Measurable locales, commutative von Neumann algebras, and measure theory

Dmitri Pavlov (Texas Tech University, Lubbock)

These slides: `https://dmitripavlov.org/denton.pdf`

Gelfand-type duality for commutative von Neumann algebras.
Journal of Pure and Applied Algebra 226:4 (2022), 106884, 1–53.
arXiv:2005.05284

Main theorem

Theorem (P.) The following categories are equivalent.

$$
\begin{array}{ccccccc}
\mathrm{HStonean} & \underset{\mathbf{Sp}}{\overset{\mathbf{\Omega}}{\rightleftarrows}} & \mathrm{HStoneanLoc} & \underset{\mathbf{Ideal}}{\overset{\mathbf{COpen}}{\rightleftarrows}} & \mathrm{MLoc} & \underset{\mathbf{ProjLoc}}{\overset{\mathbf{L}^\infty}{\rightleftarrows}} & \mathrm{CVNA}^{\mathrm{op}} \\
 & & & & \mathbf{ML}\uparrow\;\downarrow\mathbf{Spec} & & \\
 & & & & \mathrm{CSLEMS}. & &
\end{array}
$$

- HStonean (Dixmier): hyperstonean topological spaces and open maps.
- HStoneanLoc: hyperstonean locales and open maps.
- MLoc: measurable locales (opposite category of complete Boolean algebras admitting a measure).
- $\mathrm{CVNA}^{\mathrm{op}}$: opposite category of commutative von Neumann algebras and normal *-homomorphisms.
- CSLEMS: compact strictly localizable enhanced measurable spaces. (The category for measure theory.)

Categories of measure theory

What is a good category for measure theory? Possible answers:

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Measurable spaces. Objects: (X, M), X: set, $M \subset 2^X$: σ-algebra (measurable subsets).
  Morphisms: measurable maps $f\colon (X, M) \to (X', M')$: $f\colon X \to X'$ (maps of sets), $m' \in M' \Rightarrow f^{-1}m' \in M$.

  Defect: does not identify morphisms equal almost everywhere.

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Measurable spaces. Objects: (X, M), X: set, $M \subset 2^X$: σ-algebra (measurable subsets).
  Morphisms: measurable maps $f\colon (X, M) \to (X', M')$: $f\colon X \to X'$ (maps of sets), $m' \in M' \Rightarrow f^{-1}m' \in M$.

  Defect: does not identify morphisms equal almost everywhere.

- Measure spaces: Objects: (X, M, μ), $\mu\colon M \to [0, \infty]$: measure.
  Morphisms: $[f]_\sim\colon (X, M, \mu) \to (X', M', \mu')$, $f\colon X \to X'$; $f \sim f'$ if $\mu\{x \in X \mid f(x) \neq f'(x)\} = 0$ (equality a.e.).

  Major defect: composition does not respect $\sim$.

  Minor defect: μ is not always given (e.g., smooth manifold).

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Measurable spaces. Objects: (X, M), X: set, $M \subset 2^X$: σ-algebra (measurable subsets).
  Morphisms: measurable maps $f\colon (X, M) \to (X', M')$: $f\colon X \to X'$ (maps of sets), $m' \in M' \Rightarrow f^{-1}m' \in M$.

  Defect: does not identify morphisms equal almost everywhere.
- Measure spaces: Objects: (X, M, μ), $\mu\colon M \to [0, \infty]$: measure.
  Morphisms: $[f]_\sim\colon (X, M, \mu) \to (X', M', \mu')$, $f\colon X \to X'$; $f \sim f'$ if $\mu\{x \in X \mid f(x) \neq f'(x)\} = 0$ (equality a.e.).

  Major defect: composition does not respect $\sim$.

  Minor defect: μ is not always given (e.g., smooth manifold).
- Enhanced measurable spaces: Objects: (X, M, N), $N \subset M$: σ-ideal of negligible subsets (closed under countable unions, passage to subsets: $n_2 \subset n_1 \in N \Rightarrow n_2 \in N$).
  Morphisms: $[f]_\sim$; $f \sim f'$ if $\{x \in X \mid f(x) \neq f'(x)\} \in N$.

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Measure spaces: Objects: (X, M, μ), $\mu\colon M \to [0, \infty]$: measure.
  Morphisms: $[f]_\sim\colon (X, M, \mu) \to (X', M', \mu')$, $f\colon X \to X'$;
  $f \sim f'$ if $\mu\{x \in X \mid f(x) \neq f'(x)\} = 0$ (equality a.e.).

  Major defect: composition does not respect $\sim$.

  Minor defect: μ is not always given (e.g., smooth manifold).
- Enhanced measurable spaces: Objects: (X, M, N), $N \subset M$:
  σ-ideal of negligible subsets (closed under countable unions, passage to subsets: $n_2 \subset n_1 \in N \Rightarrow n_2 \in N$).
  Morphisms: $[f]_\sim$; $f \sim f'$ if $\{x \in X \mid f(x) \neq f'(x)\} \in N$.

  Major defect: composition does not respect $\sim$.

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Enhanced measurable spaces: Objects: (X, M, N), $N \subset M$: σ-ideal of negligible subsets (closed under countable unions, passage to subsets: $n_2 \subset n_1 \in N \Rightarrow n_2 \in N$).
  Morphisms: $[f]_\sim$; $f \sim f'$ if $\{x \in X \mid f(x) \neq f'(x)\} \in N$.

  Major defect: composition does not respect $\sim$.
  $g \sim g' \Rightarrow g \circ f \sim g' \circ f$: requires $f^{-1}\{x \mid g(x) \neq g'(x)\} \in N$.

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Enhanced measurable spaces: Objects: (X, M, N), $N \subset M$: σ-ideal of negligible subsets (closed under countable unions, passage to subsets: $n_2 \subset n_1 \in N \Rightarrow n_2 \in N$).

  Major defect: composition does not respect $\sim$. $g \sim g' \Rightarrow g \circ f \sim g' \circ f$: requires $f^{-1}\{x \mid g(x) \neq g'(x)\} \in N$.

  Morphisms $[f]_\sim \colon (X, M, \mu) \to (X', M', \mu')$, $f \colon X \to X'$; $m' \in M' \Rightarrow f^{-1}m' \in M$ and $n' \in N' \Rightarrow f^{-1}n' \in N$.

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Enhanced measurable spaces: Objects: (X, M, N), $N \subset M$: σ-ideal of negligible subsets (closed under countable unions, passage to subsets: $n_2 \subset n_1 \in N \Rightarrow n_2 \in N$).

  Major defect: composition does not respect $\sim$.
  $g \sim g' \Rightarrow g \circ f \sim g' \circ f$: requires $f^{-1}\{x \mid g(x) \neq g'(x)\} \in N$.

  Morphisms $[f]_\sim \colon (X, M, \mu) \to (X', M', \mu')$, $f \colon X \to X'$;
  $m' \in M' \Rightarrow f^{-1}m' \in M$ and $n' \in N' \Rightarrow f^{-1}n' \in N$.

  Example: Real measurable functions on X are morphisms $(X, M, N) \to (\mathbf{R}, \mathrm{Borel}, \{\emptyset\})$.

  Example: $(\mathbf{R}, \mathrm{Lebesgue}, \mathrm{Lebesgue}_{\mu=0}) \to (\mathbf{R}, \mathrm{Borel}, \{\emptyset\})$ is not invertible.

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Enhanced measurable spaces: Objects: (X, M, N), $N \subset M$: σ-ideal of negligible subsets (closed under countable unions, passage to subsets: $n_2 \subset n_1 \in N \Rightarrow n_2 \in N$).
  $g \sim g' \Rightarrow g \circ f \sim g' \circ f$: requires $f^{-1}\{x \mid g(x) \neq g'(x)\} \in N$.

  Morphisms $[f]_\sim \colon (X, M, \mu) \to (X', M', \mu')$, $f \colon X \to X'$; $m' \in M' \Rightarrow f^{-1}m' \in M$ and $n' \in N' \Rightarrow f^{-1}n' \in N$.

  Example: Real measurable functions on X are morphisms $(X, M, N) \to (\mathbf{R}, \mathrm{Borel}, \{\emptyset\})$.

  Example: $(\mathbf{R}, \mathrm{Lebesgue}, \mathrm{Lebesgue}_{\mu=0}) \to (\mathbf{R}, \mathrm{Borel}, \{\emptyset\})$ is not invertible.

  Defect (Fremlin): $\exists f \colon X \to X$, $X = 2^{\mathbf{R}}$, $\forall x \colon f(x) \neq x$, but: $m \in M_X \Rightarrow f^{-1}m \oplus m \in N_X$.

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Enhanced measurable spaces: Objects: (X, M, N), $N \subset M$: σ-ideal of negligible subsets (closed under countable unions, passage to subsets: $n_2 \subset n_1 \in N \Rightarrow n_2 \in N$).
  $g \sim g' \Rightarrow g \circ f \sim g' \circ f$: requires $f^{-1}\{x \mid g(x) \neq g'(x)\} \in N$.

  Morphisms $[f]_\sim \colon (X, M, \mu) \to (X', M', \mu')$, $f \colon X \to X'$; $m' \in M' \Rightarrow f^{-1}m' \in M$ and $n' \in N' \Rightarrow f^{-1}n' \in N$.

  Defect (Fremlin): $\exists f \colon X \to X$, $X = 2^{\mathbf{R}}$, $\forall x \colon f(x) \neq x$, but: $m \in M_X \Rightarrow f^{-1}m \oplus m \in N_X$.

  Morphisms: $[f]_\approx$; $f \approx f'$ if $(m \in M' \Rightarrow f^{-1}m \oplus g^{-1}m \in N)$ (weak equality almost everywhere).

Categories of measure theory

What is a good category for measure theory? Possible answers:

- Enhanced measurable spaces: Objects: (X, M, N), $N \subset M$: σ-ideal of negligible subsets (closed under countable unions, passage to subsets: $n_2 \subset n_1 \in N \Rightarrow n_2 \in N$).
  $g \sim g' \Rightarrow g \circ f \sim g' \circ f$: requires $f^{-1}\{x \mid g(x) \neq g'(x)\} \in N$.

  Morphisms $[f]_\sim\colon (X, M, \mu) \to (X', M', \mu')$, $f\colon X \to X'$; $m' \in M' \Rightarrow f^{-1}m' \in M$ and $n' \in N' \Rightarrow f^{-1}n' \in N$.

  Defect (Fremlin): $\exists f\colon X \to X$, $X = 2^{\mathbf{R}}$, $\forall x\colon f(x) \neq x$, but: $m \in M_X \Rightarrow f^{-1}m \oplus m \in N_X$.

  Morphisms: $[f]_\approx$; $f \approx f'$ if $(m \in M' \Rightarrow f^{-1}m \oplus g^{-1}m \in N)$ (weak equality almost everywhere).
  $\sim \Rightarrow \approx$: always.
  $\approx \Rightarrow \sim$: if (X', M', N') is countably separated, e.g., $(\mathbf{R}, \text{Borel}, \{\emptyset\})$.

A category for measure theory

Objects: enhanced measurable spaces (X, M, N);

- X: set
- M: σ-algebra of measurable subsets of X
- $N \subset M$: σ-ideal of negligible subsets of X

Morphisms $(X, M, N) \to (X', M', N')$: $[f]_\approx$

- $f\colon X \to X'$ map of sets
- $m' \in M' \Rightarrow f^{-1}m' \in M$
- $n' \in N' \Rightarrow f^{-1}n' \in N$
- $f \approx g$ if $m' \in M' \Rightarrow f^{-1}m' \oplus g^{-1}m' \in N$.

A category for measure theory

Objects: enhanced measurable spaces (X, M, N);

- X: set
- M: σ-algebra of measurable subsets of X
- $N \subset M$: σ-ideal of negligible subsets of X

Morphisms $(X, M, N) \to (X', M', N')$: $[f]_\approx$

- $f\colon X \to X'$ map of sets
- $m' \in M' \Rightarrow f^{-1}m' \in M$
- $n' \in N' \Rightarrow f^{-1}n' \in N$
- $f \approx g$ if $m' \in M' \Rightarrow f^{-1}m' \oplus g^{-1}m' \in N$.

Defect: Too many objects; all of measure theory fails.

A category for measure theory

Objects: enhanced measurable spaces (X, M, N);

- X: set
- M: σ-algebra of measurable subsets of X
- $N \subset M$: σ-ideal of negligible subsets of X

Morphisms $(X, M, N) \to (X', M', N')$: $[f]_\approx$

- $f\colon X \to X'$ map of sets
- $m' \in M' \Rightarrow f^{-1}m' \in M$
- $n' \in N' \Rightarrow f^{-1}n' \in N$
- $f \approx g$ if $m' \in M' \Rightarrow f^{-1}m' \oplus g^{-1}m' \in N$.

Defect: Too many objects; all of measure theory fails.

Theorem (I. Segal, 1950.) (X, M, N) satisfies Radon–Nikodym
$\iff (X, M, N)$ satisfies Riesz representation theorem $(L^1)^* \cong L^\infty$
$\iff L^\infty(X, M, N)$ is a von Neumann algebra
$\iff M/N$ is a complete Boolean algebra admitting a measure

A category for measure theory

Objects: enhanced measurable spaces (X, M, N);

- $N \subset M$: σ-ideal of negligible subsets of X

Morphisms $(X, M, N) \to (X', M', N')$: $[f]_\approx$

- $f\colon X \to X'$ map of sets
- $m' \in M' \Rightarrow f^{-1}m' \in M$
- $n' \in N' \Rightarrow f^{-1}n' \in N$
- $f \approx g$ if $m' \in M' \Rightarrow f^{-1}m' \oplus g^{-1}m' \in N$.

Theorem (I. Segal, 1950.) (X, M, N) satisfies Radon–Nikodym
$\iff$ (X, M, N) satisfies Riesz representation theorem $(L^1)^* \cong L^\infty$
$\iff$ $L^\infty(X, M, N)$ is a von Neumann algebra
$\iff$ M/N is a complete Boolean algebra admitting a measure

Definition (I. Segal): (X, M, N) is localizable if M/N is a complete Boolean algebra that admits a faithful measure.

A category for measure theory

Objects: localizable enhanced measurable spaces (X, M, N);

- M/N is a complete Boolean algebra that admits a faithful measure

Morphisms $(X, M, N) \to (X', M', N')$: $[f]_\approx$

- $f\colon X \to X'$ map of sets
- $m' \in M' \Rightarrow f^{-1}m' \in M$
- $n' \in N' \Rightarrow f^{-1}n' \in N$
- $f \approx g$ if $m' \in M' \Rightarrow f^{-1}m' \oplus g^{-1}m' \in N$.

A category for measure theory

Objects: localizable enhanced measurable spaces (X, M, N);

- M/N is a complete Boolean algebra that admits a faithful measure

Morphisms $(X, M, N) \to (X', M', N')$: $[f]_\approx$

- $f\colon X \to X'$ map of sets
- $m' \in M' \Rightarrow f^{-1}m' \in M$
- $n' \in N' \Rightarrow f^{-1}n' \in N$
- $f \approx g$ if $m' \in M' \Rightarrow f^{-1}m' \oplus g^{-1}m' \in N$.

Three defects: $f\colon (X, M, N) \to (X', M', N')$ is a morphism

- $\exists f$: $[f^{-1}]\colon M'/N' \to M/N$ is discontinuous.
- $\exists f$ such that $[f^{-1}]$ is invertible, but f is not.
- $\exists f$: f does not have a measurable image.

A category for measure theory

Three defects: $f\colon (X, M, N) \to (X', M', N')$ is a morphism

- $\exists f$: $[f^{-1}]\colon M'/N' \to M/N$ is discontinuous.
- $\exists f$ such that $[f^{-1}]$ is invertible, but f is not.
- $\exists f$: f does not have a measurable image.

Definition: (X, M, N) is strictly localizable if
$(X, M, N) = (\coprod_i X_i, \prod_i M_i, \prod_i N_i)$, where (X_i, M_i, N_i) is σ-finite.

Definition (Marczewski, 1953): (X, M, N) is compact if

- $\exists$ compact class $K \subset M$: $\forall m \in M \setminus N\colon \exists k \in K \setminus N\colon k \subset m$.

$K \subset M$ is a compact class if

- $\forall K' \subset K\colon (\forall K'' \subset_{\text{finite}} K'\colon \bigcap K'' \neq \emptyset) \Rightarrow \bigcap K' \neq \emptyset$.

Example: measurable spaces with a Radon measure are compact and strictly localizable.

A category for measure theory

Three defects: $f\colon (X,M,N)\to(X',M',N')$ is a morphism

- $\exists f$: $[f^{-1}]\colon M'/N'\to M/N$ is discontinuous.
- $\exists f$ such that $[f^{-1}]$ is invertible, but f is not.
- $\exists f$: f does not have a measurable image.

Definition: (X,M,N) is strictly localizable if
$(X,M,N)=(\coprod_i X_i,\prod_i M_i,\prod_i N_i)$, where (X_i,M_i,N_i) is σ-finite.

Definition (Marczewski, 1953): (X,M,N) is compact if

- $\exists$ compact class $K\subset M$: $\forall m\in M\setminus N\colon \exists k\in K\setminus N\colon k\subset m$.

$K\subset M$ is a compact class if

- $\forall K'\subset K\colon (\forall K''\subset_{\mathrm{finite}} K'\colon \bigcap K''\neq\emptyset)\Rightarrow\bigcap K'\neq\emptyset$.

Example: measurable spaces with a Radon measure are compact and strictly localizable.

Proposition (Fremlin): (X,M,N) compact, (X',M',N') strictly localizable $\Rightarrow$ the measurable image of $f\colon X\to X'$ exists.

The category for measure theory

Objects: enhanced measurable spaces (X, M, N)

- X: set; M: σ-algebra; N: σ-ideal
- (X, M, N) is strictly localizable ($\coprod$ σ-finite)
- (X, M, N) is compact (like Radon measures)

Morphisms: $[f]_\approx$ (weak equality almost everywhere)

- $m' \in M' \Rightarrow f^{-1}m' \in M$; $n' \in N' \Rightarrow f^{-1}n' \in N$
- $f \approx f'$ if for all $m' \in M'$: $f^{-1}m' \oplus f'^{-1}m' \in N$

The category for measure theory

Objects: enhanced measurable spaces (X, M, N)

- X: set; M: σ-algebra; N: σ-ideal
- (X, M, N) is strictly localizable ($\coprod$ σ-finite)
- (X, M, N) is compact (like Radon measures)

Morphisms: $[f]_\approx$ (weak equality almost everywhere)

- $m' \in M' \Rightarrow f^{-1}m' \in M$; $n' \in N' \Rightarrow f^{-1}n' \in N$
- $f \approx f'$ if for all $m' \in M'$: $f^{-1}m' \oplus f'^{-1}m' \in N$

An equivalent category: measurable locales

Definition: $\mathrm{MLoc} = \mathrm{LBAlg}^{\mathrm{op}}$; LBAlg: localizable Boolean algebras:

- Objects: Dedekind-complete and admit a faithful measure.
- Morphisms: continuous homomorphisms.

The category for measure theory

Objects: enhanced measurable spaces (X, M, N)

- X: set; M: σ-algebra; N: σ-ideal
- (X, M, N) is strictly localizable ($\coprod$ σ-finite)
- (X, M, N) is compact (like Radon measures)

Morphisms: $[f]_\approx$ (weak equality almost everywhere)

Definition: $\mathrm{MLoc} = \mathrm{LBAlg}^{\mathrm{op}}$; LBAlg: localizable Boolean algebras:

- Objects: Dedekind-complete and admit a faithful measure.
- Morphisms: continuous homomorphisms.

CSLEMS $\to$ MLoc: $(X, M, N) \mapsto M/N$; $[f]_\approx \mapsto f^{-1}$ (Fremlin).
CSLEMS $\to$ $\mathrm{CVNA}^{\mathrm{op}}$: $(X, M, N) \mapsto \mathrm{L}^\infty(X, M, N)$;
CSLEMS $\to$ HStonean: Gelfand spectrum of $\mathrm{L}^\infty(X, M, N)$
MLoc $\to$ CSLEMS: Loomis–Sikorski, 1948: X: Stone spectrum; N: meager; M: Baire (meager $\oplus$ open)
CSLEMS $\to$ MLoc $\to$ CSLEMS: requires the theorems of von Neumann–Maharam (1958) and Ionescu Tulcea (1965).

Souvenirs to take home

- CSLEMS: compact strictly localizable enhanced measurable spaces.
- Equivalent to: (2) measurable locales, (3) commutative von Neumann algebras, (4) hyperstonean locales / (5) spaces.

Measure theory wants to be (point) free:

- MLoc $\to$ Locale: full (!) subcategory
- HStoneanLoc $\to$ LocaleOpen: full subcategory

CSLEMS is a closed monoidal category (for VNA: Kornell, 2012)

- $\otimes$: measure-theoretic (not categorical) product.
- $\mathbf{Hom}(X, Y) = Y^X$: enhanced measurable space of equivalence classes of measurable maps.
- enhancements etc. crucial for the existence of **Hom**.
- evaluation morphism: $X \otimes \mathbf{Hom}(X, Y) \to Y$.
- adjunction property: $X \to \mathbf{Hom}(Y, Z) \iff X \otimes Y \to Z$.
- Aumann, 1960: negative results for the non-enhanced case.

Future work

- Measurable Serre–Swan theorem; applications: Haar measure, spectral theorem.
- Pushforwards/pullbacks for L^p-spaces and disintegration theorems.
- Measurable correspondences; measurable Markov category.

Future work

- Measurable Serre–Swan theorem; applications: Haar measure, spectral theorem.
- Pushforwards/pullbacks for L^p-spaces and disintegration theorems.
- Measurable correspondences; measurable Markov category.

Thank you!