \xrightarrow{\mathcal{N}} Cat$

$\mathcal{U}$, $\langle \ell \rangle$, $[m_1], \dots, [m_\ell]$

Ob: $(M,\ P: M \to \langle \ell \rangle,\ C^1, \dots, C^d : \begin{array}{c} M \times \mathcal{U} \\ \downarrow \\ \mathcal{U} \end{array})$; $inf > core(M,P,C)$

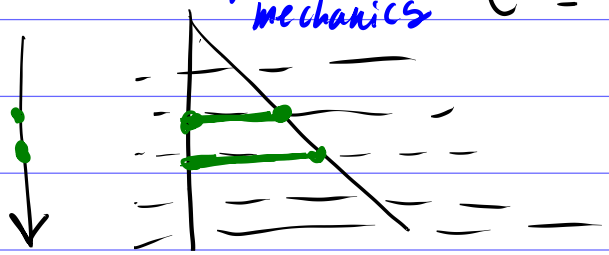
transversal, compact, globular } in smooth families over $\mathcal{U}$

Mor: $(M,P,C) \xrightarrow{f} (M',P',C')$: open emb $M \xrightarrow{f} M'$ compat w/ P, C $f^*P' = P$, $f^*C' = C$

Example $d = 1$: $\mathcal{U} = \mathbb{R}_{>0}$, $M = \mathbb{R}$ — time evolution op

quantum mechanics

$C^1 = (0\ ;\ t)$



Thus: $t \mapsto \mathcal{F}(I_{[0,t]})$ is smooth

$t \in \mathcal{U} = \mathbb{R} \to M$

$\Rightarrow \exists$ infinitesimal generator (Hamiltonian)

$M = \mathbb{R}$

$FFT_{d,V} = \underbrace{\mathbb{R}\mathrm{Hom}(\mathrm{Bord}_d,}_{\text{Smooth } \infty\text{-groupoid}} \mathcal{D})$

Geometric structures on bordisms

a) Without smooth families

Def A d-dimensional geometric structure is an ∞-sheaf $\mathrm{Emb}_d^{op} \xrightarrow{G} \mathrm{sSet}$

Def Emb_d: Site. Ob: smooth d-manifolds
Mor: open embeddings (good open covers)

Equivalent sites: 1) manifolds $\leftrightarrow$ cartesian spaces
2) open embeddings $\leftrightarrow$ local diffeomorphisms. (etale maps)

Examples a) Riemannian metrics
$M \longmapsto$ set of Riem metrics on M
$M \hookrightarrow N \longmapsto$ restriction map

b) other differential-geometric structures.
conformal, symplectic, contact, complex, Kähler, ...
curvature >0, ...

c) $M \longmapsto C^\infty(M, T)$
T: fixed (σ-model)

d) topological structures: orientation

f) higher gauge fields: B-field, C-field, RR field (∞-groupoid)

e) $M \longmapsto \{$groupoid of principal G-bundles w/ connection over M and ∇-preserving isos.$\}$
$\mathbb{B}_\nabla G$
(gauge fields)

b) With smooth families

Def $\mathbb{U} \in \mathrm{StCart} \to \mathrm{Cart}$, $\mathrm{SCart}^{p|q}$, $(C^\infty\text{-loci}_{\mathrm{Cart}})$, $\mathbb{R}^n \times \mathrm{Spec}\, W$

structured cartesian spaces

$\mathrm{DCart} : \mathrm{Sym}_{C^\infty}\left(\mathbb{R}^{n_k} \to \cdots \to \mathbb{R}^{n_0}\right)$

Def A d-dimensional geometric structure is an ∞-sheaf $\mathbb{F}\mathrm{Emb}_d^{op} \to \mathrm{sSet}$

Def $\mathbb{F}\mathrm{Emb}_d$: Site.

Ob: smooth d-manifolds

Mor: families of fiberwise open embeddings

$\mathbb{R}^d \times \mathbb{U} \to \mathbb{U}$

$M \to \mathbb{U}$ submersion w/ d-dim fibers

$M \xrightarrow{f} N$, $M \to \mathbb{U}$, $N \to V$, $\mathbb{U} \xrightarrow{g} V$, $M \xrightarrow{p} \mathbb{U}$, C^∞

$f : M_{\mathbb{U}} \hookrightarrow N_{g(\mathbb{U})}$ open embedding

$\left\{ M_i \xrightarrow{f_i} N,\ \mathbb{U}_i \xrightarrow{g_i} V \right\}$ is a covering family if f_i and g_i are open embeddings and $\{f_i : M_i \hookrightarrow N\}$ is a cov. family of N, $\bigcup_i f_i M_i = N$

Ex a) Fiberwise Riemannian metric (a metric on V)

$M \quad V \hookrightarrow TM \quad V$

$\downarrow p \quad \downarrow T_p \quad \downarrow$ vertical

$\mathbb{U} \quad T\mathbb{U} \quad M$ tangent bundle

c) $M \to \mathbb{U} \longmapsto \sigma\text{-model } C^\infty(M, T)$

d) fiberwise topological structures

e) fiberwise principal G-bundles with connection. on M $\neq$ prin G-bun W/M

Bordism categories with geometric structures

Input data

$d \geq 0$: $S : \mathrm{Emb}_d^{op} \to \mathrm{Set}$

$$\mathrm{Bord}_d^S : \mathrm{StCart}^{op} \times \Gamma^{op} \times (\Delta^{\times d})^{op} \to \mathrm{sSet} \xrightarrow{\text{diag}} \mathrm{sCat}$$

$$U, \quad \langle \ell \rangle, \quad [m_1], \dots, [m_j]$$

Ob: $(M, P : M \to \langle \ell \rangle, C^1, \dots, C^d : M \times_U U \to U)$

transversal, compact, globular } in smooth families over U

$$\coprod_{(M,P,C)} S\begin{pmatrix} M \times U \\ \downarrow \\ U \end{pmatrix} \quad \in \mathrm{Ob}\,\mathrm{Bord}_d(U, \langle \ell \rangle, [m])$$

Mor:

$$\coprod_{(M,P,C) \xrightarrow{f} (M',P',C')} S\begin{pmatrix} M' \times U \\ \downarrow \\ U \end{pmatrix} \quad \in \mathrm{Mor}\,\mathrm{Bord}_d$$

f^* sends $S\begin{pmatrix} M \times U \\ \downarrow \\ U \end{pmatrix}_{(M,P,C)}$ to $S\begin{pmatrix} M'' \times U \\ \downarrow \\ U \end{pmatrix}_{(M'',P'',C'')}$, with $(M'',P'',C'') = (M',P',C')$

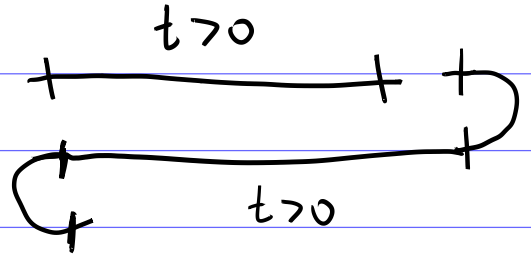
s, t: $\mathrm{Mor} \rightrightarrows \mathrm{Ob} = \coprod_{(M'',P'',C'')} S\begin{pmatrix} M'' \times U \\ \downarrow \\ U \end{pmatrix}$

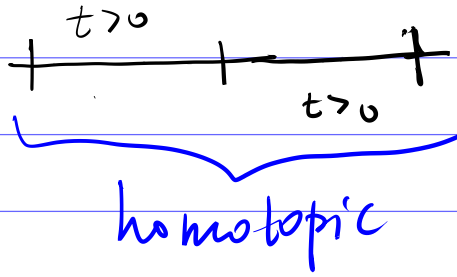
$$S = \mathbb{1} \qquad \mathbb{1}\begin{pmatrix} T \\ \downarrow \\ U \end{pmatrix} = \Delta^0$$

$$\mathrm{Ob} = \coprod_{(M,P,C)} S\begin{pmatrix} M \times U \\ \downarrow \\ U \end{pmatrix} = \coprod_{(M,P,C)} \Delta^0 = \{(M,P,C)\}$$

Isotopies of bordisms

Defects: dual and inverse bordisms do not work as expected





$\Rightarrow$ Want a cylinder of length $t>0$ to be in the same connected component as id = cylinder of length 0.

$\Rightarrow$ An isotopy of bordisms.

Isotopies of bordisms can move cuts around (smoothly).

$\rightsquigarrow$ $sSet \to PSh(Cart, sSet)$

Def $PSh(Cart, sSet)$ is equipped w/ transferred model structure. $PSh(Cart, sSet) \xrightarrow{C^\infty\text{-sing}} sSet_{KQ}$, $F \mapsto (n \mapsto F(\Delta^n)_n)$

$C^\infty sSet$

Weak equivalences = shape equivalences. ? Čech equival.

Def $\mathbb{B}\mathrm{ord}_d : \left(\mathrm{St}\,\mathrm{Cart} \times \Gamma \times \Delta^{\times d}\right)^{op} \longrightarrow C^\infty\mathrm{sSet}$

bold = Fraktur = enriched over $C^\infty\mathrm{sSet}$

controls smooth families (and concordances)

controls isotopies.

$$\mathbb{B}\mathrm{ord}_d\left(U, \langle b\rangle, [m_2], \dots, [m_d]\right)\left(L \xrightarrow{b} \mathrm{Cart}\right)$$

isotopies parametrized by L

$$= \mathbb{B}\mathrm{ord}_d\left(U \times L, \langle b\rangle, [m_1], \dots, [m_d]\right)$$

Ex $d = 1$

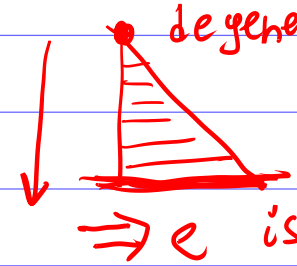
nondegenerate $\mapsto e : V \to V$, $e \circ e = e$

degenerate $= id \longmapsto id_V$

$V = \mathrm{Vect}$

$\longmapsto V$

$\rightsquigarrow$ with isotopies : $L = \mathbb{R}$



id, e are in a smooth family

Def $\mathbb{F}\mathrm{Emb}_d$: site enriched in $C^\infty\mathrm{Set}$

Ob: same as in $\mathbb{F}\mathrm{Emb}_d$

$$\mathbb{F}\mathrm{Emb}_d\begin{pmatrix} T & T' \\ \downarrow & \downarrow \\ U & U' \end{pmatrix}(L) = \mathbb{F}\mathrm{Emb}_d\begin{pmatrix} T\times L & \dashrightarrow & T' \\ \downarrow & & \downarrow \\ U\times L & \longrightarrow & U' \\ & \searrow^{\mathrm{proj}} & \uparrow \\ & & U \end{pmatrix}$$

($L \to \mathrm{Cart}$)

Def Enriched geometric structure: $\mathcal{S} : \mathbb{F}\mathrm{Emb}_d^{op} \to C^\infty\mathrm{sSet}$

∞-sheaf

($C^\infty\mathrm{Set}$-enriched functor)

Def Enriched bordism category w/ geo. struct. and isotopies

$d \geq 0$, $\mathcal{S} : \mathbb{F}\mathrm{Emb}_d^{op} \to C^\infty\mathrm{sSet}$

$$\mathbb{B}\mathrm{ord}_d^{\mathcal{S}} : \mathrm{St}\,\mathrm{Cart} \times \Gamma \times \Delta^{\times d} \longrightarrow C^\infty\mathrm{sSet} \hookrightarrow C^\infty\mathrm{sCat}$$

$$Ob(L) = \coprod_{(M,C,P) \in \mathbf{Bord}_d(U,\langle\ell\rangle,[m])} \mathbf{S}\left(\begin{array}{c} M \times U \\ \downarrow \\ U \end{array}\right)(L)$$

$\ell \in \mathrm{Cart}$

$$Mor(L) = \coprod_{(M,C,P) \to (M',C',P')} \mathbf{S}\left(\begin{array}{c} M' \times U \\ \downarrow \\ U \end{array}\right)(L)$$

"isotopies" (d-thin) of geometric structures

concordances of geometric structures

Example σ-model $X \in Man$ spacetime

$$\mathbf{S} : \mathbf{FEmb}_d^{op} \to C^\infty \mathrm{sSet}$$

$$T \downarrow U \longmapsto \left(L \in \mathrm{Cart} \longmapsto \begin{cases} T \times L \xrightarrow{f} X \\ \forall u \in U: \\ T_u \times L \xrightarrow{f_u} X \text{ factors through } W \end{cases} \right)$$

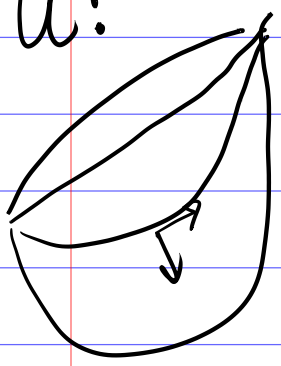
W a d-dimen. manifold

$\approx$ rank $f_u \leq d$

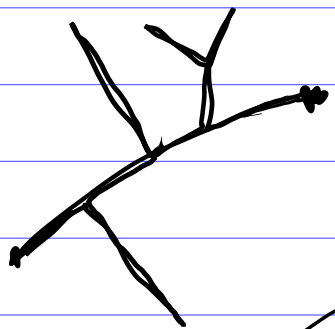
$=$ d-thin homotopy

$d = 1$

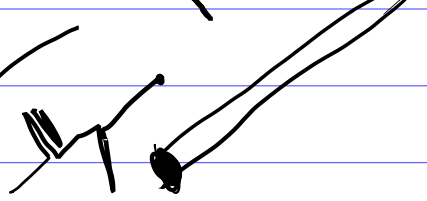
U:



L:



$\times$



smooth homotopy $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$X \times [0,1] \xrightarrow{h} Y$$

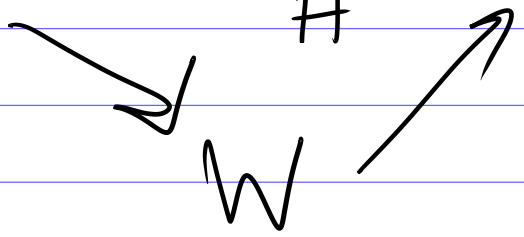
$$h(-,0) = f$$

$$h(-,1) = g$$

d-thin homotopy: 1) rank $h \leq d$

OR: 2) $X \times [0,1] \xrightarrow{h} Y$

2) $\Rightarrow$ 1)



$\dim W = d$.

Reference for 1-dim:

Smooth 1-dimensional topological field theories are vector bundles with conn

Berwick-Evans, P.

Ex

$p: [0,1] \longrightarrow X$

$h: [0,1] \times [0,1] \longrightarrow X$

$h(x,t) = p(x \cdot t)$

$h(x,1) = p$

$h(x,0) = p(0)$

Ex $p: [0,1] \to X$

$q: [0,2] \to X$ $\quad q(x) = \begin{cases} p(x), & x \le 1 \\ p(2-x), & x \ge 1 \end{cases}$

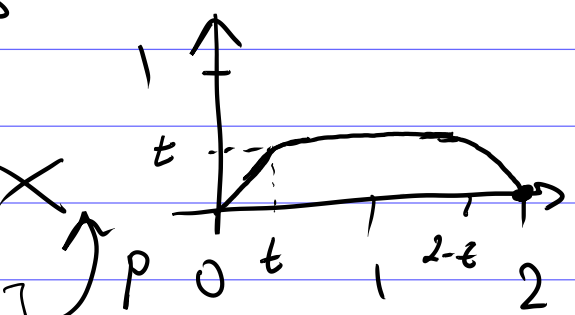
$q(0) = q(2)$

q is a loop.

$h: [0,2] \times [0,1] \to X$

$h(-,t): [0,2] \to X$

$h(-,1) = q$ $\quad h(-,0) = p(0)$ $\quad r: [0,2] \to [0,1]$



Def A k-morphism $f: A \to B$ in an (∞, d)-cat $\mathcal{C}$

($k=0$: sym. mon.) is dualizable

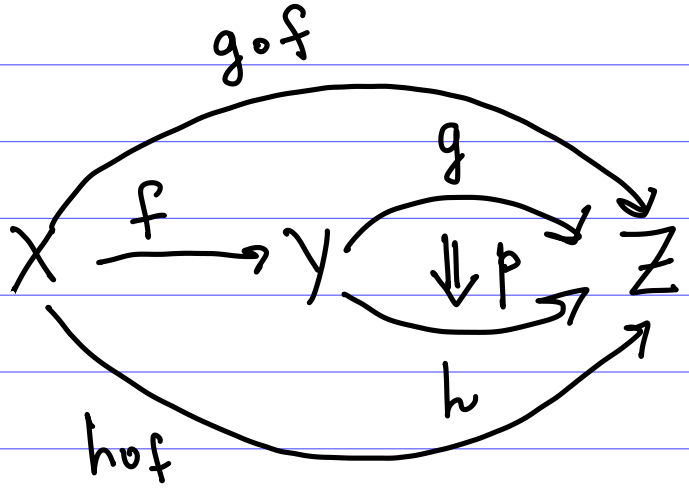
if $\exists\, g: B \to A$ (the dual morphism)

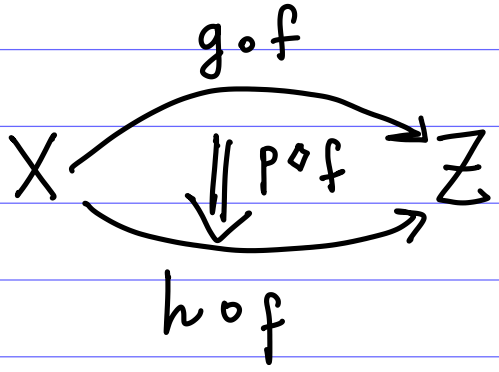
$\exists\, \underset{\text{(unit)}}{\eta}: id_A \to g \circ f$, $\underset{\text{(counit)}}{\varepsilon}: f \circ g \to id_B$

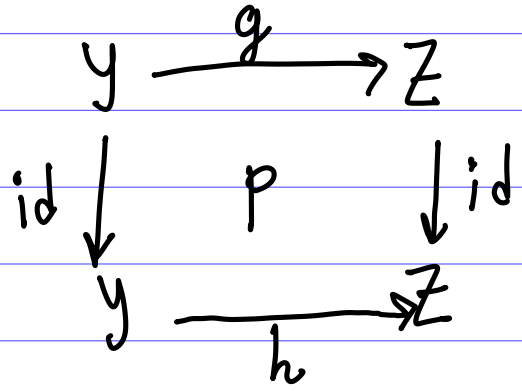
such that the triangle identities are satisfied:

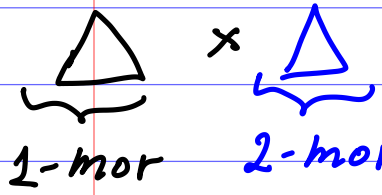
$$f \simeq f \circ id_A \xrightarrow{f \circ \eta} f \circ g \circ f \xrightarrow{\varepsilon \circ f} id_B \circ f \simeq f$$

$$id_f$$

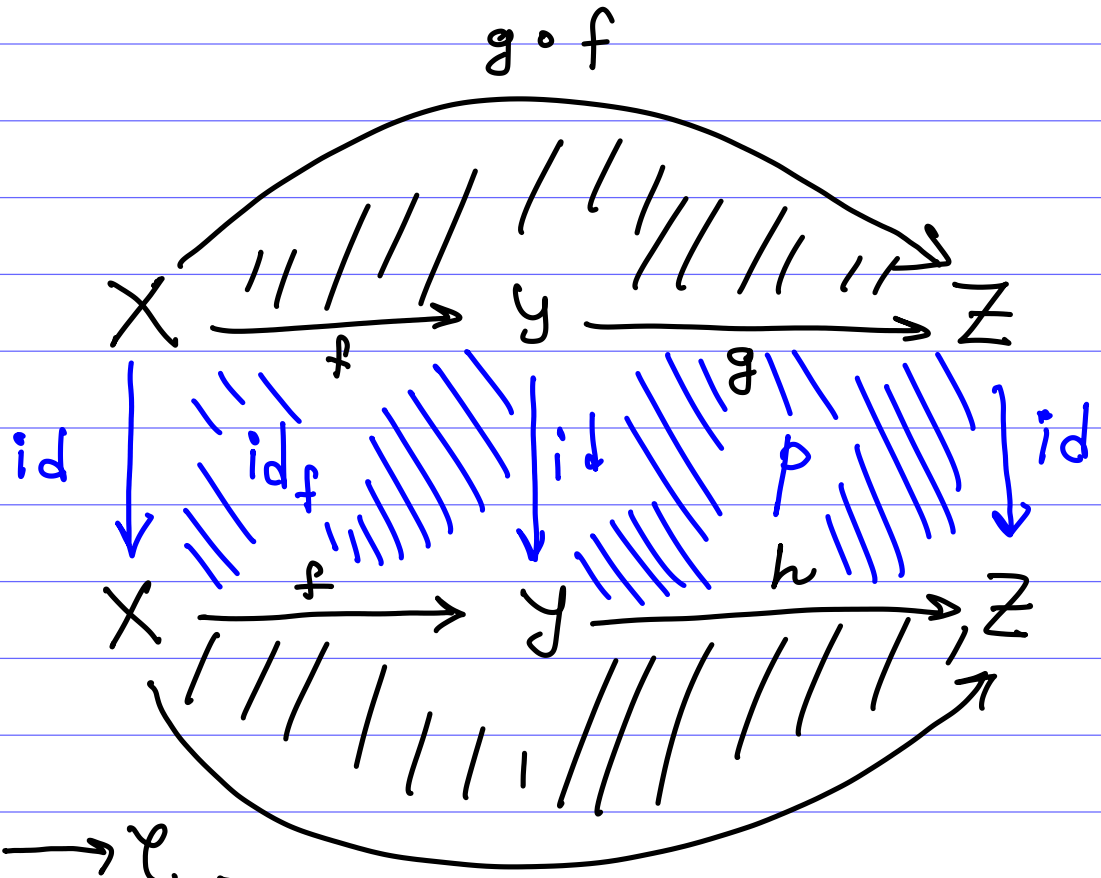






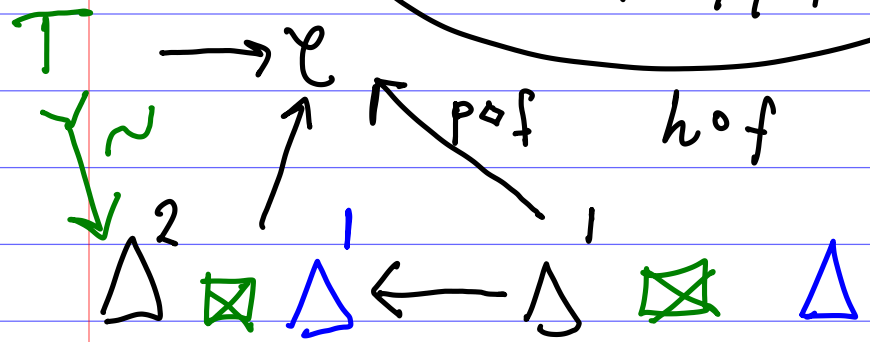


$(\Delta \times \Delta)^{op} \to sSet$



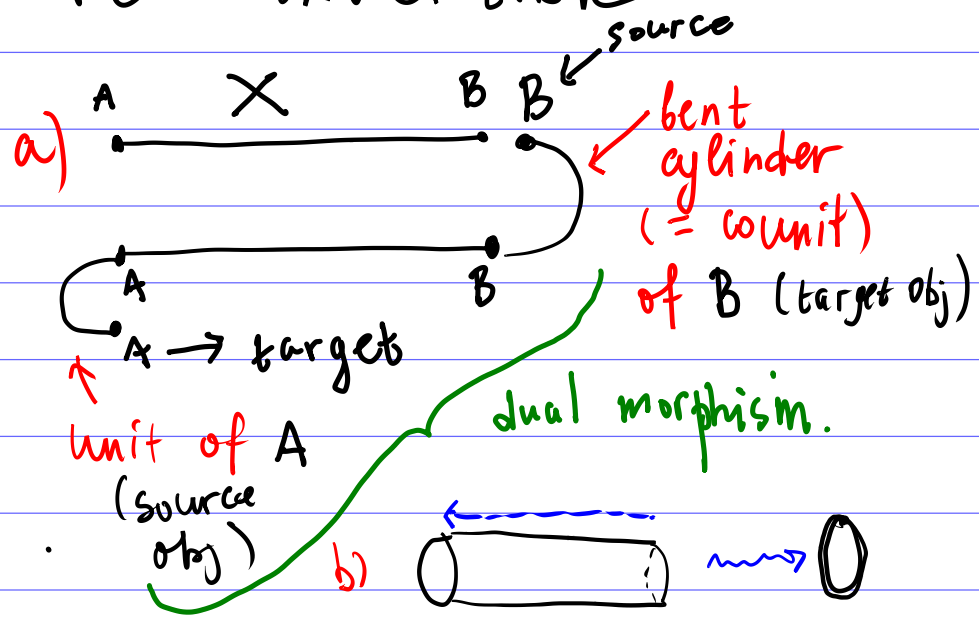
(globularity)

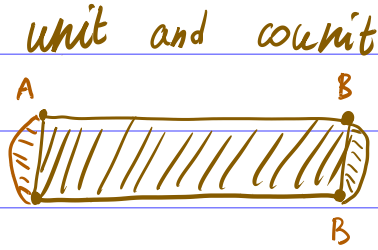
"tent"



Proposition a) All k-morphisms $(0 \le k < d)$ in Bord_d^S are dualizable. ↘ ≤?

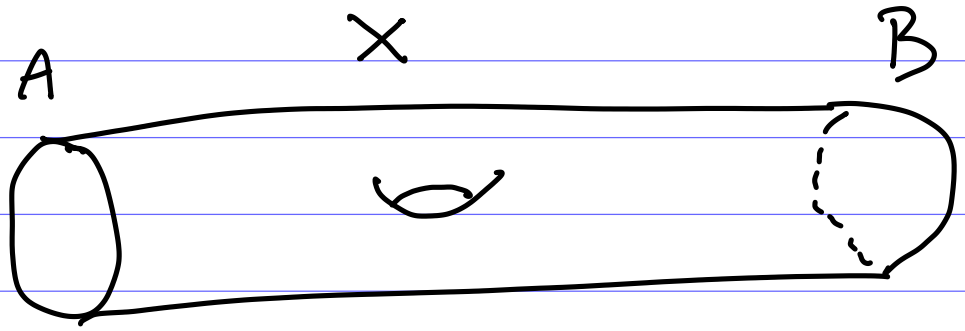
b) All cylinders on k-morphisms $(0 \le k < d)$ are invertible

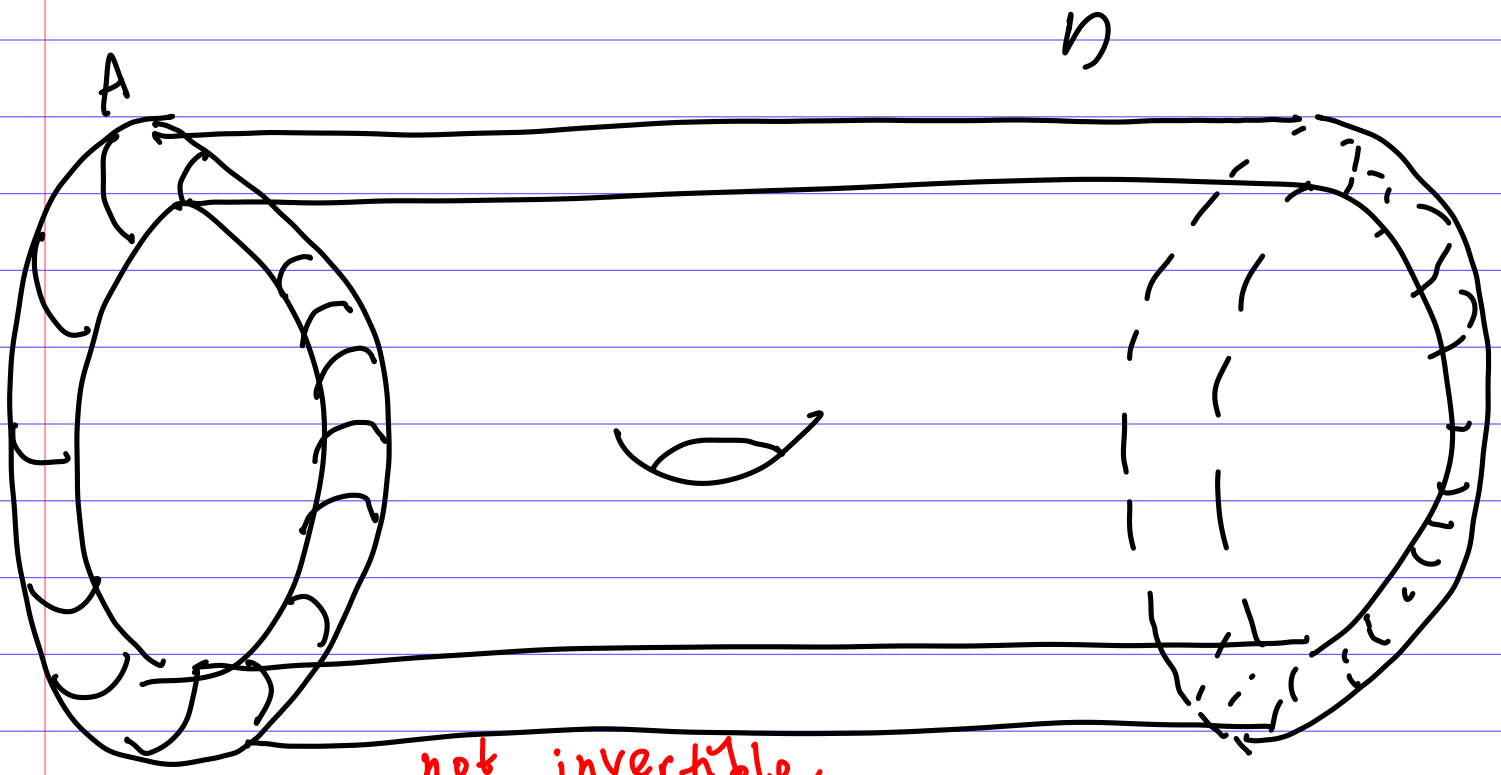




Def A d-dimensional functorial field theory with geometric structure S and a (fibrant) category of values V is a morphism $\mathrm{Bord}_d^S \longrightarrow V$

Prop Any FFT factors as $\mathrm{Bord}_d^S \longrightarrow V^{f.d.}$ ↖ fully dualizable

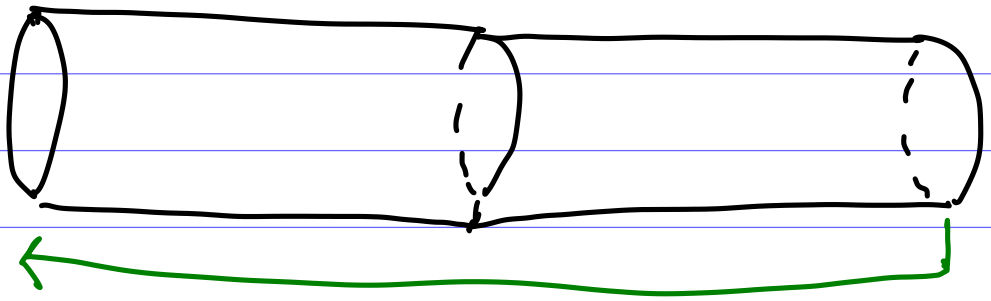




not invertible

η fills in the space between the two cylinders.

b) A cylinder on a $(d-1)$-bordism is invertible.
(also works for h-cobordisms)



R-isotopy

$R \in \mathrm{CRing}$

Example Dualizable objects in $(\mathrm{Mod}_R, \otimes_R, R)$

nLab: dualizable module $\equiv$ finitely generated projective modules

$\equiv$ direct summand of R^n

$\equiv$ retracts of R^n

$M^* = \mathrm{Hom}_R(M, R)$

M^*: Bra-vectors; M: Ket-vectors

counit map: $M^* \otimes_R M = \mathrm{Hom}_R(M,R) \otimes_R M \xrightarrow{ev} R$, $f, m \mapsto f(m)$ $\quad \langle -|- \rangle$

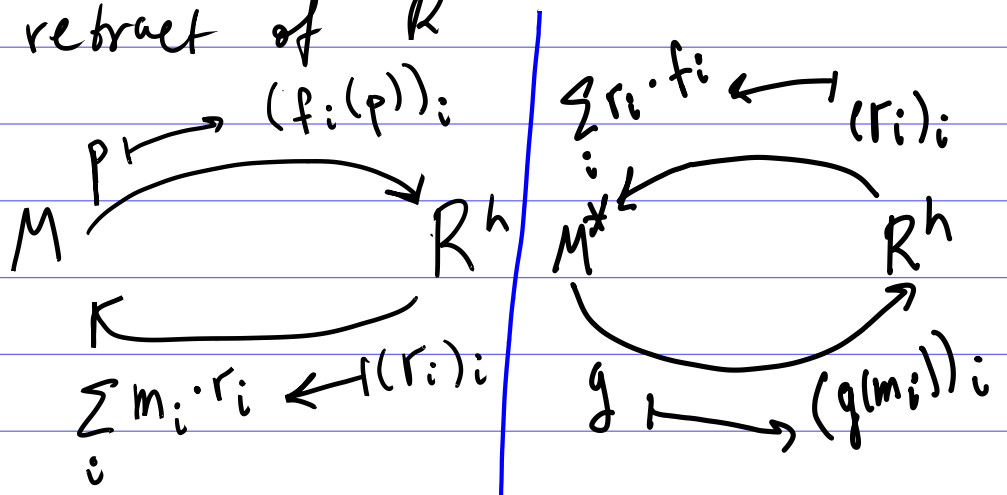
unit map $R \longrightarrow M \otimes_R \mathrm{Hom}_R(M,R)$

$1 \longmapsto \sum_i m_i \otimes f_i$

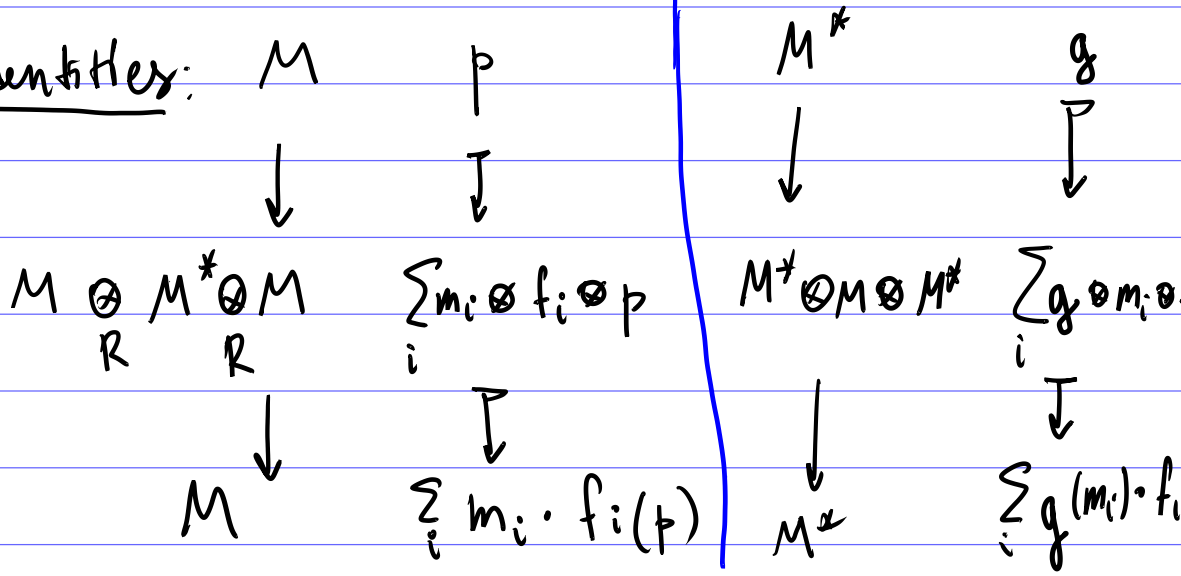
$1 \mapsto \sum_i e_i^* \otimes e_i$

$\|e_i^* \otimes e_i\| = 1$

Recall: M is a retract of R^h



Triangle identities:



d = 2: Separable algebras

Alg_k, Bimod, Int : what are the fully dualizable obj.?

Prop a) $A \in Alg_k$ is dualizable triangle: exercise

$A^* = A^{op}$

unit: $A^* \otimes_k A \to k$ (right mult, left mult) counit: $k \to A \otimes A^*$ (right, left)

b) $A \in Alg_k$ is fully dualizable $\Leftrightarrow$ A is separable

A is dualizable as a k-module

A is separable: $\exists\, e = \sum_i x_i \otimes_k y_i$: A-central and

$e \in A \otimes A$ $\quad \sum_i x_i \cdot y_i = 1_A$ —

$\forall a \in A$: $a \cdot e = e \cdot a$

Galois theory

K/k sep $\Leftrightarrow$ ${}_k K$ sep

$A \otimes A \xrightarrow{\mu} A$

Ex $A = End(k^n) = M_n(k)$

$e = \sum_i e_{i,i} \otimes e_{i,i}$

$\mu(e) = \sum_i e_{i,i} = \mathbb{1}$

k field: A sep $\Leftrightarrow$ $A \cong \bigoplus_i End(k^{n_i})$.

Ex Clifford algebras are separable.

Recall • $d \geq 0$: dimension

• FEmb_d: site; Ob: $T \downarrow U$ submfd; Mor: $\begin{array}{ccc} T_1 & \xrightarrow{\text{fib.}} & T_2 \\ \downarrow & & \downarrow \\ U_1 & \longrightarrow & U_2 \end{array}$

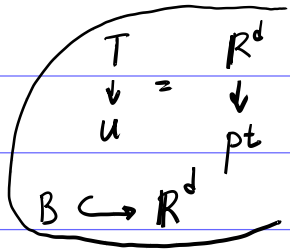
covering fam: on T_i

enriched over $C^\infty\mathrm{Set} = \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{Set})_{\text{shape}}$

• $GS_d = \mathrm{Fun}_{enr}(\mathrm{FEmb}_d^{op}, C^\infty\mathrm{sSet})_{\check{C}ech}$ geo. str.

• $C^\infty\mathrm{Cat}^{\otimes}_{\infty,d} = \mathrm{Fun}(\mathrm{Cart}^{op} \times \Gamma^{op} \times (\Delta^{\times d})^{op}, C^\infty\mathrm{sSet})_{\check{C}ech, Segal, CSS, glob}$

Theorem 1 (Locality; GCH Part I)



• The functor

$$\mathrm{Bord}_d : GS_d \longrightarrow C^\infty\mathrm{Cat}^{\otimes}_{\infty,d}$$

is a left Quillen functor.

• In particular, Bord_d preserves homotopy colimits. (Bord_d is an ∞-cosheaf)

• The right adjoint $(-)_d : C^\infty\mathrm{Cat}^{\otimes}_{\infty,d} \longrightarrow GS_d$

$$V_d\left(\begin{array}{c} T \\ \downarrow \\ U \end{array}\right) = \mathrm{map}_{\infty\mathrm{Set}}\left(\mathrm{Bord}_d^{y(T\to U)}, V\right)$$

• In particular, $\mathrm{FFT}^{S}_{d,V} = \mathbb{R}\mathrm{map}(\mathrm{Bord}_d^S, V)$ } difficult to compute

V is fibrant $= s\mathrm{Set}(\mathrm{Bord}_d^S, V) \cong \underbrace{s\mathrm{Set}}_{\text{mapping simplicial set in } GS_d}(S, V_d)$ } easier to compute

How to compute V_d?

Theorem 2 (GCH, Part II)

T ↓ U

- $d \geq 0$, $\mathcal{V} \in C^\infty \mathrm{Cat}^{\otimes}_{\infty,d}$ fully dualizable

$$\mathcal{V}_d\left(\begin{array}{c}\mathbb{R}^d \times U \\ \downarrow \\ U\end{array}\right) \xrightarrow[\text{evaluate at points}]{\simeq} \mathcal{V}(U)^{\text{f.d.}, \times}$$

discard noninvertible k-morphisms, $0 < k \leq d$ (*)

$$\| \qquad \text{evaluate at } [0]^d \in \Delta^{\times d}$$

$$\mathcal{V}^{\text{f.d.}}(U, \langle 1 \rangle, [0]^d)$$

Ex $W\left(\begin{array}{c}\mathbb{R}^d \times U \\ \downarrow \\ U\end{array}\right) = \Omega^{\leq d}_{fw}\left(\begin{array}{c}\mathbb{R}^d \times U \\ \downarrow \\ U\end{array}\right)$; $\mathcal{V} = B^d U(1)$; $W \to \mathcal{V}$, $\omega \in \Omega^d_{fw}$, $B \mapsto \exp\left(\int_B \omega\right)$

$\exp\left(\frac{i}{\hbar} S(\varphi)\right) \in U(1)$ prequantum action functional

- Practical recipe to compute $W \simeq \mathcal{V}_d$ as a presheaf of some forms

1) Guess W based on (*)

2) Guess a map $W \xrightarrow{\text{e.g., integrate some forms}} \mathcal{V}_d$ ← the only place where field theories appear

3) Prove: for any $U \in \mathrm{Cart}$: $W\left(\begin{array}{c}\mathbb{R}^d \times U \\ \downarrow \\ U\end{array}\right) \to \mathcal{V}_d\left(\begin{array}{c}\mathbb{R}^d \times U \\ \downarrow \\ U\end{array}\right)$

easy in practice

does not talk about functorial field theories!

$$\begin{array}{ccc} W\left(\begin{array}{c}\mathbb{R}^d \times U \\ \downarrow \\ U\end{array}\right) & \to & \mathcal{V}_d\left(\begin{array}{c}\mathbb{R}^d \times U \\ \downarrow \\ U\end{array}\right) \\ & \searrow^{\sim} & \# \downarrow \mathrm{ev} \\ & & \mathcal{V}(U)^{\text{f.d.}, \times} \end{array}$$

How to compute $\mathrm{sSet}(S, W)$? ← can also be computed directly

The evaluation map

$$\mathrm{map}\left(\mathbb{B}\mathrm{ord}_d^{\mathbb{R}^d \times U \downarrow U}, V\right) \xrightarrow{ev} \mathrm{map}\left(\mathbb{B}\mathrm{ord}_d^{\mathbb{R}^d \times U \downarrow U}(U, \langle 1 \rangle, [0]^d), V(U, \langle 1 \rangle, [0]^d)\right)$$

$(U, \langle 1 \rangle, [0]^d)$

U-indexed families of objects

$V(U, \langle 1 \rangle, [0]^d)$: ∞-groupoid of U-indexed families of objects and invertible k-morphisms, $k>0$ (requires V to be fibrant, more precisely, complete)

evaluate at 0

$\langle m \rangle \in \Gamma$ ⇝ m-tuples of multipliable elements

$\langle 1 \rangle \in \Gamma$ ⇝ single bordisms

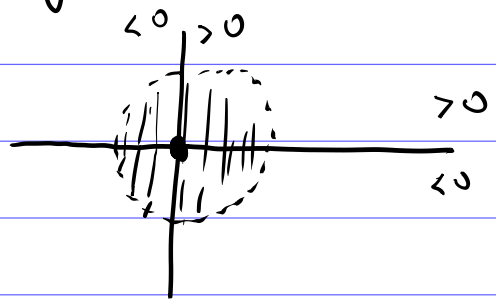
$$0 \in \mathbb{B}\mathrm{ord}_d^{\mathbb{R}^d \times U \downarrow U}(U, \langle 1 \rangle, [0]^d)$$

$V(U, \langle 1 \rangle, [0]^d)$

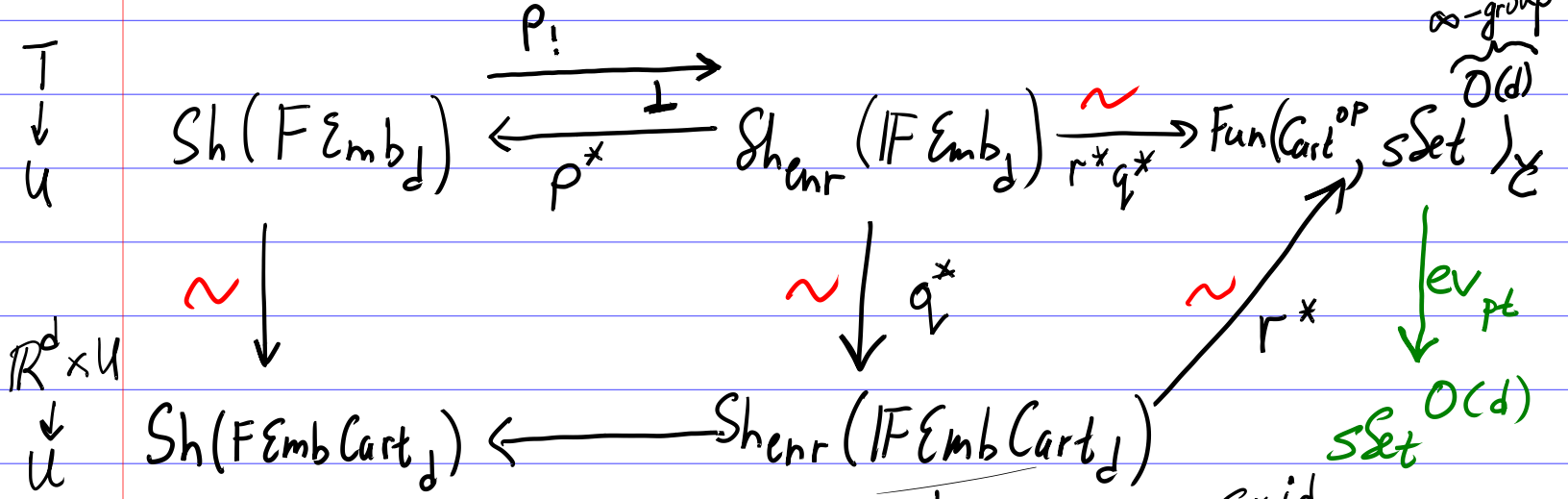
M = ambient manifold $= \mathbb{R}^d$

C: U-indexed family of cut grids in $\mathbb{R}^d$

P: $M \to \langle 1 \rangle$, $M \mapsto 1^M$ $\qquad d = 2$



Proposition $Sh_{(an)}(\mathcal{C}) := Fun_{(enr)}(\mathcal{C}^{op}, (C^\infty/_s Set)_{\check{C}ech})$



Proof $O(d) \xhookrightarrow{\sim} End\left(\begin{smallmatrix}\mathbb{R}^d \times U \\ \downarrow \\ U\end{smallmatrix}\right)$ (enr)

$G : \mathbb{R}^d \to \mathbb{R}^d \mapsto$ (G x id)

$\mathbb{R}^d \times U \to \mathbb{R}^d \times U$

$U \xrightarrow{id} U$

is a weak equivalence by the smooth analogue of the Kister-Mazur theorem. ∎

Typical calculation:

1) Start with $S \in Sh(FEmb_d)$.

2) Compute $\mathcal{S} := r^* q^* p_! S$

3) Now $\mathbb{R}map(S, W)$

$\simeq \mathbb{R}map(\mathcal{S}, r^* q^* W)$.

Example 1

1-dimensional oriented Riemannian field theories

(quantum mechanics!)

$$S\begin{pmatrix}\mathbb{R}\times U\\ \downarrow\\ U\end{pmatrix} = \left\{\text{fiberwise Riem. metr. on } \begin{pmatrix}\mathbb{R}\times U\\ \downarrow\\ U\end{pmatrix}\right\}$$

Want to compute: $S = \overbrace{r^* q^*}^{\text{equiv}} \underbrace{p_!}_{\text{cocont}} S = C_1 S$

1) $S = \mathrm{Riem}_1 \xleftarrow{\sim} \mathrm{Riem}_1^{\text{finite length}} = R_1^f$

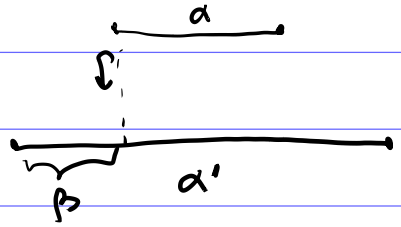
2) $S = C_1 \mathrm{Riem}_1 = C_1 \operatorname{hocolim} y\begin{pmatrix}\mathbb{R}\times U\\ \downarrow\\ U\end{pmatrix}$

$\underbrace{\begin{pmatrix}\mathbb{R}\times U\\ \downarrow\\ U\end{pmatrix} \to R_1^f}$

equivalent to:

Ob: $(U, \alpha: U \to (0,\infty))$

Mor: $(h: U \to U',\ \beta: U \to [0,\infty))$



$$= C_1 \operatorname*{hocolim}_{\alpha;\beta} y\begin{pmatrix}\mathbb{R}\times U\\ \downarrow\\ U\end{pmatrix}$$

$$= \operatorname*{hocolim}_{\alpha;\beta} C_1\, y\begin{pmatrix}\mathbb{R}\times U\\ \downarrow\\ U\end{pmatrix}$$

$$= \operatorname*{hocolim}_{\alpha;\beta} \mathbb{Z}/2 \times y(U) = \mathbb{Z}/2 \times \operatorname*{hocolim}_{\alpha;\beta} y(U)$$

3) Compute hocolim as the Grothendieck construction

$$W \in_{Cart} \xmapsto{E} \begin{cases} \text{Ob: } A: W \to (0,\infty) \\ \text{Mor}(A,A'): \quad B: W \to [0,\infty) \\ \qquad\qquad A+B \leq A' \end{cases}$$

4) $F: E_W \to B(C^\infty(W,\mathbb{R}))$

$A \longmapsto *$

$B \longmapsto B$

Quillen's Theorem A: F is a final functor $\Rightarrow$ NF is a weak eq.

$$C_1 R_1 \simeq \mathbb{Z}/2 \times B\mathbb{R}$$

Corollary 1-dimensional oriented Riemannian FFTs

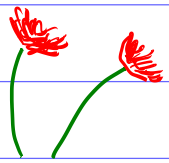
$\text{Bord}_1^{\text{Riem}} \to V$

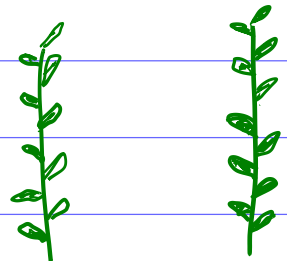
are $B\mathbb{R} \to V^X$ (morphisms in $Sh(Cart)$).

May 12: no lecture

May 19: possibly lecture (?)

Harvest time!





$\iota_!\, y\begin{pmatrix} T \\ \downarrow \\ U \end{pmatrix} = y\begin{pmatrix} T \\ \downarrow \\ U \end{pmatrix}$ $\quad S = \iota_! S'$

Recall: $FFT_{d,S,V} = \mathbb{R}map(Bord_d^S, V)$

$\iota_!$

$Sh(FEmb_d) \rightleftarrows Sh(FEmb_d)$ $\quad \iota^*$

$\simeq \mathbb{R}map(S, V_d^\times) \times$ (sheaves on $FEmb_d$)

$\times \simeq \mathbb{R}map(S', \iota^* V_d^\times)$

Recipe to compute $V_d^\times$: $\quad V_d^\times\begin{pmatrix} T \\ \downarrow \\ U \end{pmatrix} = FFT_{d, y\left(\begin{smallmatrix} T \\ \downarrow \\ U \end{smallmatrix}\right), V}$

1) Guess $W \to V_d^\times$

2) Check

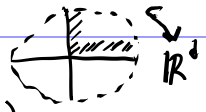
w.e. of sSet

$$W\begin{pmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{pmatrix} \xrightarrow{\sim} V^{f.d.,\times}(U)$$

$$W\begin{pmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{pmatrix} \to V_d^\times\begin{pmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{pmatrix} \xrightarrow{\sim} V^{f.d.,\times}(U) \quad \text{(ev)}$$

connected 0-bordisms

$d = 2$



$map(S, W)$

Recipe to compute $\mathbb{R}map(S, V_!^\times) \simeq \mathbb{R}map(S, W)$

or: to construct points in ↑ (i.e., field theories)

1) (Optional) Pick a resolution $W \to W'$, replace W by W'

2a) Compute natural operations $S \to W$ $\quad Emb(\mathbb{R}^d) \simeq GL(d)$

or:

2b) Apply $Sh(FEmb_d) \xrightarrow{\sim} Sh(Cart, sSet^{O(d)})$

and compute $O(d)$-equivariant maps of stacks over Cart

Example $V = \mathbb{B}^d \mathcal{U}(1)$

single k-morphism for $0 \le k < d$; d-morphisms $= \mathcal{U}(1)$

(the target for prequantum field theories)

$\exp(\frac{i}{\hbar} S(\varphi)) \in \mathcal{U}(1)$

$A = \mathcal{U}(1)$

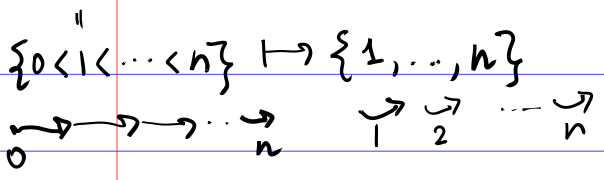
Recall: $\Gamma = \mathrm{FinSet}_*^{op} \simeq \mathrm{FinSetPart}^{op}$

A: abelian Lie group

$\Delta \to \Gamma$

$[n] \mapsto \langle n \rangle$

$\{0 < 1 < \cdots < n\} \mapsto \{1, \dots, n\}$



$\Gamma \times \Delta^{\times d} \xrightarrow{\mathbb{B}^d \mathbb{Z}} \mathrm{sSet}$, with $\mathbb{Z} : \Gamma \to \mathrm{sSet}$

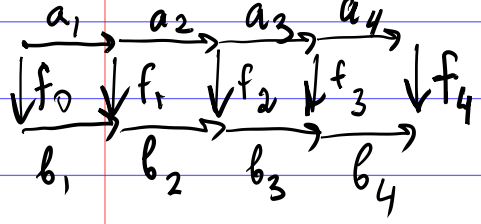
$\Gamma \times \Gamma^{\times d} = \Gamma^{\times(d+1)} \xrightarrow{\otimes} \Gamma$

$\langle 1 \rangle, [0], \dots [0] \mapsto \langle 0 \rangle$

naive (nonfibrant) delooping: $\ell \times m_1 \times \cdots \times m_d$

$\langle \ell \rangle, [m_1], \dots, [m_d] \longmapsto \mathcal{U}(1)$

$d = 1$: $[m_1] \mapsto$ Rezk



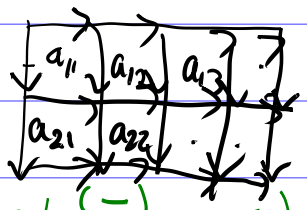
- Enforce globularity
- Enforce completeness:

$U, \langle \ell \rangle, [m_1], \dots, [m_d] \mapsto C^{d-*}\left(\coprod_i [m_i], C^\infty(U, A) \right)^{\times \ell}$

Want to define $W \longrightarrow V_d^X$

$\simeq \mathbb{R}\mathrm{map}(\mathbb{B}\mathrm{ord}_d^{(-)}, V)$

$d = 2$



$$\mathbb{B}^d C^\infty_{\mathrm{fconst}}\left(\begin{matrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{matrix}, A \right) \longrightarrow \left[\mathbb{B}\mathrm{ord}_d^{\mathbb{R}^d \times U}, \mathbb{B}^d A \right]$$

$$\mathbb{B}^d C^\infty(U, A) \otimes \mathbb{B}\mathrm{ord}_d^{\mathbb{R}^d \times U} \longrightarrow \mathbb{B}^d A$$

$$\mathbb{B}^d C^\infty(U, A) \times \mathbb{B}\mathrm{ord}_d(V, \langle \ell \rangle, [m_1], \dots [m_d]) \to \mathbb{B}^d A(V, \langle \ell \rangle, [m_1], \dots)$$

on objects: $\mathbb{B}^d C^\infty(U, A) \times \mathbb{B}\mathrm{ord}_d(V, \langle 1 \rangle, [0], \dots, [0]) \to \mathbb{B}^d C^\infty(V, A)$

$W : \mathbb{F}\mathrm{Emb}_d^{op} \to \mathrm{sSet}$

$$W\left(\begin{matrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{matrix} \right) \simeq V^X(U) \simeq \mathbb{B}^d C^\infty(U, \mathcal{U}(1))$$

$$\begin{matrix} T \\ \downarrow \\ U \end{matrix} \longmapsto \mathbb{B}^d C^\infty_{\mathrm{fconst}}(T, \mathcal{U}(1))$$

Another computation of V_d^X $\mathfrak{a}$ = Lie algebra of A

$A = U(1)$: $\mathfrak{a} = iR$

W = injective resolution of $\mathbb{B}^d A$:

$$\begin{array}{ccccccccc} & & 0 & & 1 & & d-1 & & d \\ \begin{smallmatrix} T \\ \downarrow \\ u \end{smallmatrix} \longmapsto \Gamma\mathcal{U}\Big(& \Omega^d_u\left(\begin{smallmatrix} T \\ \downarrow \\ u \end{smallmatrix}, \mathfrak{a}\right) & \leftarrow & \Omega^{d-1} & \leftarrow \cdots \leftarrow & \Omega^1_u\left(\begin{smallmatrix} T \\ \downarrow \\ u \end{smallmatrix}, \mathfrak{a}\right) & \leftarrow & C^\infty(T, A) \Big) \end{array}$$

always closed because $\dim_u\left(\begin{smallmatrix} T \\ \downarrow \\ u \end{smallmatrix}\right) = d$

Deligne complex

dlog

$Ch_{\geq 0} \xrightarrow{\Gamma} sAb \xrightarrow{\mathcal{U}} sSet$

$\exp: \mathfrak{a} \to A$

$C^\infty_{\text{f const}}(T, A)[d] \xrightarrow{\sim} \Omega^{\leq d}_u\left(\begin{smallmatrix} T \\ \downarrow \\ u \end{smallmatrix}, A\right)$

Poincaré lemma

$$W\left(\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}\right) \times \mathrm{Bord}_d(V, \langle 1\rangle, [1], \ldots, [1]) \longrightarrow \underset{\text{Ob: } C^\infty(V,A)}{\mathbb{B}^d A(V)}$$

$$\omega \in \Omega^d_u\left(\begin{smallmatrix} T \\ \downarrow \\ u \end{smallmatrix}\right), \quad B: \boxed{} \hookrightarrow \begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix} \longmapsto \exp\left(\int_B \omega\right).$$

$$W \longrightarrow V_d^X = \mathbb{R}\mathrm{map}\left(\mathrm{Bord}_d^{(-)}, V\right)$$

$$\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}: \quad W\left(\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}\right) \to \mathbb{R}\mathrm{map}\left(\mathrm{Bord}_d^{\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}}, V\right)$$

$$W\left(\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}\right) \times \mathrm{Bord}_d^{\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}} \longrightarrow V$$

$W\left(\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}\right)$: sSet; $\mathrm{Bord}_d^{\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}}$: $\mathbb{R}^d \times u \to u$

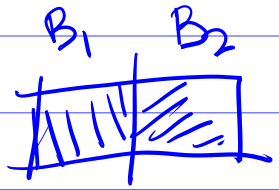
$$W\left(\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}\right) \times \mathrm{Bord}_d^{\begin{smallmatrix} \mathbb{R}^d \times u \\ \downarrow \\ u \end{smallmatrix}}\left(V, \langle 1\rangle, [1], [1], \ldots, [1]\right) \to \underbrace{V(V, \langle 1\rangle, [1])}_{\text{V-ind fam of d-morph in } \mathbb{B}^d(u(1))}$$

$$\mathcal{U}\left(\Omega^d_U\begin{pmatrix}\mathbb{R}^d\times Y\\ \downarrow\\ U\end{pmatrix}\leftarrow\cdots\right)\times \{V\to U\} \longrightarrow \left(C^\infty(V,U(1)),\ \text{isos}\right)$$

Ob:
$$\underbrace{\Omega^d_U\begin{pmatrix}\mathbb{R}^d\times U\\ \downarrow\\ U\end{pmatrix}}_{\omega}\times\left\{V\xrightarrow{f}U\supset B\right\}\longrightarrow C^\infty(V,U(1))$$

$$\longmapsto \exp\left(\int_B f^*\omega\right)$$

$$f^*\omega\in\Omega^d_V\begin{pmatrix}\mathbb{R}^d\times V\\ \downarrow\\ V\end{pmatrix}$$



$$\int_B f^*\omega\in C^\infty(V,i\mathbb{R})$$

$$\exp\left(\int_{B_1\cup B_2} f^*\omega\right)=\exp\left(\int_{B_1} f^*\omega+\int_{B_2} f^*\omega\right)$$

$$\exp\left(\int_B f^*\omega\right)\in C^\infty(V,U(1))=\exp\left(\int_{B_1} f^*\omega\right)\exp\left(\int_{B_1} f^*\omega\right)$$

$$\exp\left(\frac{i}{\hbar}S(\cdots)\right)=\exp\left(\frac{i}{\hbar}\int_B\mathcal{L}\right)$$

unit of action

$\mathbb{R}\mathrm{map}(S,\mathbb{B}^dU(1))=\hat{H}^d(S)$ differential cohomology

Ex $V = Vect_n$

$V_1^{\times} =$ $\rightarrow \left(fVect_n^{f\nabla}, \nabla\text{-preserving isos} \right)$

$\dim(\text{fiber}) = 1$

$\simeq \left(tVect_n, \text{trivial maps} \right)$

$$\left(W, \overset{f: V \to U}{B} \right) \longmapsto \|_B^W$$

No lecture on Sunday, May 12

Possibly no lecture on May 17.

S: $\begin{array}{c} T \\ \downarrow \\ U \end{array} \longmapsto$ the set of fiberwise flat Riemannian metrics on $\begin{array}{c} T \\ \downarrow \\ U \end{array}$

(unenriched geo str.)

S

$Sh(FEmb_d) \underset{i^*}{\overset{i_!}{\rightleftarrows}} Sh(IFEmb_d) \xrightarrow[r^*]{\sim} Sh(Cart, Set^{Ob})$

$\sim \downarrow$

$Sh(FEmb_{Cart_d}) \underset{i^*}{\overset{i_!}{\rightleftarrows}} Sh(IFEmb_d) \xrightarrow{\sim}$

$r^* i_!$ homotopy cocontinuous functor

Compute $r^* i_! = \mathcal{C}$ applied to S

1) co-Yoneda: $S = \underset{\left(\begin{smallmatrix} T \\ \downarrow \\ U \end{smallmatrix}\right) \to S}{\text{hocolim}} \; y\left(\begin{smallmatrix} T \\ \downarrow \\ U \end{smallmatrix}\right)$

good open cover argument

$= \underset{\left(\begin{smallmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{smallmatrix}\right) \to S}{\text{hocolim}} \; y\left(\begin{smallmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{smallmatrix}\right)$

can require $\cong$ open balls in $\mathbb{R}^d$ w/ met. of radius $r > 0$ st.

$= \operatorname{hocolim} \mathcal{G}\begin{pmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{pmatrix}$

Mor:

$$\begin{pmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{pmatrix} \longrightarrow S$$

isometry

$$\begin{pmatrix} \mathbb{R}^d \times V \\ \downarrow \\ V \end{pmatrix}$$

Ob: $r : U \to (0, \infty)$

Mor: $r_1 \to r_2$

$r_1 : U \to (0, \infty)$

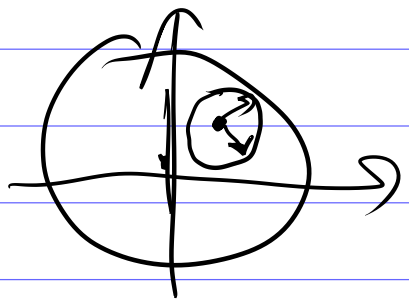
$r_2 : V \to (0, \infty)$

$f : U \to V$

$g : U \longrightarrow \mathrm{Euc}(d) = \mathrm{Aut}(\mathbb{R}^d, \langle -, - \rangle) = \mathbb{R}^d \rtimes O(d)$

translations
+ rotations

$B_{r_1} \hookrightarrow B_{r_2}$



such that the image of B_{r_1} under g is inside B_{r_2}

$$= \operatorname*{hocolim}_{(r,(f,g))} y\begin{pmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{pmatrix} \simeq S$$

$$\mathcal{C}(S) = \widetilde{\widetilde{\mathcal{C}}}(\operatorname{hocolim})$$

$$= \operatorname*{hocolim}_{(r,(f,g))} \underbrace{\mathcal{C}}_{r^* U_!} \begin{pmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{pmatrix}$$

$$= \operatorname*{hocolim}_{(r,(f,g))} \underbrace{O(d) \times}\ y(U)$$

$$r^* \begin{pmatrix} \mathbb{R}^d \times U \\ \downarrow \\ U \end{pmatrix} = \left(\begin{array}{c} V \\ \uparrow \\ \text{Cart} \end{array} \mapsto \mathbb{F}\mathrm{Emb}_d\begin{pmatrix} \mathbb{R}^d \times V & \mathbb{R}^d \times U \\ \downarrow & \downarrow \\ V & U \end{pmatrix} \right)$$

$$\simeq C^\infty(V, W) \times \mathbb{E}\mathrm{mb}(\mathbb{R}^d, \mathbb{R}^d)$$

$$\underset{\text{Kister-Mazur}}{\simeq} C^\infty(V, U) \times O(d)$$

$\simeq O(d) \times \mathrm{hocolim}\, Y(U)$

$(r_1 (f, g))$

$F: I \to Cat_{Thomason} \xrightarrow{N} sSet$

Thomason's theorem: $\mathrm{hocolim}\, N \circ F \simeq N(\mathrm{Groth.\ of}\ F)$

$\simeq O(d) \times \left(W \mapsto N(\dots) \right)$

Grothendieck construction

$F: I \xrightarrow{Cat} Set$

= category of elements

Ob: $(i, x \in F(i))$

Mor: $(i, x) \to (i', x')$

$i \xrightarrow{f} i'$

$F(f)(x) = x'$

Ob: $r: U \to (0, \infty)$

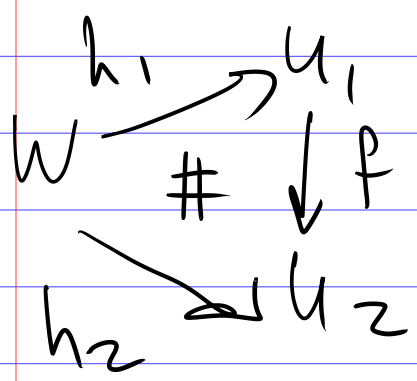
$h: W \to U$

Mor: $(r_1, f_1) \to (r_2, f_2)$

$r_1: U_1 \to (0, \infty)$ $h_1: W \to U_1$

$r_2: U_2 \to (0, \infty)$ $h_2: W \to U_2$

$\begin{cases} f: U_1 \to U_2 \\ g: U_1 \to Euc(d) \end{cases}$



The image of the open ball B_{r_1} under g is inside B_{r_2}

Simplify: $(r, h) \longmapsto r \cdot h : W \to (0, \infty)$

$(f, g) : (r_1, h_1) \to (r_2, h_2) \mapsto g \circ h_1 : W \to Euc(d)$

$\simeq O(d) \times (W \longmapsto \mathcal{N}(\cdots))$

Ob: $R : W \to (0, \infty)$

Mor: $G : W \to Euc(d)$

$R_1 \to R_2$ The image of B_{R_1} under G is contained in B_{R_2}

$\xrightarrow[\sim]{\Pi} O(d) \times (W \longmapsto B(C^\infty(W, Euc(d))))$

$R \longmapsto *$

$G : R_1 \to R_2 \longmapsto G$

Claim: Π is a final functor, hence $N\Pi$ is a weak equivalence

$$sGrp \underset{\text{loop group } \Omega}{\overset{B}{\underset{\longleftarrow}{\longrightarrow}}} sSet_* \qquad \Omega = Map(S^1_*, -)$$

Π is a final functor:

Quillen Theorem A:

$\mathcal{C} \xrightarrow{F} \mathcal{D}$ is ∞-final

$\Leftrightarrow \forall d \in \mathcal{D}$: d/F has weakly contractible nerve

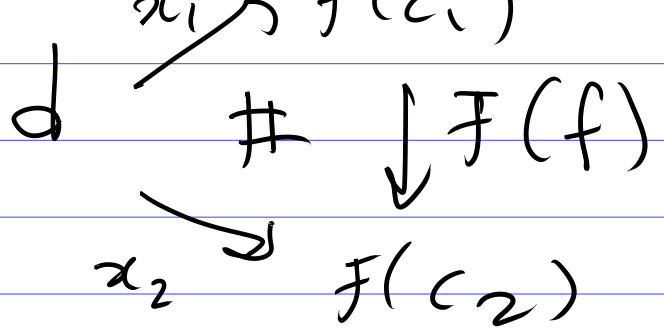
Comma category

d/F: Ob: (c, x) $c \in \mathcal{C}$

$x: d \to F(c)$

Mor $(c_1, x_1) \to (c_2, x_2)$

$f: c_1 \to c_2$



$R \longmapsto^{\Pi} *$

$G : R_1 \to R_2 \longmapsto^{\Pi} G$

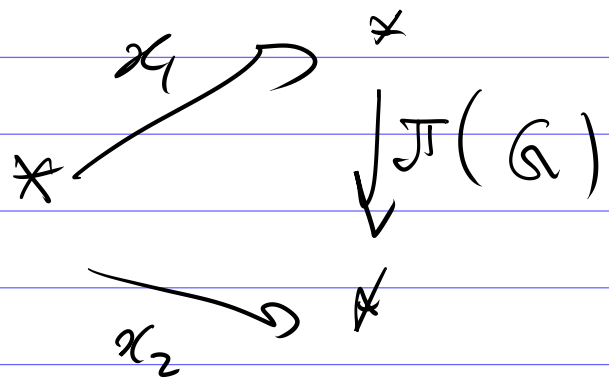
$*/\Pi$: poset

Ob: $x : W \to \underbrace{Euc(d)}_{\text{group}}$

$R : W \to (0, \infty)$,

Mor: $(R_1, x_1) \longrightarrow (R_2 \to x_2)$

$G : R_1 \longrightarrow R_2$



$G \circ x_1 = x_2$

$\Rightarrow G = x_2 \circ x_1^{-1}$

(R, x) = open ball of rad. R in position x inside $\mathbb{R}^d$.

morphisms = inclusions

Claim: $*/\Pi$ is filtered $\Rightarrow N(*/\Pi)$ is weakly contr.

A poset is filtered

if $\forall x, y \quad \exists z$:

$x \leq z$ and $y \leq z$

Any two open balls in $\mathbb{R}^d$

are contained in a

bigger open ball

$\simeq O(d) \times B\,\mathrm{Euc}(d)$

$\simeq \mathcal{C}(S) = r^* \iota_! S \simeq$

Corollary $FFT_{d,S,V}$

$$= \mathbb{R}\mathrm{map}\left(\mathrm{Bord}_d^S, V\right) \qquad C^\infty\mathrm{Cat}_{\infty,d}^{\otimes}$$

$$\overset{GCH}{\simeq} \mathbb{R}\mathrm{map}\left(S, \iota^* V_d^X\right) \qquad \mathrm{Sh}(\mathrm{Emb}_d)$$

$$\overset{GCH}{\simeq} \mathbb{R}\mathrm{map}\left(\iota_! S, V_d^X\right)$$

$$\simeq \mathbb{R}\mathrm{map}\left(r^*\iota_! S, r^* V_d^X\right) \qquad \mathrm{Sh}(\mathrm{Cart}, \mathrm{sSet}) \overset{O(d)}{\cdots}$$

$$\simeq \mathbb{R}\mathrm{map}\left(O(d) \times B\mathrm{Euc}(d), r^* V_d^X\right)$$

$$\simeq \mathbb{R}\mathrm{map}\left(B\mathrm{Euc}(d), r^* V_d^X\right) \qquad \mathrm{Sh}(\mathrm{Cart})$$

$$\simeq \mathbb{R}\mathrm{map}\left(\mathrm{Euc}(d), \Omega_X r^* V_d^X\right) \qquad \mathrm{Sh}(\mathrm{Cart}, \mathrm{sGrp})$$

$\simeq$ homotopy coherent representations

$$\mathrm{Euc}(d) \longrightarrow \mathrm{Aut}(X) \qquad X \to X \circlearrowleft X$$

$\mathrm{Sh}(\mathrm{Cart}, \mathrm{sGrp})$ $\qquad X \in V^{fd}$ ∎

Riemannian $\rightarrow$ Lorentzian

Poincaré $(d) \longrightarrow$ Aut (X)

"elementary particle"