Coadmissibility of colored cooperads in monoidal model categories

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Contents

Abstract. We construct a model structure on the category of coalgebras over a cooperad in a combinatorial monoidal model category satisfying some mild additional conditions. We verify these assumptions for a large variety of examples, recovering many existing results and obtaining new ones, such as a model structure on oplax monoidal functors.

1 Introduction

2 Cotransfer of model structures

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ to a model category $\mathcal{D}$, one can ask for a model structure on $\mathcal{C}$ whose weak equivalences are created by the functor. For right adjoint functors, with fibrations created by the functor (i.e., *transferred* model structures), a criterion for the existence of such model structures was given by Kan, see, for example, Lemma 2.12 in Barwick [LR] or Theorem 11.3.2 in Hirschhorn [ModCat]. In this purely expository section we review the analog of this result for left adjoint functors F, with cofibrations created by the functor F, i.e., *cotransferred* model structures.

Several results for the existence of cotransferred model structures are found in the literature. Apart from numerous results for the case of abelian categories (e.g., chain complexes), one should mention the theorems of Hess (Corollary 5.15 in [HHGE], a general result using Postnikov presentations), and Hess and Shipley (Theorem 5.8 in [Comonad], an application of the previous result for coalgebras over a comonad). A recent result by Makkai and Rosický (Remark 3.8 in [CellCat]), combined with the Smith recognition theorem, immediately yields a rather general existence criterion, stated and proved below as the cotransfer theorem 2.2. Bayeh, Hess, Karpova, Kędziorek, Riehl, and Shipley [LeftInd] give further applications of the results by Hess and Shipley, and in Theorem 2.23 they give an exposition of the cotransfer theorem 2.2, though the proof there is rather indirect.

We start by formalizing the definition of a (co)transferred structure.

Definition 2.1. Given a right (left) adjoint functor $\mathcal{C} \to \mathcal{D}$ and a model structure on $\mathcal{D}$, the *(co)transferred model structure* on $\mathcal{C}$, if it exists, is the unique model structure whose weak equivalences and (co)fibrations (hence also acyclic (co)fibrations) are precisely those maps that are mapped by F to weak equivalences and (co)fibrations in $\mathcal{D}$.

Typically, the only nontrivial part in constructing a transferred model structure is to prove that cobase changes of acyclic cofibrations in $\mathcal{C}$ are weak equivalences in $\mathcal{C}$. In a similar way, the only nontrivial part in constructing a cotransferred model structure usually amounts to proving that maps with a right lifting property with respect to all cofibrations in $\mathcal{C}$ are weak equivalences in $\mathcal{C}$.

The following result offers a formalization of this thesis for the case of combinatorial model categories. It is a direct consequence of the Smith recognition theorem, a theorem by Makkai and Paré (inclusion of accessible categories and functors into all categories and functors creates PIE-limits), and a similar recent result by Makkai and Rosický about combinatorial categories, i.e., locally presentable categories equipped with a weakly saturated class of morphisms generated by a set, for which the forgetful functor also creates PIE-limits.

Cotransfer theorem 2.2. (Makkai, Paré, Rosický, Smith.) Suppose $F: \mathcal{C} \to \mathcal{D}$ is a cocontinuous functor between locally presentable categories, where $\mathcal{D}$ is equipped with a combinatorial model structure. If the maps that have the right lifting property with respect to all cofibrations in $\mathcal{C}$ are weak equivalences in $\mathcal{C}$, then the cotransferred structure on $\mathcal{C}$ exists and is combinatorial. Used in 2.0*, 2.0*, 2.4, 2.4, 3.0*, 3.9*.

Proof. We use the Smith recognition theorem, see Theorem 1.7, Propositions 1.15 and 1.19 in Beke [ShHMC], Proposition A.2.6.8 in Lurie [HTT], or Proposition 2.2 in Barwick [LR]. The category $\mathcal{C}$ is locally presentable by assumption. The class of weak equivalences in $\mathcal{C}$ satisfies the 2-out-of-3 property because so does its image under F and it is an accessible subcategory of the category of morphisms in $\mathcal{C}$ by Theorem 5.1.6 in Makkai and Paré [AccCat] or by Corollary A.2.6.5 in Lurie [HTT]. The class of (acyclic) cofibrations in $\mathcal{C}$ is closed under weak saturation because the functor F is cocontinuous. Furthermore, by Remark 3.8 in Makkai and Rosický [CellCat], the resulting class of (acyclic) cofibrations in $\mathcal{C}$ is the weak saturation of a set of morphisms. The remaining condition is the lifting property that appears in the statement. ▮

Example 2.3. Suppose $F: \mathcal{C} \to \mathcal{D}$ is a fully faithful cocontinuous functor between locally presentable categories, where $\mathcal{D}$ is equipped with a combinatorial model structure. If the weak saturation of the cofibrations in $\mathcal{C}$ consists of all cofibrations in $\mathcal{D}$, then the cotransferred model structure on $\mathcal{C}$ along F exists and is combinatorial. We have recovered Theorem 2.1 in Haraguchi [Corefl] for the locally presentable case.

Counterexample 2.4. We demonstrate that the lifting property is essential for the validity of the cotransfer theorem 2.2. Consider the fully faithful inclusion of simplicial groups into simplicial monoids. Its right adjoint takes the (levelwise) simplicial group of invertible elements (alias units) and can be computed by taking the pullback of the diagram $M \times M \to M \times M \leftarrow 1$, where the first map is $x, y \mapsto xy, yx$ and the second map is $* \mapsto 1, 1$. Equip simplicial monoids with the model structure transferred from simplicial sets. All conditions of the cotransfer theorem 2.2 are satisfied for the inclusion functor except for the lifting property. We claim that the cotransferred model structure on simplicial groups does not exist. Indeed, if such a model structure existed, the right adjoint must map acyclic fibrations to weak equivalences. Kan's fibrant replacement functor Ex^∞ for simplicial sets preserves finite limits. In particular, it lifts to a fibrant replacement functor on simplicial monoids. Furthermore, preservation of finite limits implies that the units of a fibrant replacement can be computed as the fibrant replacement of units, in particular, they are weakly equivalent to the original units. It remains to construct a weakly contractible simplicial monoid (not necessarily fibrant) whose units are not weakly contractible. We construct such a monoid as the nerve of a strict monoidal category. The monoid of objects is $\mathbf{Z}/2$. The monoid of morphisms is the submonoid of $\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}_{\geq 0}$ consisting of triples (a, b, x) such that $x > 0$ if $a \neq b$. The source and target of (a, b, x) are a and b respectively. The identity map is $a \mapsto (a, a, 0)$ and the composition map is $(b, c, y) \circ (a, b, x) = (a, c, x + y)$. The units of this strict monoidal category form a discrete category on $\mathbf{Z}/2$, which is not weakly contractible.

3 Enriched cotransfer of model structures

Using a classical path object argument due to Quillen (see the last paragraph in the proof of Theorem 4 in §II.4 of [HoAlg] for the case of transferred structures), we can give a practical criterion to verify the lifting condition in the cotransfer theorem 2.2 using enrichments.

For a general expository account of enriched categories we refer the reader to Chapter 6 in Borceux [CatAlg]. We recall several facts that are of particular importance to us.

Notation 3.1. Henceforth $\mathcal{V}$ is a locally presentable closed symmetric monoidal category.

A $\mathcal{V}$-enriched category $\mathcal{D}$ (henceforth simply a $\mathcal{V}$-category) is $\mathcal{V}$-*(co)complete* if it admits small $\mathcal{V}$-weighted (co)limits (henceforth simply $\mathcal{V}$-(co)limits). By Theorem 6.6.14 in Borceux [CatAlg] this amounts to saying that the underlying category of $\mathcal{D}$ is (co)complete, $\mathcal{D}$ is (co)powered over $\mathcal{V}$, and any enriched corepresentable (representable) functor sends ordinary (co)limits to limits.

A $\mathcal{V}$-functor between $\mathcal{V}$-(co)complete $\mathcal{V}$-categories is $\mathcal{V}$-*(co)continuous* if it preserves small $\mathcal{V}$-(co)limits. By Corollary 6.6.15 in Borceux [CatAlg], a $\mathcal{V}$-functor is $\mathcal{V}$-(co)continuous if and only if it preserves $\mathcal{V}$-(co)powerings and its underlying functor is (co)continuous. In particular, right (left) $\mathcal{V}$-adjoint $\mathcal{V}$-functors are $\mathcal{V}$-(co)continuous $\mathcal{V}$-functors. By Theorem 6.7.6 in Borceux [CatAlg] a $\mathcal{V}$-functor between $\mathcal{V}$-(co)complete $\mathcal{V}$-categories is right (left) $\mathcal{V}$-adjoint if and only if it preserves $\mathcal{V}$-(co)powerings and its underlying ordinary functor is a right (left) adjoint.

The following definition of monoidal and enriched model categories is standard, except that we prefer to deal with unit axioms separately. It implies that any symmetric monoidal model category is canonically enriched over itself.

Definition 3.2. A *symmetric monoidal model category* is a symmetric monoidal category $\mathcal{V}$ equipped with a model structure such that the monoidal product $\otimes: \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ is a left Quillen bifunctor. For any such $\mathcal{V}$ a $\mathcal{V}$-*enriched model category* (henceforth simply a $\mathcal{V}$-model category) is a $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete $\mathcal{V}$-category $\mathcal{C}$ equipped with a model structure such that the $\mathcal{V}$-copowering $\otimes: \mathcal{V}\times\mathcal{C}\to\mathcal{C}$ is a left Quillen bifunctor (equivalently, the powering $\mathcal{V}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{C}$ is a right Quillen bifunctor, equivalently, the enriched hom $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{V}$ is a right Quillen bifunctor). A *left (right) Quillen* $\mathcal{V}$*-functor* is a left (right) $\mathcal{V}$-adjoint $\mathcal{V}$-functor whose underlying functor is a left (right) Quillen functor.

Definition 3.3. If $F: \mathcal{C}\to\mathcal{D}$ is a right (left) $\mathcal{V}$-adjoint $\mathcal{V}$-functor and $\mathcal{D}$ is equipped with a $\mathcal{V}$-model structure, then the *(co)transferred* $\mathcal{V}$*-model structure* (if it exists) is the unique $\mathcal{V}$-model structure on $\mathcal{C}$ such that the underlying model structure on $\mathcal{C}$ is (co)transferred from $\mathcal{D}$ along F.

Proposition 3.4. Suppose $F: \mathcal{C}\to\mathcal{D}$ is a $\mathcal{V}$-functor between $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete $\mathcal{V}$-categories and $\mathcal{D}$ is equipped with a $\mathcal{V}$-model structure. If the (co)transferred model structure on $\mathcal{C}$ exists, then it is also a (co)transferred $\mathcal{V}$-model structure.

Proof. The only thing to prove is that the $\mathcal{V}$-powering $\mathcal{V}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{C}$ (respectively the $\mathcal{V}$-copowering $\mathcal{V}\times\mathcal{C}\to\mathcal{C}$) is a right (left) Quillen bifunctor. Indeed, (co)fibrations and acyclic (co)fibrations in $\mathcal{C}$ are created by the $\mathcal{V}$-(co)continuous $\mathcal{V}$-functor F, which in particular preserves $\mathcal{V}$-(co)powerings. This immediately reduces the right (left) Quillen bifunctor axiom for the (co)powering of $\mathcal{C}$ to that of $\mathcal{D}$. ∎

Definition 3.5. An *interval* in a symmetric monoidal model category $\mathcal{V}$ is a factorization $1\sqcup 1\to\hat{1}\to 1$ of the codiagonal $1\sqcup 1\to 1$ of the monoidal unit 1 in $\mathcal{V}$ such that the map $1\sqcup 1\to\hat{1}$ is a cofibration and the map $\hat{1}\to 1$ is a weak equivalence in $\mathcal{V}$. In other words, an interval is a cylinder object for the monoidal unit in $\mathcal{V}$.

Remark 3.6. An interval can be constructed by factoring the codiagonal map $1\sqcup 1\to 1$ as a cofibration followed by an acyclic fibration. Furthermore, using the lifting property of acyclic fibrations with respect to cofibrations one can construct a morphism from any interval to such an interval. (A morphism of intervals is simply a map of intermediate objects with the two obvious triangles commuting.) The 2-out-of-3 property shows that the resulting morphism is a weak equivalence, thus any two intervals can be connected by a single zigzag of weak equivalences.

Unit axiom 3.7. A $\mathcal{V}$-model category $\mathcal{D}$ satisfies the *unit axiom* for some interval $1\sqcup 1\to\hat{1}\to 1$ in $\mathcal{V}$ if for any cofibrant object X in $\mathcal{D}$ the morphism $(\hat{1}\to 1)\otimes X$ is a weak equivalence in $\mathcal{D}$.

Remark 3.8. It is unclear whether the unit axiom for some interval implies the unit axiom for all other intervals. This is true if we require $\hat{1}$ to be cofibrant, which usually holds in practice.

Enriched cotransfer theorem 3.9. Suppose $F: \mathcal{C} \to \mathcal{D}$ is a $\mathcal{V}$-cocontinuous $\mathcal{V}$-functor between locally presentable $\mathcal{V}$-categories. Furthermore, suppose that $\mathcal{V}$ is a monoidal model category with an interval $\hat{1}$, $\mathcal{D}$ is equipped with a combinatorial $\mathcal{V}$-model structure satisfying the unit axiom, and $\mathcal{C}$ admits an endofunctor $Q: \mathcal{C} \to \mathcal{C}$ with a natural transformation $q: Q \to \mathrm{id}$ such that FQ lands in cofibrant objects in $\mathcal{D}$ and the natural transformation $F(q)$ has weak equivalences in $\mathcal{D}$ as its components. Then the cotransferred combinatorial $\mathcal{V}$-model structure on $\mathcal{C}$ exists. Used in 5.2*.

Proof. Using the cotransfer theorem 2.2, the only condition to verify is that a morphism $f: X \to Y$ in $\mathcal{C}$ that has a right lifting property with respect to all cofibrations in $\mathcal{C}$ is a weak equivalence in $\mathcal{C}$. We consider the commutative square

$$\begin{tikzcd} QX \arrow[r, "q_X"] \arrow[d, "Qf"] & X \arrow[d, "f"] \\ QY \arrow[r, "q_Y"] & Y. \end{tikzcd}$$

Using the cofibrancy of QY and the right lifting property of f, we construct a diagonal arrow $d: QY \to X$ such that $fd = q_Y$.

Quillen's cylinder object argument considers the commutative diagram

$$\begin{tikzcd} QX \sqcup QX \arrow[r, "(d \circ Qf, q_X)"] \arrow[d] & X \arrow[d, "f"] \\ \hat{1} \otimes QX \arrow[r, "f q_X p"] & Y, \end{tikzcd}$$

where $p = (\hat{1} \to 1) \otimes QX$ is a weak equivalence by the unit axiom. The left map is a cofibration because QX is cofibrant and $1 \sqcup 1 \to \hat{1}$ is a cofibration, so we can construct a diagonal arrow $e: \hat{1} \otimes QX \to X$ (not to be confused with the map $q_X p$) that makes both triangles commutative.

The two component maps $QX \to \hat{1} \otimes QX$ of the left arrow are weak equivalences and yield the same map in the homotopy category. Indeed, postcomposing with the weak equivalence p gives us two identity maps $QX \to QX$. If instead we postcompose with the diagonal map e, we can use the commutativity of the upper triangle to deduce that the two components of the upper arrow yield the same map in the homotopy category. The second component q_X is a weak equivalence, hence so is the first component $d \circ Qf$.

Looking now at the original square diagram we see that in a composable triple (Qf, d, f) the two compositions $fd = q_Y$ and $d \circ Qf$ are weak equivalences, hence all three maps Qf, d, and f are weak equivalences by the 2-out-of-6 property. In particular, the map f is a weak equivalence, as desired. ∎

4 Lax comonads

Although the result of the previous section is quite general, it says nothing about how to obtain a cofibrant replacement (Q, q) that satisfies the given properties. Of course, one can always use the small object argument to factor $\emptyset \to X$ as a cofibration $\emptyset \to QX$ and a map $QX \to X$ with a right lifting property with respect to all cofibrations, however, there is no way to establish that the latter map is a weak equivalence. A particularly important example of a left adjoint functor along which one might want to cotransfer a model structure is given by the forgetful functor from the category of coalgebras over a comonad in a category $\mathcal{D}$ to $\mathcal{D}$. In this section we give a criterion that allows us to construct the pair (Q, q) in such a setting. In fact, we allow for a slightly more general notion of a *lax comonad*, which allows us to treat the case of coalgebras over cooperads, as explained below.

An important class of comonads is given by cooperads. Recall that a coooperad is an operad in the opposite category. Operads can be defined as monoids in the category of symmetric sequences with the substitution product and cooperads can be similarly defined as comonoids in the category of symmetric sequences with the *decomposition product* (defined in a dual fashion). If one wants to use the definition of (co)operads as (co)monoids in symmetric sequences, there is a subtlety involved: symmetric sequences form a monoidal category if the monoidal product preserves coproducts in each argument, which is usually satisfied in practice. However, for cooperads one would have to require the monoidal product to preserve products in each variable, which often fails in practice. For example, already in the case of a cartesian monoidal structure such a condition would imply that the natural transformation $A \times (B \times C) \to (A \times B) \times (A \times C)$ is an isomorphism, in particular, for the case when $B = C$ is the terminal object, the diagonal map $A \to A \times A$ is an isomorphism for any A, which fails unless the underlying category is the terminal category.

In the absence of such commutation conditions, one can still write down a candidate for the substitution product $X \circ Y \circ Z$ of three symmetric sequences together with two noninvertible maps $X \circ Y \circ Z \to (X \circ Y) \circ Z$ and $X \circ Y \circ Z \to X \circ (Y \circ Z)$. Such a structure (together with maps for higher arities) can be organized into a *(normal) oplax monoidal category*, as defined by Day and Street [Lax]. (Normal refers to the fact that in arity 1 the multiplication map $\mathcal{D}^1 \to \mathcal{D}$ is the identity map, whereas oplax refers to the direction of arrows for the associator maps; a lax monoidal category has maps going in the opposite direction.) (Normal) oplax monoidal categories can be concisely defined as (normal) oplax monoids (i.e., strong monoidal oplax functors from the category of finite ordered sets with the disjoint union) in the (large) monoidal 2-category of categories equipped with the cartesian product. Operads can then be defined as monoids (with the appropriately generalized unbiased definition of monoids) in this normal oplax monoidal category. See Ching [Monoid] for more details about this definition of operads. Similarly, cooperads can be defined as comonoids in the normal *lax* monoidal category of symmetric sequences equipped with the decomposition product. All of this generalizes to the colored case. Gambino and Joyal [Colored] give a definition of colored operads as monoids in the appropriate monoidal category. The above remarks about (op)lax monoidal categories also apply in this case. We now briefly recall the relevant definitions of comonoids and comodules in a lax monoidal category.

Definition 4.1. A *lax monoidal category* is a strong monoidal lax functor from the category of finite ordered sets with disjoint union to the large 2-category of categories, functors, and natural transformations with the cartesian product. A lax monoidal category is *normal* if identity maps are mapped to identity functors and the associator maps are identities whenever one of the original maps is an identity.

Remark 4.2. Concretely, a lax monoidal category is a category $\mathcal{D}$ equipped with multiplication functors $\mu_n \colon \mathcal{D}^n \to \mathcal{D}$ for all $n \geq 0$ and associator transformations $a_f \colon \mu_n \mu_f \to \mu_m$, where $f \colon m \to n$ is a morphism of finite ordered sets and μ_f denotes $\prod_{i \in n} \mu_{f^{-1}(i)}$. The associator maps have to satisfy the obvious compatibility condition for composable triples of morphisms. Normality means that $\mu_1 = \mathrm{id}_{\mathcal{D}}$, $a_{\mathrm{id}_m} = \mathrm{id}$, $a_{m \to 1} = \mathrm{id}$.

Lax bicategories are defined in the obvious way as the many-objects analog of lax monoidal categories, in the same way as bicategories are the many-objects analog of monoidal categories. Thus a lax bicategory with one object is essentially the same thing as a lax monoidal category. The primary example of a lax bicategory one should keep in mind is the lax bicategory of sets, symmetric collections between sets (with the composition given by the decomposition product), and their transformations. For a set W, the category of W-colored cooperads can then be defined as the category of comonoids in the lax monoidal category of endomorphisms of W in this lax bicategory.

Definition 4.3. A *lax bicategory* is given by a class of objects O, a hom category $\mathrm{Map}(X_0, X_1)$ for any pair of objects (X_0, X_1), composition functors $\circ_X\colon \mathrm{Map}(X_{n-1}, X_n) \times \cdots \times \mathrm{Map}(X_0, X_1) \to \mathrm{Map}(X_0, X_n)$ for any finite family of objects $X_0, \ldots, X_n$ $(n \geq 0)$, and associator transformations $\circ_{X_{n_0}, X_{n_1}, \ldots, X_{n_m}}(\circ_{X_{n_{m-1}}, \ldots, X_{n_m}} \times \cdots \times \circ_{X_{n_0}, \ldots, X_{n_1}}) \to \circ_{X_0, X_1, \ldots, X_{n_m}}$ that satisfy the obvious compaitibility condition.

We now define comonoids in a lax monoidal category. Again, the primary example to keep in mind is that of colored cooperads.

Definition 4.4. A *comonoid* in a lax monoidal category $\mathcal{D}$ is an object M in $\mathcal{D}$ together with comultiplication maps $m_n\colon M \to \mu_n(M, \ldots, M)$ such that the coassociativity property holds: the composition $M \to \mu_m(M, \ldots, M) \to \mu_m(\mu_{n_0}(M, \ldots, M), \ldots, \mu_{n_{m-1}}(M, \ldots, M)) \to \mu_N(M, \ldots, M)$ equals the map m_N for any $m \geq 0$ and any m-tuple n, where $n_i \geq 0$ and $N = \sum_i n_i$. A comonoid is *normal* if $m_1 = \mathrm{id}_M$.

Remark 4.5. The maps m_n for $n \leq 2$ together with coassociativity conditions for $N \leq 3$ are sufficient to define comonoids, see Proposition 3.4 in Ching [Monoid].

Definition 4.6. A *left comodule* over a comonoid O in a lax monoidal category $\mathcal{D}$ is an object L in $\mathcal{D}$ together with comultiplication map $l_n\colon L \to \mu_{n+1}(M, \ldots, M, L)$ (with n copies of M) such that the coassociativity condition is satisfied: the composition

$$\begin{aligned} L \to \mu_{m+1}(M, \ldots, M, L) &\to \mu_{m+1}(\mu_{n_0}(M, \ldots, M), \ldots, \mu_{n_{m-1}}(M, \ldots, M), \mu_{n_m+1}(M, \ldots, M, L)) \\ &\to \mu_{N+1}(M, \ldots, M, L) \end{aligned}$$

equals the map l_N for any $m \geq 0$ and any $(m+1)$-tuple n, where $n_i \geq 0$ and $N = \sum_i n_i$. A comodule is *normal* if $l_0 = \mathrm{id}_L$. Morphisms of comodules are morphisms of the underlying objects that preserve the structure strictly.

Remark 4.7. The maps l_n for $n \leq 1$ together with coassociativity conditions for $N \leq 2$ are sufficient to define comodules, see Remark 3.7 in Ching [Monoid]. Note that there is a shift in indexing for l because we don't count the object L when we enumerate terms, which explains occasional $+1$.

For similar reasons one fails to obtain a (co)monad in the classical sense from a (co)operad when the commutation conditions are not met. Instead, one gets what we call an *oplax monad* for an operad and a *lax comonad* for a cooperad. Henceforth we focus on comonads. The idea behind the following definition is that for a cooperad O the resulting normal lax comonad T has components of the form $T_n(X) = O \circ \cdots \circ O \circ X$, with n copies of O (n can be any finite ordered set) and $\circ$ denoting the corresponding $(n+1)$-fold monoidal product in the normal lax monoidal category of symmetric sequences with the decomposition product. Morphisms of finite ordered sets (e.g., $2 \to 1$) induce natural transformations of the corresponding components (e.g., $T_1 \to T_2$), which play the role of comultiplication morphisms. The oplax structure $T_{m+n} \to T_m \circ T_n$ is induced by the oplax structure on the monoidal category.

Definition 4.8. A *lax comonad* on a category $\mathcal{D}$ is a lax monoidal functor T from the opposite category of finite ordered sets equipped with the disjoint union to the category of endofunctors on $\mathcal{D}$ equipped with the composition. A lax comonad is *normal* if the lax unit morphism $\mathrm{id} \to T_\emptyset$ is an isomorphism.

Remark 4.9. Concretely, a lax comonad is a sequence of functors $T_n\colon \mathcal{D} \to \mathcal{D}$ for $n \geq 0$ together with comultiplication maps $T_m \leftarrow T_n$ for $m \to n$ that satisfy the coassociativity condition and the associator maps $T_m \circ T_n \to T_{m+n}$ that are compatible with the comultiplication maps using the associators.

Proposition 4.10. Any comonoid O in a lax monoidal category $\mathcal{D}$ induces a normal lax comonad T^O defined as follows: $T^O_n(-) = \mu_{n+1}(O, \ldots, O, -) = O \circ \cdots \circ O \circ -$, $T^O_f\colon T^O_m \leftarrow T^O_n$ is the composition $a_f \circ \mu_{n+1}(m_{f^{-1}(1)}, \ldots, m_{f^{-1}(n)}, -)$, and the lax maps $T_m \circ T_n \to T_{m+n}$ are the associator maps $a_{m,n+1}$.

Definition 4.11. A *coalgebra* over a lax comonad T in a category $\mathcal{D}$ is an object L in $\mathcal{D}$ together with comultiplication maps $l_n\colon L \to T_n L$ that satisfiy the coassociativity relation: the composition of

$$L \xrightarrow{l_m} T_m L \xrightarrow{T_m(l_{n_m})} T_m(T_{n_m} L) \xrightarrow{T_f(T_{n_m} L)} T_n T_{n_m} L \xrightarrow{a_{m,n}} T_N L$$

equals l_N. A coalgebra is *normal* if $l_0\colon X \to T_0 X$ coincides with the lax unit morphism of the comonad. (In particular, if the comonad is normal, l_0 is the identity map.) Morphisms of coalgebras are morphisms of the underlying objects that preserve the structure strictly.

Proposition 4.12. Given a left module L over a comonoid O in a lax monoidal category $\mathcal{D}$, we get a coalgebra L over the lax comonad T^O by taking the same underlying object and the comultiplication morphisms given by those of the left module.

Proposition 4.13. The category of coalgebras over a lax comonad T in a cocomplete category $\mathcal{D}$ is itself cocomplete and the forgetful functor creates colimits.

Proof. Given a small indexing category I, consider an I-shaped diagram L of coalgebras in $\mathcal{D}$. Apply the forgetful functor U to L and denote by X the colimit of the resulting diagram UL in $\mathcal{D}$. We equip X with a structure of a T-coalgebra as follows. The map $X \to T_n(X)$ is defined using the universal property of maps out of X: the component $UL_i \to T_n(X)$ is the composition $UL_i \to T_n(UL_i) \to T_n(X)$, where the first map is the structure map of L_i and the second map is induced by the inclusion $UL_i \to X$. The coassociativity relation is likewise verified on individual components of X. It remains to prove that any cocone $L_i \to Y$ yields a unique morphism $X \to Y$. On the level of underlying objects we get a unique morphism by the universal property of colimits in $\mathcal{D}$. The colinearity condition is again verified on individual components of X. ▮

Definition 4.14. An lax comonad is *accessible* if its individual components T_n are accessible functors.

Proposition 4.15. The category of coalgebras over an accessible lax comonad in $\mathcal{D}$ is accessible or locally presentable if $\mathcal{D}$ is.

Proof. As established before, colimits in the category of coalgebras are created by the forgetful functor to $\mathcal{D}$, so it is enough to establish accessibility. Theorem 5.1.6 in Makkai and Paré [AccCat] proves that the forgetful functor from the 2-category of accessible categories, accessible functors, and natural transformations to the 2-category of categories, functors, and natural transformations creates 2-limits weighted by the 2-category of categories. (2-limits are homotopy limits in this setting.) Thus it is sufficient to construct the category of coalgebras as such a 2-limit. This is a straightforward classical construction, indeed Makkai and Paré already observe it for coalgebras over comonads in §5.1.1 of their book. Details for the case of algebras over monads can be found in Theorem 2.78 in Adámek and Rosický [LPAC], for example. ▮

We finish with a brief discussion of $\mathcal{V}$-enriched lax comonads and coalgebras over them.

Definition 4.16. A lax $\mathcal{V}$-comonad on a $\mathcal{V}$-category $\mathcal{D}$ is defined in the same way as an ordinary lax comonad, with $\mathcal{V}$-endofunctors on $\mathcal{D}$ replacing ordinary endofunctors on $\mathcal{D}$. Coalgebras over a lax $\mathcal{V}$-comonad are defined in exactly the same way, with the involved morphisms taken from the underlying category of $\mathcal{D}$. The enriched hom between coalgebras is defined as the subobject of the enriched hom between the underlying objects by imposing the axioms of coalgebras in the obvious fashion.

Proposition 4.17. The $\mathcal{V}$-category $\mathcal{C}$ of coalgebras over a lax $\mathcal{V}$-comonad T in a $\mathcal{V}$-cocomplete $\mathcal{V}$-category $\mathcal{D}$ is $\mathcal{V}$-cocomplete and the forgetful $\mathcal{V}$-functor creates $\mathcal{V}$-colimits. If $\mathcal{D}$ is locally presentable, then so is $\mathcal{C}$.

Proof. The proof reproduces verbatim the proof for ordinary categories. For expository purposes we briefly discuss the copowering of coalgebras over $\mathcal{V}$. Given a coalgebra $(L, L \to T_n L)$ over T and an object $E \in \mathcal{V}$, we have $E \otimes (L, L \to T_n L) = (E \otimes L, E \otimes L \to E \otimes T_n L \to T_n(E \otimes L))$, where the last map comes from the structure of a $\mathcal{V}$-functor on T_n. ▮

5 Lax comonadic cotransfer

Armed with the tools developed in the previous section, we can now formulate and prove a cotransfer theorem for the category of coalgebras over a lax $\mathcal{V}$-comonad.

Definition 5.1. Given a lax $\mathcal{V}$-comonad T on a $\mathcal{V}$-category $\mathcal{D}$, an endofunctor $Q\colon \mathcal{D} \to \mathcal{D}$, and a natural transformation $q\colon Q \to \mathrm{id}_{\mathcal{D}}$, a *commutator* for (T, Q, q) is a family E of natural transformations $E_n\colon QT_n \to T_nQ$ that satisfies the additional assumptions of coassociativity for Q and colinearity for q. The coassociativity condition requires that for any coalgebra X over T the composition of

$$QX \xrightarrow{Ql_n} QT_nX \xrightarrow{E_n} T_nQX \longrightarrow T_nT_mQX$$

must be equal to

$$QX \xrightarrow{Ql_n} QT_nX \xrightarrow{E_n} T_nQX \xrightarrow{T_nQl_m} T_nQT_mX \xrightarrow{T_nE_m} T_nT_mQX.$$

The colinearity condition requires the compositions of

$$QX \xrightarrow{q_X} X \xrightarrow{l_n} T_nX \qquad \text{and} \qquad QX \xrightarrow{Ql_n} QT_nX \xrightarrow{E_nX} T_nQX \xrightarrow{T_nq_X} T_nX$$

to coincide.

Enriched lax comonadic cotransfer theorem 5.2. Suppose T is an accessible lax $\mathcal{V}$-comonad on a locally presentable $\mathcal{V}$-category $\mathcal{D}$. Furthermore, suppose that $\mathcal{V}$ is a symmetric monoidal model category with an interval $\hat{1}$, $\mathcal{D}$ is a combinatorial $\mathcal{V}$-model category that satisfies the unit axiom, $Q\colon \mathcal{D} \to \mathcal{D}$ is a functor that lands in cofibrant objects, $q\colon Q \to \mathrm{id}$ is a natural weak equivalence, and E is a commutator for (T, Q, q). Then the $\mathcal{V}$-category $\mathcal{C}$ of coalgebras over T in $\mathcal{D}$ admits a cotransferred combinatorial $\mathcal{V}$-model structure along the forgetful $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$.

Proof. We verify the conditions of the enriched cotransfer theorem 3.9. As established in the previous section, the forgetful $\mathcal{V}$-functor is a $\mathcal{V}$-cocontinuous $\mathcal{V}$-functor between locally presentable $\mathcal{V}$-categories. It remains to construct the pair (Q, q) (abusing notation we denote them by the same letters). Given a coalgebra $(X, X \to T_nX)$ over T the functor Q maps it to $(QX, QX \to QT_nX \to T_nQX)$, where the last map comes from the commutator. The latter triple is a coalgebra over T by definition of a commutator. The natural transformation $q\colon Q \to \mathrm{id}$ on $\mathcal{C}$ is induced by the corresponding natural transformation on $\mathcal{D}$. Its components are morphisms of coalgebras again by definition of a commutator. By construction, FQ lands in cofibrant objects of $\mathcal{D}$ and $F(q)$ is a natural weak equivalence in $\mathcal{D}$. ▮

6 The lax comonad of a cooperad

In this section we examine lax comonads induced by cooperads in a symmetric monoidal category. See Ching [Monoid] for a detailed description of the dual case (operads). See also Gambino and Joyal [Colored] for a thorough exposition of colored operads. A brief overview can be found in §9.1 of the author and Scholbach's [Operads].

Definition 6.1. Given a finite sequence $(V_0, \ldots, V_m)$ of sets of colors, the symmetric monoidal groupoid Forest_V of *V-forests* has as objects chains $I_0 \to \cdots \to I_m$ of maps of finite sets together with maps $I_k \to V_k$ for all $0 \le k \le m$. Isomorphisms from I to I' are families of bijections $I_k \to I'_k$ that make the corresponding triangles commute. The symmetric monoidal structure is given by the componentwise disjoint union: $(I \otimes I')_k = I_k \sqcup I'_k$. We refer to the full (nonmonoidal) subgroupoid of objects for which I_m is a singleton as *V-trees*. If $m = 1$ we talk about the symmetric monoidal groupoid of *V-multicorollas* and the groupoid of *V-corollas* respectively. If $m = 0$ we talk about *V-multicolors* and *V-colors* respectively, the latter is just the discrete groupoid on V.

Remark 6.2. The canonical inclusion of V-trees into V-forests exhibits the target as the free symmetric monoidal groupoid on the source. In particular, the category of strong monoidal functors from V-forests to $\mathcal{V}$ is canonically equivalent to the category of functors from V-trees to $\mathcal{V}$ via the restriction functor. We refer to these equivalent categories as the category of *$\mathcal{V}$-valued V-collections* and denote them by $\mathrm{Coll}_V(\mathcal{V})$.

Definition 6.3. The lax bicategory of *$\mathcal{V}$-valued colored collections* has sets as objects. The category of morphisms from V_0 to V_1 is the category $\mathrm{Coll}_{V_0,V_1}(\mathcal{V})$ of $\mathcal{V}$-valued (V_0, V_1)-collections. The composition morphism associated to a sequence of objects V is given by the *decomposition product* $\mathrm{Coll}_{V_{m-1},V_m} \times \cdots \times \mathrm{Coll}_{V_0,V_1} \to \mathrm{Coll}_{V_0,V_m}$ computed as the composition of the functor $\mathrm{Coll}_{V_{m-1},V_m} \times \cdots \times \mathrm{Coll}_{V_0,V_1} \to \mathrm{Coll}_V$ induced by the functor $\mathrm{Tree}_V \to \mathrm{Cor}_{V_{n-1},V_n} \times \cdots \times \mathrm{Cor}_{V_0,V_1}$ and the right Kan extension functor $\mathrm{Coll}_V(\mathcal{V}) \to \mathrm{Coll}_{V_0,V_n}$ induced by the functor $\mathrm{Tree}_V \to \mathrm{Cor}_{V_0,V_n}$ that discards the intermediate components of a V-tree.

Proposition 6.4. Given a colored cooperad O its lax comonad can be computed as explained in the proof.

Proof. The comonad T_O induced by the cooperad O sends an object $D \in D^W$ to the family

$$v \mapsto \lim_{\bar{w} \in W^*} O_{\bar{w},v} \otimes \bigotimes_i D_{\bar{w}_i},$$

where W^* denotes the groupoid of sequences in W (objects are elements of the free monoid on W, i.e., finite sequences of elements in W, and (iso)morphisms are permutations that turn the source into target). We remark that a limit over W^* can be computed as the product over isomorphism classes in W^*, where for a class represented by some sequence w we take the $\mathrm{Aut}(w)$-fixed points of the value of the diagram on w, where $\mathrm{Aut}(w)$ is itself the product of symmetric groups of cardinalities equal to the number of indices in w with the given value. The counit of T_O is the natural transformation $TD \to D$ that projects to the component $O_{(v),v} \otimes D_v$ indexed by $\bar{w} = (v)$ and then applies the counit map $O_{(v),v} \to 1$ of the colored cooperad O. The comultiplication of T_O is the natural transformation $TD \to TTD = (v \mapsto \lim_{\bar{w} \in W^*} O_{\bar{w},v} \otimes \bigotimes_i \lim_{\bar{x} \in W^*} O_{\bar{x},\bar{w}_i} \otimes \bigotimes_j D_{\bar{x}_j})$ whose component indexed by v is the map $\lim_{\bar{y} \in W^*} O_{\bar{y},v} \otimes \bigotimes_i D_{\bar{y}_i} \to \lim_{\bar{w} \in W^*} O_{\bar{w},v} \otimes \bigotimes_i \lim_{\bar{x}_i \in W^*} O_{\bar{x}_i,\bar{w}_i} \otimes \bigotimes_j D_{\bar{x}_{i,j}})$ (we renamed the indexing variable on the left to avoid collisions and added an index to $\bar{x}$ for convenience). ∎

The monad induced by an operad in a closed symmetric monoidal category is finitely accessible, in fact, it preserves sifted colimits because the free functor is a colimit of terms of the form $O_n \otimes X^{\otimes n}$, and the latter expression preserves sifted colimits in X. Filtered colimits do not in general commute with infinite products, so one cannot expect the lax comonad of a cooperad to be finitely accessible. However, things improve once one starts looking at λ-filtered colimits for higher λ.

Proposition 6.5. The lax comonad induced by a W-colored cooperad in a closed symmetric monoidal locally λ-presentable category is κ-accessible, where κ is uncountable, $\kappa \ge \lambda$, and $\kappa > |W|$.

Proof. The monoidal product with a fixed object as well as invariants under the action of a finite group both preserve κ-filtered colimits for any κ, so it is sufficient to ensure that the infinite products used in the definition of the lax comonad of a cooperad also preserve κ-filtered colimits. These infinite products are indexed by the set V of isomorphism classes of W-colored trees of some fixed depth. The set V has the same cardinality as W if W is infinite, or the countable cardinality for a finite nonempty W.

Recall that in a λ-presentable category $\mathcal{D}$ small products distribute over λ-filtered colimits, i.e., for any small family $F\colon I \to \mathrm{Fun}(J_i, \mathcal{D})$ of λ-filtered diagrams in $\mathcal{D}$ (here I is a discrete set and J is a function from I to the (large) category of λ-filtered categories) the canonical morphism

$$\operatorname*{colim}_{\bar{\jmath}\in\prod_i J_i} \prod_i F(i)(\bar{\jmath}(i)) \to \prod_i \operatorname*{colim}_{j\in J_i} F(i)(j)$$

is an isomorphism. This is seen directly for sets, whereas the case of an arbitrary locally λ-presentable category is reduced to that of sets by taking homs from an arbitrary λ-small object, commuting it past products and colimits (the category $\prod_i J_i$ is λ-filtered), and observing that isomorphisms in a locally λ-presentable category can be detected by corepresentable functors of λ-small objects.

Thus to establish κ-accessibility for $\kappa \geq \lambda$ it suffices to make the diagonal functor $J \to J^V$ cofinal so that the J^V-indexed colimit can be replaced by the J-indexed colimit, which implies κ-accessibility. The cofinality follows as soon as any V-indexed family of elements in J has an upper bound, which is true whenever J is μ-filtered, where μ is the successor of the cardinality of V. Thus the accessibility index of the induced comonad is bounded from above by the maximum of the accessibility index of $\mathcal{D}$ and the successor of the cardinality of V. ∎

7 Enriched cooperadic cotransfer

The conditions on the comonad imposed by the above theorem can be verified with relative ease when the comonad is induced by a colored cooperad. Recall that colored cooperads can be defined as colored operads in the opposite category.

Proposition 7.1.

Proof. Similarly, the natural transformation $E\colon QT \to TQ$ is constructed by expanding the definition of T_O and defining the invidual components as the compositions $Q\left(O_{\bar{w},v} \otimes \bigotimes_i D_{\bar{w}_i}\right) \to QO_{\bar{w},v} \otimes \bigotimes_i QD_{\bar{w}_i} \to O_{\bar{w},v} \otimes \bigotimes_i QD_{\bar{w}_i}$, where the first map comes from the oplax monoidal structure on Q and the second map is induced by $q_{O_{\bar{w},v}}$. ∎

Enriched cooperadic admissibility criterion 7.2. Suppose $\mathcal{V}$, $\mathcal{D}$, Q, and q are as in the above two theorems and $\mathcal{D}$ is equipped with a symmetric monoidal $\mathcal{V}$-enriched model structure. If Q is oplax symmetric monoidal, then for any W-colored cooperad O in $\mathcal{D}$ the category of coalgebras over O in $\mathcal{D}$ admits a cotransferred model structure.

Proof. All properties of $\mathcal{D}$ under consideration immediately imply the same properties for $\mathcal{D}^W$. ∎

8 Oplax monoidal cofibrant replacement functors

Proposition 8.1. A symmetric monoidal model category whose objects are cofibrant admits an oplax monoidal cofibrant replacement, namely, the identity functor with the identity natural transformation.

Examples 8.2. The cartesian model categories of simplicial sets, simplicial presheaves with the injective model structure.

Proposition 8.3. If $\mathcal{C}$ is a symmetric monoidal category and $\mathcal{D}$ is a combinatorial symmetric monoidal model category with an oplax monoidal cofibrant replacement (Q, q), then the projective model structure on $\mathcal{D}^{\mathcal{C}}$ is a symmetric monoidal model category (via Day convolution) with a componentwise oplax monoidal cofibrant replacement.

Proof. Generating (acyclic) cofibrations of $\mathcal{D}^{\mathcal{C}}$ can be obtained by tensoring corepresentable functors of objects of $\mathcal{C}$ with generating (acyclic) cofibrations of $\mathcal{D}$. This immediately implies the pushout product axiom. Likewise, the unit axiom of $\mathcal{D}$ implies the unit axiom for $\mathcal{D}^{\mathcal{C}}$. Given an oplax monoidal cofibrant replacement (Q, q) for $\mathcal{D}$, we construct one for $\mathcal{D}^{\mathcal{C}}$ as follows. ∎

Example 8.4. If $\mathcal{D}$ is a combinatorial symmetric monoidal model category with an oplax monoidal cofibrant replacement (Q, q) such that Q is also oplax monoidal with respect to coproducts, then the combinatorial symmetric monoidal model category of symmetric sequences in $\mathcal{D}$ admits an oplax monoidal cofibrant replacement.

Proposition 8.5. Suppose $F: \mathcal{C} \to \mathcal{D}$ is a strong monoidal left Quillen functor between combinatorial symmetric monoidal model categories such that the model structure on $\mathcal{D}$ is transferred along the right adjoint G of F.

Proof. We construct the functor Q as the composition of the bar construction functor and the weighted colimit functor with respect to a cofibrant replacement W of the constant weight on the monoidal unit. We require W to be equipped with a structure of a (componentwise) cocommutative comonoid, which gives us a symmetric oplax monoidal structure on the W-weighted colimit functor. The bar construction is oplax monoidal: ∎

We also state the trivial case of left Bousfield localizations, which, however, has nice consequences.

Proposition 8.6. If a symmetric monoidal model category admits an oplax monoidal cofibrant replacement functor, then so does any of its monoidal left Bousfield localizations.

Example 8.7. Model categories of symmetric spectra are obtained as left Bousfield localizations of R-modules, where R is a commutative monoid in symmetric sequences in some combinatorial symmetric monoidal model category $\mathcal{D}$. Typically, R is taken to be the free commutative monoid on a symmetric sequence concentrated in degree 1, where it is given by some object in $\mathcal{D}$. Thus symmetric R-spectra in $\mathcal{D}$ admit an oplax monoidal cofibrant replacement as soon as $\mathcal{D}$ does. In particular, simplicial symmetric spectra and motivic symmetric spectra admit such a replacement.

9 References

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