Brown representability via concordance

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Abstract. We prove that concordance classes of fully extended functorial field theories in the sense of Segal, Atiyah, and Freed are representable by a (unique) classifying space. Our proof relies on a refined version of the Brown representability theorem: given a sheaf of spaces on the site of smooth manifolds, we show that the concordance classes extracted from this sheaf are representable by homotopy classes of maps into a unique classifying space. As an added benefit, we can rederive in one line a large variety of classical representability results: de Rham cohomology, singular cohomology, vector bundles, K-theory, Chern character as a morphism of E-infinity ring spectra, Quinn's model for cobordism, equivariant de Rham theory and equivariant K-theory, Haefliger structures, etc.

Introduction

The classical Brown representability theorem shows that a contravariant functor from the homotopy category of pointed CW-complexes to the category of pointed sets is representable if and only if it is half-exact, i.e., sends coproducts to products and homotopy pushouts to weak pullbacks (meaning only the existence condition for pullbacks is satisfied). Heller established a variant of this result: a contravariant functor from the homotopy category of CW-complexes to the category of sets that admits a group structure is representable if and only if it is half-exact. The representing space is always unique up to a homotopy equivalence, and we refer to it as the *classifying space*.

Although this theorem is a powerful result, it suffers from a substantial defect: the proof of representability uses an inductive cellular construction, so there is no explicit formula to compute the classifying space. For instance, if we apply the theorem to the functor that sends a connected pointed CW-complex X to the pointed set of isomorphism classes of n-dimensional vector bundles over X, there is no simple way to deduce that the resulting classifying space is weakly equivalent to $\mathrm{BO}(n)$. In contrast, our result gives a simple formula for the classifying space that immediately yields $\mathrm{BO}(n)$ in this situation.

Another disadvantage is that algebraic structures on the values of the functor do not immediately give rise to a corresponding algebraic structure on the classifying space. For instance, finite-dimensional vector bundles over X form a groupoid equipped with two monoidal structures (direct sum and tensor product) that commute up to a coherent homotopy. We refer to such a structure as an E_∞*-rig* (i.e., ring without negative elements). However, the Brown representability theorem does not allow us to deduce that the classifying space $\coprod_{n\geq 0}\mathrm{BO}(n)$ is also an E_∞-rig. In contrast, our result shows that any algebraic structure present on the values of the original functor is inherited by its classifying space. In particular, this immediately equips $\coprod_{n\geq 0}\mathrm{BO}(n)$ with a structure of an E_∞-rig. The range of allowed algebraic structures is essentially arbitrary and includes spaces, ∞-groups, ∞-monoids, E_∞-rigs and E_∞-rings, connective spectra, algebras over topological or simplicial operads, connective modules over connective ring spectra, E_∞-spaces, connective chain complexes, connective dg-modules over a connective dga. We remark that connectivity assumptions are essential here, without them the result fails, as explained in Remark 13.3.

A third important point of distinction concerns the domain of the functor. Just like Heller, we do not require our spaces to be pointed or connected. More importantly, the requirement that the functor is defined for all CW-complexes and continuous maps between them is often too restrictive. For instance, one would like to show that the de Rham cohomology is representable by substituting the functor that sends a smooth manifold X to the nth de Rham cohomology of X, an abelian group. However, we need X to be a smooth manifold for this. Kreck and Singhof in [HCTM] established a variant of Brown's representability tbeorem in the context of stable homotopy theory: any sequence of contravariant functors from the homotopy category of manifolds to abelian groups equipped with a sequence of Mayer-Vietoris connecting homomorphisms is induced by a unique (up to a weak equivalence) spectrum. In their result manifolds (and their maps) can be either smooth, PL, or topological. Our setup is similar, but we work in the context of unstable homotopy theory, which, however, includes connective spectra, as explained above.

In order to allow for the above improvements, we must alter the nature of the input data supplied to the Brown representability theorem. Consider the following examples of inputs to the Brown representability theorem:

- a smooth manifold X is sent to the nth de Rham cohomology of X, i.e., closed differential n-forms on X modulo exact forms;
- a space X is sent to the nth singular cohomology of X, i.e., closed singular n-cochains modulo exact n-cochains;
- a smooth manifold X is sent to the set of equivalence classes of vector bundles with connection modulo isomorphisms that ignore the connections;

In all cases the resulting set is constructed as a set of equivalences classes of certain objects. Although it might not be immediately obvious, all three equivalences relations can be described in a uniform abstract fashion as *concordance*. Given a sheaf F (of sets or spaces) on the site of smooth manifolds, two sections a and b of F over X (i.e., elements or points in $F(X)$) are *concordant* if there is a section c of F over $X\times\mathbf{R}$ whose pullbacks to $X\times 0$ and $X\times 1$ are a and b respectively. For instance, two closed differential n-forms a and b on a smooth manifold X are concordant if there is a closed differential n-form c on $X\times\mathbf{R}$ whose pullbacks to $X\times 0$ and $X\times 1$ are equal to a respectively b. As shown in Proposition 1.2, this is equivalent to requiring that the form $a-b$ is exact. In the same vein, two vector bundles V and W with connections over X are concordant if there is a vector bundle with connection over $X\times\mathbf{R}$ whose fibers over $X \times 0$ and $X \times 1$ are isomorphic (via a connection-preserving map) to V and W, and such an equivalence amounts to V and W becoming isomorphic after we discard the connections.

We will revisit these examples in more detail in §1.1 and §3.1. Now we state the main definition and theorem.

maindef **Definition 0.1.** Suppose $F\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ is a simplicial presheaf. We define

$$\mathcal{C}F := (X \mapsto \operatorname{hocolim}_{n\in\Delta^{\mathrm{op}}} F(\mathbf{\Delta}^n \times X))$$

for a smooth manifold X. The simplicial set $\mathfrak{C}F := \mathcal{C}F(\mathrm{pt})$ is the *concordance space* of F. We have a natural morphism

$$\mathcal{C}F(X) \to \mathrm{Map}(\mathfrak{C}X, \mathfrak{C}F),$$

see Definition 9.3 for details. (Here $\mathfrak{C}X \simeq \mathrm{Sing}(X)$, so $\mathrm{Map}(\mathfrak{C}X, \mathfrak{C}F) \simeq \mathrm{Sing}\,\mathrm{Map}(X, |\mathfrak{C}F|)$, where the right Map is for topological spaces.)

mainthm **Theorem 0.2.** Suppose $F\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ is a simplicial presheaf that satisfies the homotopy descent property. The natural morphism

$$\mathcal{C}F(X) \to \mathrm{Map}(\mathfrak{C}X, \mathfrak{C}F)$$

is a weak equivalence. Used in I.5*, 3.17*, 7.5*.

Remark 0.3. If we take the induced map on the connected components, the left side $\mathcal{C}F(X)$ becomes $F[X]$, the set of concordance classes of sections of F over X, whereas the right side becomes $[X, \mathfrak{C}F]$, the set of homotopy classes of maps from X to $\mathfrak{C}F$. Madsen and Weiss in [Mumford] prove this result from the special case of sheaves of sets. We generalize their result by allowing sheaves of spaces instead of sets, and by considering the mapping space instead of the set of homotopy classes of maps.

Remark 0.4. As mentioned above, one can replace the target category of simplicial sets by any model category of algebraic nature, see Theorem 13.1 for details. Bunke, Nikolaus, and Völkl in [DiffSpec] prove this statement when X is compact and the target is a stable ∞-category. We do not allow arbitrary stable ∞-categories as a target, and indeed, for noncompact X the main theorem is false in such generality, as explained in Remark 13.3.

Remark 0.5. Similarly, the source category of smooth manifolds can be replaced by topological or PL-manifolds, CW-complexes or polyhedra, or even arbitrary topological spaces if we use numerable covers, see §12.

We emphasize that Theorem 0.2 not only establishes representability, like the Brown representability theorem, but also gives a convenient explicit formula to compute $\mathfrak{C}F$, namely $\mathfrak{C}F = \operatorname{hocolim}_{n\in\Delta^{\mathrm{op}}} F(\mathbf{\Delta}^n)$. This formula enables us to establish a relation between differential-geometric constructions (such as differential forms and connections) and homotopy-theoretic constructions (such as Eilenberg—MacLane spaces and topological K-theory spectra), and allows us to easily prove (or reprove in classical examples) various representability results, as demonstrated by numerous examples in the first part of this article.

Outline 0.6. The part is divided into two parts. The first part uses the main theorem as a black box to discuss a variety of applications, including de Rham cohomology in all of its incarnations, singular cohomology, various flavors of K-theory, cobordism, as well as the more exotic examples of Haefliger structures, factorization algebras, etc. We also discuss the original motiviation for this paper, the representability of conordance classes of (nontopological) quantum field theories. The second part contains the proof of main theorem. Our tools are sheaf- and homotopy-theoretic, and depend on a variety of tools, including homotopy (co)limits, explicit descripting of weak equivalence using sphere fillings, partitions of unity, etc.

Prerequisites 0.7. We assume familiarity with basic homotopy theory, including simplicial sets and their homotopy (co)limits, Quillen model categories, and elementary theory of smooth manifolds, including smooth partitions of unity. Some familiarity with homotopy descent might be helpful, but we review all the necessary definitions.

Notation and conventions 0.8. We use the prefix ∞- or the adjectives "homotopy" or "derived" to refer to homotopy-invariant constructions in any model for ∞-categories, e.g., model categories. All ∞-categories used in this paper are presentable, in particular, they can be (and are) presented by combinatorial model categories. However, except for examples our proofs are model-independent, so any other model can be used, such as quasicategories or simplicial categories. Throughout the paper we work extensively with functors of the form $\mathrm{Man}^{\mathrm{op}} \to T$, where Man is a small site of manifolds. Except for Theorem 13.1, our target category is the category of *homotopy spaces*, which we present using simplicial sets with the Kan—Quillen model structure. When we want to emphasize that we are working with the strict 1-categorical model, we refer to such functors as *presheaves*. On the other hand, when we want to emphasize that we are working with them in a model-independent fashion, we use the term *prestack*. Finally, by a *stack* we mean a prestack that satisfies the homotopy descent condition. One can use other models for prestacks, such as Grothendieck fibrations in simplicial sets over Man, in place of presheaves, see Heuts and Moerdijk [LeftFib] for an overview of the relevant model structure and two different Quillen equivalences to simplicial presheaves. The category of simplicial spaces with weak equivalences given by maps whose realization is a weak equivalence of spaces is denoted sSp.

Examples and applications

1 Differential forms

forms 1.1. ***Closed differential forms and the de Rham theorem*** Used in I.0*.

This section considers one of the most elementary examples of an input to the main theorem, the sheaf of closed differential n-forms for some $n \geq 0$.

We start our analysis by identifying the concordance relation on closed n-forms.

concforms **Proposition 1.2.** Two closed differential n-forms on a smooth manifold X are concordant if and only if they differ by an exact form. Hence, concordances classes of closed differential n-forms on X are in bijection with the nth de Rham cohomology of X. Used in I.0*, 1.3*, 3.17*.

Proof. Given closed differential n-forms ψ and ω on X, together with a concordance c, which is a closed n-form on $X \times \mathbf{R}$, we construct an $(n-1)$-form χ on X such that $d\chi = \psi - \omega$. We set $\chi = \int_{X\times[0,1]\to X} c$, where the integrand c is restricted to $X \times [0,1] \subset X \times \mathbf{R}$. By the fiberwise Stokes theorem (see Greub, Halperin, and van Stone [CCC], Chapter VII, Problem 4(iii)), we have $d\chi = \int_{X\times[0,1]\to X} dc + (-1)^{n+1}(-\psi + \omega)$. Since $dc = 0$, the first term vanishes, and adjusting the sign of χ as necessary, we see that $d\chi = \psi - \omega$. In the other direction, if we are given χ, then we can construct c as the form $(1-t)p^*\psi + tp^*\omega - dt \wedge p^*\chi$. The first two terms interpolate between ψ and ω, ensuring that we have a concordance of desired type. The last term pulls back to zero when restricted to $X \times 0$ or $X \times 1$ and ensures that the form is closed. ∎

deRhamEM **Proposition 1.3.** For any $n \geq 0$ there is a canonical weak equivalence $\mathfrak{C}\Omega^n_{\mathrm{cl}} \to \mathrm{K}(\mathbf{R}, n)$.

Proof. Eilenberg—MacLane spaces are uniquely determined by the property of having a unique nonvanishing homotopy group. The homotopy groups of $\mathfrak{C}\Omega^n_{\mathrm{cl}}$ can be computed using Proposition 14.1 as the pointed concordance classes of closed differential n-forms over $\mathfrak{S}^k$. A minor subtlety arises from the fact that concordance classes are pointed, which in the only nontrivial case $n = 1$ amounts to having an additional concordance from the zero 1-form and the pullback of the 1-form on $\mathfrak{S}^k$ to some fixed basepoint. However, any two closed 1-forms on $\mathbf{\Delta}^1$ are concordant, so the 1-form part can be thrown out. By Proposition 1.2, concordance classes of closed n-forms on $\mathfrak{S}^k$ for $n > 0$ are in bijection with the nth de Rham cohomology of $\mathfrak{S}^k$, which completes the proof in this case. For $n = 0$ we see directly that $\mathfrak{C}\Omega^0_{\mathrm{cl}} = \mathbf{R}$ as a discrete abelian group. ∎

Corollary 1.4. The nth de Rham cohomology is representable by $\mathrm{K}(\mathbf{R}, n)$, i.e., is canonically isomorphic to the nth ordinary cohomology with real coefficients.

1.5. ***Weil's de Rham descent theorem***

In this section we elaborate on the previous example and show how the machinery of concordances allows us to recover Weil's de Rham descent theorem [deRham], which states that the prestack Ω that sends a smooth manifold S to the real cochain complex $\Omega^\bullet(S)$ of smooth differential forms on S is a stack.

The target category T is the category of nonnegative real chain complexes equipped with quasi-isomorphisms. (Of course, one can obtain statements for simplicial sets or spectra by applying the Dold—Kan or Eilenberg—MacLane functors, both of which preserve homotopy limits.) For this choice of target, $\mathcal{C}\Omega^n_{\mathrm{cl}}(S)$ can be computed (by presenting the realization as the totalization of a bicomplex, for example) as the complex $\Omega^n_{\mathrm{cl}}(S) \leftarrow \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^1) \leftarrow \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^2) \leftarrow \cdots$ with the differentials given by alternating sum of restrictions to individual faces.

coforms **Proposition 1.6.** There is a canonical weak equivalence of prestacks $\mathcal{C}\Omega^n_{\mathrm{cl}} \to \Omega^{\leq n}$, where $\Omega^{\leq n}(S) = \Omega^n_{\mathrm{cl}}(S) \leftarrow \Omega^{n-1}(S) \leftarrow \cdots \leftarrow \Omega^0(S)$. Used in 1.12*, 4.5*.

Remark 1.7. An interesting observation to be made here is that this construction recovers the chain complex of differential forms from *closed* differential forms and their concordances.

Proof. The morphism $\Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k) \to \Omega^{n-k}(S)$ is given by the fiberwise integration (pushforward) with respect to the projection $S \times \mathbf{\Delta}^k \to S$, where the integral is taken over the compact interiors (affine coordinates are nonnegative) of $\mathbf{\Delta}^k$. For $k = 0$ nothing happens, so we indeed land in closed forms in this case. That this is a chain map is precisely the content of the fiberwise Stokes theorem for the case of smooth simplices. The map $\Omega^{n-k}(S) \to \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k)$ in the other direction is constructed by pulling back along $S \times \mathbf{\Delta}^k \to S$ and taking the product with the translation invariant closed form on $\mathbf{\Delta}^k$ that integrates to 1.

By construction, the composition $\Omega^{n-k}(S) \to \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k) \to \Omega^{n-k}(S)$ is the identity map. The other composition $\Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k) \to \Omega^{n-k}(S) \to \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k)$ is homotopic to the identity map via the chain homotopy $\Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k) \to \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^{k+1})$ that can be constructed explicitly. The chain homotopy condition requires the sum of the compositions of $\Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k) \to \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^{k+1}) \to \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k)$ and $\Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k) \to \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^{k-1}) \to \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k)$ to be equal to the difference of the identity map and the composition $\Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k) \to \Omega^{n-k}(S) \to \Omega^n_{\mathrm{cl}}(S \times \mathbf{\Delta}^k)$. ∎

Corollary 1.8. The prestack $\Omega^{\le n}$ is a stack.

Proof. The previous proposition proved that $\Omega^{\le n}$ is weakly equivalent to $\mathcal{C}\Omega^n_{\mathrm{cl}}$, our main theorem says that the functor $\mathcal{C}$ preserves stacks, and Ω^n_{cl} is a sheaf of sets, hence also a stack of nonnegative chain complexes concentrated in degree 0. ∎

CechdeRham **Corollary 1.9.** (Weil's Čech—de Rham descent theorem.) The prestack $\Omega = (S \mapsto \Omega^0(S) \to \Omega^1(S) \to \cdots)$ of real *cochain* complexes is a stack. Used in 1.12*.

Proof. This follows immediately from the previous corollary: take the homotopy truncation $\Omega^{\le n}$ of Ω and use the fact that a morphism of real cochain complexes is a quasiisomorphism if and only if its homotopy truncation above any level is a quasiisomorphism of *chain* complexes (with the reversed grading) and the fact that the homotopy truncation functor at any level preserves homotopy limits. ∎

Recall that the Eilenberg—MacLane functor H can be upgraded to a functor (a zigzag of Quillen equivalences) from cdgas (commutative differential graded algebras) to $\mathrm{H}\mathbf{R}$-algebras, i.e., E_∞-ring spectra under the E_∞-ring spectrum $\mathrm{H}\mathbf{R}$.

EideRham **Corollary 1.10.** (E_∞ de Rham theorem.) For a smooth manifold S there is a natural weak equivalence of E_∞-algebras over $\mathrm{H}\mathbf{R}$

$$\mathrm{H}(\Omega(S)) \to \mathrm{Hom}(\Sigma^\infty \mathfrak{C}S, \mathrm{H}\mathbf{R}),$$

where $\mathrm{H}(\Omega(S))$ is the Eilenberg—MacLane E_∞-ring spectrum of the real cdga of differential forms on S. Equivalently, there is a natural weak equivalence of real cdgas

$$\Omega(S) \to \mathrm{Hom}(\mathfrak{C}S, \mathbf{R}),$$

where the latter term denotes the cdga of real singular cochains on S. Used in 1.12*.

Proof. The forgetful functor from real cdgas to real cochain complexes creates (i.e., preserves and reflects) homotopy limits. Thus $S \mapsto \Omega(S)$ is a stack of real cdgas because its underlying prestack of real cochain complexes satisfies descent by the above corollary. Furthermore, Ω is concordance-invariant. The representability of concordance-invariant stacks immediately implies the desired result. ∎

eqdeRham 1.11. ***Equivariant de Rham cohomology*** Used in 3.19*.

The purpose of this section is to illustrate that the main theorem produces interesting results not only for representable stacks (i.e., manifolds), but also for other stacks, e.g., quotients by group actions.

Consider a smooth manifold X with an action of a Lie group G. Apart from the traditional stacky quotient $X/\!\!/G$ we also consider the *connection quotient* $X/\!\!/^\nabla G$, defined as $(\mathrm{TBun}^\nabla_G \times X)/\!\!/G$. There is a canonical map $X/\!\!/^\nabla G \to X/\!\!/G$, which becomes a weak equivalence after applying $\mathfrak{C}$, as follows from the homotopy cocontinuity of $\mathfrak{C}$ and the contractibility of $\mathfrak{C}\mathrm{TBun}^\nabla_G$.

In the terminology of Definition 7.22 in Freed and Hopkins [CWFAHT], the stack $X/\!\!/^\nabla G$ is called the *simplicial Borel quotient* and is denoted $(X_G)_\nabla$.

Theorem 7.28 in Freed and Hopkins [CWFAHT] shows that $\mathrm{Map}(X/\!\!/^\nabla G, \Omega)$ is quasiisomorphic (or even isomorphic, assuming the mapping space is computed individually for each degree) to the Weil model for the G-equivariant cohomology of X, i.e., the basic subcomplex of $\Omega(X, \mathrm{Kos}\, g^*)$ with the differential $d_X + d_K$, with d_X being the de Rham differential and d_K the Koszul differential defined below. Here $\mathrm{Kos}\, g^* = \Lambda g^* \otimes \mathrm{Sym}\, g^*$ is the Koszul complex of g^*, with the differential $d_K(\Lambda^1 g^*) = \mathrm{Sym}^1 g^*$ and $d_K(\mathrm{Sym}^1 g^*) = 0$. A *basic* form is a G-invariant form that vanishes under the substitution of any vector field coming from g.

The equivariant de Rham theorem (see, for example, Theorem 2.5.1 in Guillemin and Sternberg [SSE-dRT]) states that the basic subcomplex is weakly equivalent to the mapping space $\mathrm{Map}(X^h_G, \mathrm{H}\mathbf{R})$. (One either has to apply the Dold—Kan functor to the basic subcomplex or its inverse to the mapping space, so that the weak equivalence makes sense.) Here X^h_G denotes the homotopy quotient of X by G, as a space, which we know is weakly equivalent to $\mathfrak{C}(X/\!\!/G$.

Combined together, these two results tell us that the real cochain cdga $\mathrm{Map}(X/\!\!/^{\nabla}G, \Omega)$ is weakly equivalent to $\mathrm{Map}(X^h_G, \mathrm{H}\mathbf{R})$. (Equivalently, we can work with E_∞-algebras over $\mathrm{H}\mathbf{R}$.) Here we offer a simple proof of this result that does not rely on the above theorems.

Equivariant de Rham theorem 1.12. For any smooth manifold X with an action of a Lie group G, there is a natural weak equivalence of E_∞-algebras over $\mathrm{H}\mathbf{R}$

$$\mathrm{H}(\mathrm{Map}(X/\!\!/^{\nabla}G, \Omega)) \to \mathrm{Hom}(\Sigma^\infty X^h_G, \mathrm{H}\mathbf{R}).$$

Equivalently, there is a natural weak equivalence of real cochain cdgas

$$\mathrm{Map}(X/\!\!/^{\nabla}G, \Omega) \to \mathrm{Hom}(X^h_G, \mathbf{R}),$$

where the latter term denotes the cdga of real singular cochains on X^h_G.

Proof. The same argument as in 1.10 (the functor that forgets E_∞-structures creates homotopy limits) allows us to get rid of E_∞-structures, and the same argument as in 1.9 (quasiisomorphisms of cochain complexes are detected on their homotopy truncations in all degrees) allows us to further reduce the problem to the case of Ω^n_{cl} and $\mathrm{K}(\mathbf{R}, n)$ instead of Ω and $\mathrm{H}\mathbf{R}$. Using Corollary , we deduce that the canonical map $\mathrm{Map}(X/\!\!/^{\nabla}G, \mathcal{C}\Omega^n_{\mathrm{cl}}) \to \mathrm{Map}(\mathfrak{C}(X/\!\!/^{\nabla}G), \mathfrak{C}\Omega^n_{\mathrm{cl}})$ is a weak equivalence. Finally, $\mathcal{C}\Omega^n_{\mathrm{cl}}$ is weakly equivalent to $\Omega^{\le n}$ (by 1.6) and $\mathfrak{C}(X/\!\!/^{\nabla}G)$ is weakly equivalent to X^h_G, which completes the proof. ∎

Remark 1.13. The above construction can be used to *define* the equivariant de Rham complex even if one knows nothing about the Weil model.

2 Singular cohomology

In this section we explain how singular cohomology can be recovered from a certain presheaf of abelian groups. The construction resembles Voevodsky's *presheaves with transfers*. The smooth manifold version of this idea was developed by Michael Weiss under the name of *mapping cycles* in [HWS].

We start by constructing a category MC enriched in abelian groups, whose objects are manifolds and $\mathrm{MC}(M, N)$ is the abelian group given by the evaluation at M of the sheafification of the presheaf of abelian groups on M that sends an open subset U of M to the free abelian group on the set of maps from U to N. Concretely, a mapping cycle from M to N is given by an open cover U of M and a compatible U-family of formal sums (with signs) of maps of the form $U_i \to N$.

The identity map $\mathbf{Z} \to \mathrm{Map}(M, M)$ sends 1 to the identity map $M \to M$ (for the singleton cover of M). The composition $\mathrm{Hom}(M, N) \otimes \mathrm{Hom}(L, M) \to \mathrm{Hom}(L, N)$ is the unique bilinear map canonically extended from the generators of the corresponding presheaves, where it is defined as the composition of maps of manifolds.

The ordinary category of manifolds admits a functor into MC (after we discard the abelian group structure) by sending a map $M \to N$ of manifolds to itself, considered now as an element of the sheafification using the singleton cover of M. The representable presheaves $\mathrm{MC}(-, N)$ are sheaves (of abelian groups) for any manifold M, i.e., the site MC equipped with the standard topology of open covers is subcanonical.

The following proposition is a tautological consequence of the formula $\mathfrak{C}F = \operatorname{diag} F(\mathbf{\Delta}^\bullet)$ combined with the fact that the presheaf used to define $\mathrm{MC}(M, N)$ is already a sheaf when M is cartesian, hence $\mathrm{MC}(M, N)$ is the free abelian group on maps $M \to N$ in this case, so substituting $M = \mathbf{\Delta}^n$ we get the nth term of the singular chain complex of N. Of course, there is nothing special about $\mathbf{Z}$ here: one can replace abelian groups with R-modules for any ring R, recovering singular R-chains.

Proposition 2.1. For any manifold N we have $\mathfrak{C}\mathrm{MC}(-, N) = \mathbf{Z}[\operatorname{Sing} N] \simeq \Omega^\infty(\mathrm{H}\mathbf{Z} \wedge N)$. In particular, $\mathcal{C}\mathrm{MC}(-, N)(M) =: \mathcal{C}\mathrm{MC}(M, N) \simeq \mathrm{Map}(M, \Omega^\infty(\mathrm{H}\mathbf{Z} \wedge N))$ is the bivariant homology-cohomology theory associated to the Eilenberg—MacLane spectrum (of the integers, or, more generally, any ring).

Corollary 2.2. For any manifold N the nth singular homology group of N is canonically isomorphic to the abelian group of pointed concordance classes of $\mathrm{MC}(-, N)$ over $\mathbf{S}^n_*$. Equivalently, $\mathrm{H}_n(N) \cong \ker(\mathrm{MC}[\mathbf{S}^n, N] \to \mathrm{MC}[\mathrm{pt}, N] = \mathrm{H}_0(N)) \cong \operatorname{coker}(\mathrm{MC}[\mathrm{pt}, N] \to \mathrm{MC}[\mathbf{S}^n, N])$, where $[-]$ as usual denotes concordance classes (with respect to the first argument in this case). Informally, $\mathrm{H}_n(N) \cong [\mathbf{S}^n, N]/[\mathrm{pt}, N]$.

Substituting $N = \mathbf{S}^n_*$ and using the fact that $\mathbf{Z}[\operatorname{Sing} \mathbf{S}^n_*] \simeq \mathrm{K}(\mathbf{Z}, n)$ is the nth Eilenberg—MacLane space, we obtain the following result.

Corollary 2.3. For any manifold M the nth singular cohomology group of M is canonically isomorphic to the abelian group of pointed concordance classes of $\mathrm{MC}(-, \mathbf{S}^n_*) = \operatorname{coker}(\mathrm{MC}(-, \mathrm{pt}) \to \mathrm{MC}(-, \mathbf{S}^n))$ over M. Informally, $\mathrm{H}^n(M) \cong [M, \mathbf{S}^n]/[M, \mathrm{pt}]$.

3 Bundles

bundles 3.1. ***Vector bundles with connection*** Used in I.0*.

In this section we work with either real or complex vector bundles. The notation $\mathrm{GL}(n)$ denotes the Lie group $\mathrm{GL}(\mathbf{R}, n)$, respectively, $\mathrm{GL}(\mathbf{C}, n)$ and $\mathrm{End}(n)$ the Lie monoid $\mathrm{End}(\mathbf{R}^n)$, respectively, $\mathrm{End}(\mathbf{C}^n)$.

We concentrate our attention on two stacks, both of which are constructed by applying the nerve functor to the strictification of some Grothendieck fibration in groupoids over manifolds. The stack Vect is constructed from the Grothendieck fibration of vector bundles and isomorphisms, whereas Vect^∇ is constructed from vector bundles with connection and connection-preserving isomorphisms.

As in the section on closed differential forms, we start by identifying the concordance relation on vector bundles, with or without connection.

Proposition 3.2. Two vector bundles are concordant if and only if they are isomorphic. Two vector bundles with connection are concordant if and only if they are isomorphic after discarding the connection.

Proof. We start by constructing concordances from isomorphisms. The trivial concordance solves the problem in the case without connections. Connections on a vector bundle form an affine space whose associated vector space is the space of connection 1-forms. Given two connections ∇_1 and ∇_2 on X, a concordance between them can be constructed as $p^*\nabla_1 + t(\nabla_2 - \nabla_1)$, where $p\colon X \times \mathbf{R} \to X$ is the projection map, the summand $p^*\nabla_1$ is a connection on $X \times \mathbf{R}$, and $t(\nabla_2 - \nabla_1)$ is a connection 1-form on $X \times \mathbf{R}$. In the opposite direction, given a concordance between two vector bundles (possibly with connection), we construct an isomorphism between them by choosing a connection if necessary, and taking the parallel transport isomorphism along the path $[0, 1]$ in each fiber. ▮

vectbun **Proposition 3.3.** There is a canonical weak equivalence $\mathfrak{C}\mathrm{Vect} \to \coprod_{n\ge0} \mathrm{BGL}(n)$. In particular, the latter space classifies isomorphism classes of vector bundles. Here $\mathrm{BGL}(n)$ denotes the classifying space of the simplicial group given by the singular simplicial set of $\mathrm{GL}(n)$, or, equivalently, the topological group $\mathrm{GL}(n)$. Used in 3.16*.

Proof. There are no morphisms between vector bundles of different dimensions, so we can work one dimension at a time. The formula for $\mathfrak{C}$ gives $\mathfrak{C}\mathrm{Vect} = \mathrm{hocolim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Vect}(\mathbf{\Delta}^n)$. We have natural weak equivalence $\mathrm{Vect}(\mathbf{\Delta}^n) \to \mathrm{BC}^\infty(\mathbf{\Delta}^n, \mathrm{GL}(n))$ (all vector bundles on $\mathbf{\Delta}^n$ are trivial). Therefore,

$$\mathfrak{C}\mathrm{Vect} \simeq \mathrm{hocolim}_{n\in\Delta^{\mathrm{op}}} \mathrm{BC}^\infty(\mathbf{\Delta}^n, \mathrm{GL}(n)).$$

Using the homotopy cocontinuity of B we transform this into $\mathrm{B}\,\mathrm{hocolim}_{n\in\Delta^{\mathrm{op}}} \mathrm{C}^\infty(\mathbf{\Delta}^n, \mathrm{GL}(n))$, and the inner homotopy colimit is nothing but the singular simplicial set of $\mathrm{GL}(n)$, which completes the proof. ▮

Remark 3.4. A purely grammatical adjustment of the above proofs shows that $\mathfrak{C}\mathrm{Bun}_G \simeq \mathrm{B}G$, where $\mathrm{B}G$ is the classifying space of G taken with its topology and Bun_G is the stack of principal G-bundles and their isomorphisms.

The case of $\mathfrak{C}\mathrm{Vect}^\nabla$ can be reduced to that of $\mathfrak{C}\mathrm{Vect}$. Both spaces turn out to be weakly equivalent, but only one of them can be used in the Chern—Weil machine to construct the Chern character, as we explain later.

vectbunconn **Proposition 3.5.** There is a weak equivalence $\mathfrak{C}\mathrm{Vect}^\nabla \simeq \coprod_{n\ge0} \mathrm{BGL}(n)$. Used in 3.16*.

Proof. It suffices to show that the forgetful morphism $\mathrm{Vect}^\nabla \to \mathrm{Vect}$ is mapped to a weak equivalence $\mathfrak{C}\mathrm{Vect}^\nabla \to \mathfrak{C}\mathrm{Vect}$ by the functor $\mathfrak{C}$. There is a weak equivalence from $\mathrm{Vect}^\nabla(\mathbf{\Delta}^k)$ to the nerve of the groupoid whose objects are connections on the trivial vector bundle with fiber $\mathbf{R}^n$ and morphisms are connection-preserving isomorphism. Thus the set of objects is $\Omega^1(\mathbf{\Delta}^k, \mathrm{End}(n))$ for all $n \ge 0$ and the set of all morphisms with a fixed source of dimension n is $\mathrm{Map}(\mathbf{\Delta}^k, \mathrm{GL}(n))$, which acts on the source 1-form by gauge transformations, i.e., $\alpha \cdot h = h^{-1}(dh) + h^{-1}\alpha h$. Consider the morphism

$$\mathrm{Vect}^\nabla(\mathbf{\Delta}^k) \to \mathrm{Vect}(\mathbf{\Delta}^k) = \mathrm{B}\,\mathrm{Map}(\mathbf{\Delta}^k, \mathrm{GL}(n))$$

that forgets the 1-form. As we vary k, the resulting morphism of simplicial objects in simplicial sets is a *homotopical acyclic Kan fibration*, meaning that it satisfies the homotopical analog of the sphere filling condition: given a square diagram of simplicial objects in simplicial sets of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Vect}^\nabla(\mathbf{\Delta}^\bullet) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathrm{B}\,\mathrm{Map}(\mathbf{\Delta}^\bullet, \mathrm{GL}(n)) \end{tikzcd}$$

together with a levelwise homotopy between the two compositions, we can find a diagonal arrow such that both triangles commute up to a levelwise homotopy and the composition of these homotopies is itself homotopic to the original homotopy. In our case this boils down to proving that any 1-form over $\partial\mathbf{\Delta}^n$ can be extended to a 1-form over $\mathbf{\Delta}^n$. Indeed, such a 1-form can be constructed by affine interpolation with bump functions (here we use the fact that $\mathrm{End}(n)$ is a real vector space). A crucial property of homotopical acyclic Kan fibrations is that their realization is a weak equivalence, see Theorem 4.4 in Mazel-Gee [MIC], which completes the proof. ▮

prinbun **Remark 3.6.** The case of principal G-bundles with connection is completely analogous. (Recall that the groupoid of n-dimensional vector bundles with connection is equivalent to the groupoid of principal $\mathrm{GL}(n)$-bundles with connection.) The relevant stacks are obtained by stackifying the corresponding prestacks of *trivializable* objects. For (trivializable) principal G-bundles we take $S \mapsto \mathrm{BC}^\infty(S, G)$, For (trivializable) flat G-bundles we take $S \mapsto \mathrm{C}^\infty_{\mathrm{lc}}(S, G)$. For (trivializable) principal G-bundles with connection we send S to the nerve of the groupoid whose objects are elements of $\Omega^1(S, g)$ (here g is the Lie algebra of G), the set of all morphisms with a given source $\alpha \in \Omega^1(S, g)$ is $\mathrm{C}^\infty(S, G)$, the target of such a morphism h is $\alpha \cdot h = h^*\theta + \mathrm{Ad}_{h^{-1}}\, \alpha$, where θ is the Maurer—Cartan form of G. (For matrix groups like $\mathrm{GL}(n)$ we recover the expression $h^{-1}(dh) + h^{-1}\alpha h$ used above.) Using an argument identical to the one in the previous paragraph (with g-valued forms in place of $\mathrm{End}(n)$-valued ones), we get a weak equivalence $\mathfrak{C}\mathrm{Bun}^\nabla_G \simeq \mathrm{B}(\mathfrak{C}G)$, where $\mathrm{B}(\mathfrak{C}G)$ is simply the usual classifying space of the Lie group G. Used in 5.0*.

3.7. *Virtual vector bundles and Simons—Sullivan structured vector bundles*

We now extend the above result to the case of virtual vector bundles, obtaining a smooth refinement of the spectra ko and ku. Fix a functorial model K for the homotopy group completion of E_∞-semirings (i.e., E_∞-monoids in E_∞-spaces with the smash product), the result being an E_∞-ring, i.e., a connective ring spectrum. For example, one can take the left derived functor of the left Quillen functor induced by the morphism RSpan → GrRSpan of simplicial algebraic theories, as defined in §8.2 of Cranch [InfAlgTh]; as explained in §6.8 there, vector spaces can be organized into an algebra over RSpan.

Definition 3.8. The prestack of simplicial sets $\mathrm{VVect}^\nabla_{\mathrm{pre}}$ sends a manifold S to the homotopy group completion of the groupoid of vector bundles with connection over S and connection-preserving isomorphisms. The stack VVect^∇ is the stackification of $\mathrm{VVect}^\nabla_{\mathrm{pre}}$. Its sections are *virtual vector bundles with connection.* The pair $(\mathrm{VVect}_{\mathrm{pre}}, \mathrm{VVect})$ is defined similarly.

Proposition 3.9. The concordance space of VVect^∇ (or VVect) is the connective E_∞-ring spectrum of real K-theory ko (respectively complex K-theory ku).

Proof. The morphism $\mathrm{VVect}^\nabla_{\mathrm{pre}}(\mathbf{\Delta}^\bullet) \to \mathrm{VVect}^\nabla(\mathbf{\Delta}^\bullet)$ induces a weak equivalence after realization. Indeed, $\mathrm{VVect}^\nabla_{\mathrm{pre}}(\mathbf{\Delta}^\bullet)$ can be computed explicitly as $\mathbf{Z} \times \mathrm{hocolim}_m \mathrm{B}\,\mathrm{Map}(\mathbf{\Delta}^\bullet, \mathrm{GL}(m)) = \mathrm{Map}(\mathbf{\Delta}^\bullet, \mathbf{Z} \times \mathrm{hocolim}_m \mathrm{BGL}(m))$, which consists of maps that *globally* factor through some finite stage of the homotopy colimit. After stackification we get maps that *locally* factor through some finite stage. The weak equivalence criterion boils down to proving that any section over $\mathbf{\Delta}^k$ whose restriction to $\partial\mathbf{\Delta}^k$ comes from some finite stage n, itself comes from some finite stage up to a concordance. Indeed, consider the open cover that defines the given section over $\mathbf{\Delta}^k$. We can choose a finite subcover of the interior part of $\mathbf{\Delta}^k$. After increasing n we may now assume that the restriction to the interior comes from a finite stage. The standard trick with a smooth retraction of $\mathbf{\Delta}^k$ onto $\partial\mathbf{\Delta}^k$ (with corners appropriately smoothened) provides the desired lift and concordance.

The homotopy group completion functor preserves homotopy colimits, so we can compute the realization of $\mathrm{VVect}^\nabla_{\mathrm{pre}}(\mathbf{\Delta}^\bullet)$ as the homotopy group completion of the realization of $\mathrm{Vect}^\nabla(\mathbf{\Delta}^\bullet)$. The latter was found to be $\coprod_{n\ge0} \mathrm{BGL}(n)$ above, and its homotopy group completion is by definition the connective E_∞-ring spectrum ko (respectively ku). ▮

Corollary 3.10. Homotopy classes of maps from a manifold X to ko (respectively ku) are in bijection with concordance classes of real (respectively complex) virtual vector bundles with (or without) connection. This correspondence preserves the abelian group structure on both sides.

Remark 3.11. There is a canonical morphism $\mathrm{VVect}^\nabla \to \pi_0\mathrm{K}(\mathrm{Struct})$, where Struct denotes the sheaf of *structured vector bundles* defined by Simons and Sullivan, who prove that the right side is the differential K-theory functor $\hat{\mathrm{K}}^0$. Thus it is reasonable to expect that VVect^∇ supplies a homotopy coherent version of this construction. Indeed, in a forthcoming work with Alex Kahle [Cocycles] we show that imposing the equivalence relation of *geometric concordance* on VVect^∇ recovers the differential K-theory *space* (as opposed to a single group). Geometric concordances are defined as concordances for which the Chern—Simons form vanishes, and the latter makes sense in the presence of any morphism of stacks; the Chern—Weil morphism recovers the ordinary Chern—Simons form.

3.12. ***Superconnections***

Quillen's *superconnections* give another model for K-theory classes that turns out to be more flexible for many geometric applications.

Definition 3.13. Denote by Ω the commutative super algebra of differential forms on X (the grading being degree mod 2) and by ΩW, where W is a $\mathbf{Z}/2$-graded (i.e., super) vector bundle the Ω-module $\Omega \otimes_{\Omega_0} W$ of W-valued differential forms on X. A *superconnection* ∇ on a super vector bundle V is an odd derivation $\nabla\colon \Omega \operatorname{End}(V) \to \Omega \operatorname{End}(V)$ of Ω-modules, where $\operatorname{End}(V)$ denotes the super algebra of endomorphisms of V.

Remark 3.14. For a fixed manifold X, the superconnections on X and their isomorphisms form a symmetric monoidal groupoid $\mathrm{SVect}^\nabla(X)$ with respect to the direct (Whitney) sum. Equipped with pullbacks, SVect^∇ is a stack of symmetric monoidal groupoids.

Remark 3.15. Superconnections on X form a torsor over the odd part of $\Omega \operatorname{End}(V)$, which we denote $(\Omega \operatorname{End}(V))^{\mathrm{odd}}$.

superconn **Proposition 3.16.** We have $\mathfrak{C}\mathrm{SVect}^\nabla \simeq \mathfrak{C}\mathrm{SVect} \simeq \coprod_{a,b\geq 0} \mathrm{BGL}(a) \times \mathrm{BGL}(b)$.

Proof. The canonical map $\mathfrak{C}\mathrm{SVect}^\nabla \to \mathfrak{C}\mathrm{SVect}$ is a weak equivalence by the same reasoning as in Proposition 3.5. The space $\mathfrak{C}\mathrm{SVect}$ is a disjoint union of its components corresponding to super vector bundles of fixed super dimension $a|b$, where $a \geq 0$ and $b \geq 0$ are arbitrary. Using the fact that over $\mathbf{\Delta}^n$ all bundles are trivial, such a component can be computed as $\operatorname{hocolim}_{n\in\Delta^{\mathrm{op}}} \mathrm{C}^\infty(\mathbf{\Delta}^n, \mathrm{GL}(\mathbf{R}^{a|b}))$. Maps from $\mathbf{\Delta}^n$ only see the reduced part of $\mathrm{GL}(\mathbf{R}^{a|b})$, which is $\mathrm{GL}(a) \times \mathrm{GL}(b)$. An argument identical to the one in Proposition 3.3 completes the proof. ▮

cwhom
3.17. ***The Chern—Weil homomorphism and Chern character***

The Chern—Weil homomorphism is a morphism of stacks of symmetric monoidal groupoids

$$\mathrm{CW}\colon \mathrm{Vect}^\nabla \to \Omega^{\mathrm{ev}}_{\mathrm{cl}}, \qquad (V,\nabla) \mapsto \exp(\operatorname{curv}(\nabla))$$

from the symmetric bimonoidal groupoid of complex vector bundles with connection equipped with the direct sum and tensor product to the abelian group of closed differential forms of (nonhomogeneous) even degree equipped with the addition and exterior multiplication. (As usual, we have a choice where to place the factor $2\pi i$ and signs; we elect to place it with the coefficients, so instead of integers we use the *Tate integers* $\mathbf{Z}(k) := (2\pi i)^k\mathbf{Z}$.) The source and target can be interpreted as stacks of E_∞-rig spaces, i.e., spaces equipped with homotopy coherent addition (without inverses) and multiplication that distribute in a homotopy coherent fashion. A convenient model is supplied by algebras over a simplicial algebraic theory, see §6.8 in Cranch [InfAlgTh]. (In our case the left side is 1-truncated and the right side is 0-truncated.) This enables us to apply Theorem 0.2 in its full algebraic generality and obtain a morphism of stacks of E_∞-rig spaces

$$\coprod_k \mathrm{BGL}(k) \simeq \mathfrak{C}\mathrm{Vect}^\nabla \to \mathfrak{C}\Omega^{\mathrm{ev}}_{\mathrm{cl}} \simeq \bigvee_n \mathrm{K}(\mathbf{R}, 2n).$$

We suppress a lot of additional structure in this formulation, for example, we could equip both sides with an action of $\mathbf{Z}/2$ (complex conjugation on the left and $(-1)^n$ acting on $\mathrm{K}(\mathbf{R},2n)$ on the right) and the map would be $\mathbf{Z}/2$-equivariant. This formulation also makes it easy to see the Chern—Simons form: a concordance of vector bundles with connections over X yields a concordance of the corresponding Chern—Weil forms, so by Proposition 1.2 specifies a Chern—Simons form whose differential equals the difference of the corresponding Chern—Weil forms. Similarly, higher concordances induce higher Chern—Simons forms.

We summarize our observations in the following result.

Proposition 3.18. The functor $\mathfrak{C}$ applied to the Chern—Weil homomorphism CW yields the (unstable) Chern character

$$\coprod_k \mathrm{BGL}(k) \to \bigvee_n \mathrm{K}(\mathbf{R}, 2n).$$

Further applying the homotopy group completion functor (see §8.2 in Cranch [InfAlgTh]) yields the Chern character map as a morphism of E_∞-ring spectra:

$$\mathrm{BU} \times \mathbf{Z} \to \bigvee_n \mathrm{K}(\mathbf{R}, 2n).$$

There is a similar diagram for $\mathrm{Vect}^\nabla_\mathbf{R}$ and $\mathrm{BO} \times \mathbf{Z}$.

3.19. ***Equivariant K-theory and the Atiyah—Segal completion theorem***

In this section we explain how the Atiyah—Segal completion theorem naturally fits within the realm of concordances. The basic setup is identical to that of §1.11: we have a manifold X with an action of a Lie group G. As described there, two quotients $X /\!\!/ G$ and $X /\!\!/^\nabla G$ are naturally associated to it.

Proposition 3.20. The space $\mathrm{Map}(X /\!\!/ G, \mathrm{Vect}^\nabla)$ is naturally weakly equivalent to the nerve of the groupoid of G-equivariant vector bundles with connection over X. Analogous statements hold for vector bundles without connection, flat vector bundles, and their virtual cousins. Likewise for principal bundles (with or without connection or flat).

Proof. Recall that $X /\!\!/ G$ is the homotopy colimit of the simplicial diagram $k \in \Delta^{\mathrm{op}} \mapsto G^k \times X$. The mapping space $\mathrm{Map}(-, \mathrm{Vect}^\nabla)$ turns this homotopy colimit into the corresponding homotopy limit. Since Vect^∇ is valued in homotopy 1-types, it is enough to look at terms in degree 0, 1, and 2. We get the homotopy limit $\mathrm{Vect}^\nabla(X) \rightrightarrows \mathrm{Vect}^\nabla(G \times X) \Rrightarrow \mathrm{Vect}^\nabla(G \times G \times X)$. The first term gives us the underlying vector bundle with connection, the second term gives us the action of G, and the third term ensures that it is associative. ▌

Corollary 3.21. The canonical comparison map

$$\alpha \colon \mathrm{Map}(X /\!\!/ G, \mathrm{VVect}^\nabla) \to \mathrm{Map}(\mathfrak{C}(X /\!\!/ G), \mathfrak{C}\mathrm{VVect}^\nabla) = \mathrm{Map}(X^h_G, \mathrm{ku})$$

induces the Atiyah—Segal completion map on π_0.

Proof. The left side was identified above with the equivariant K-theory of X and the right side is the K-theory of the homotopy quotient of X as a space. ▌

4 Gerbes and higher gerbes

We now turn our attention to higher bundle gerbes with connection. Take an abelian group A, typically $A = \mathrm{U}(1)$ or $A = \mathbf{C}^\times$. Also fix $n \geq 0$, the value $n = 0$ will give us smooth A-valued functions, $n = 1$ corresponds to principal A-bundles, and $n = 2$ will give us classical bundle A-gerbes. We shifted the index n by 1 compared to the classical notation in order to better stress the relationship to classifying spaces and the cohomological degree.

Definition 4.1. Fix $n \geq 0$ and an abelian Lie group A with a Lie algebra g. We construct three stacks as the stackification of the corresponding prestack of connective chain complexes, whose value on a smooth manifold S is specified below.

- For the stack $\mathrm{Grb}_n^\nabla(A)$ of *bundle* $(n-1)$*-gerbes with connection* we take the Deligne complex $\Omega^n(S,g) \leftarrow \Omega^{n-1}(S,g) \leftarrow \cdots \leftarrow \Omega^1(S,g) \leftarrow \mathrm{C}^\infty(S,A)$.
- For the stack $\mathrm{Grb}_n^\flat(A)$ of *flat bundle* $(n-1)$*-gerbes* we replace the initial term $\Omega^n(S,g)$ with $\Omega^n_{\mathrm{cl}}(S,g)$.
- For the stack $\mathrm{Grb}_n(A)$ of *bundle* $(n-1)$*-gerbes* (without connection) we take the smooth A-valued functions shifted by n, i.e., $\mathrm{C}^\infty(S,A)[n]$.

Remark 4.2. The A-valued Deligne complex is the g-valued de Rham complex truncated above degree n, but with 0-forms replaced by A-valued functions and the 0th differential $\mathrm{C}^\infty(S,A) \to \Omega^1(S,g)$ computed as the tangent map of $S \to A$. In fact, one can make sense of A-valued differential forms for any abelian Lie group A, which gives us the above complex.

Remark 4.3. For flat bundle $(n-1)$-gerbes one can also take the locally quasiisomorphic prestack

$$\mathrm{C}^\infty_{\mathrm{lc}}(S,A)[n] = \mathrm{C}^\infty(S, \mathrm{Map}(\mathrm{pt},A))[n],$$

i.e., locally constant A-valued functions shifted to degree n. In other words, flat gerbes with band A are equivalent to gerbes with band $\mathrm{Map}(\mathrm{pt},A)$, the underlying discrete group of A.

Remark 4.4. The classical theory of bundle gerbes is recovered when we take $A = \mathrm{U}(1)$ and $g = i\mathbf{R}$, the *imaginary numbers*, which are noncanonically isomorphic to $\mathbf{R}$. We could also treat complex gerbes with connection by replacing $i\mathbf{R}$ and $\mathrm{U}(1)$ with $\mathbf{C}$ and $\mathbf{C}^\times$. For $A = g = \mathbf{R}$ (or any real vector space V) one recovers the (V-valued) de Rham complex, i.e., bundle $(n-1)$-gerbes with connection are simply (V-valued) differential n-forms.

bungrb **Proposition 4.5.** For any abelian group A we have $\mathfrak{C}\mathrm{Grb}_n^\nabla(A) \simeq \mathfrak{C}\mathrm{Grb}_n(A) \simeq \mathrm{B}^n\mathfrak{C}A$ and $\mathfrak{C}\mathrm{Grb}_n^\flat(A) \simeq \mathrm{B}^n\,\mathrm{Map}(\mathrm{pt},A)$.

Proof. For Grb_n the computation is the same as for vector bundles. Indeed, $\mathrm{Grb}_n(\mathbf{\Delta}^m)$ can be computed as $\mathrm{B}^n\mathrm{C}^\infty(\mathbf{\Delta}^m, A)$ and a similar argument gives us the space $\mathrm{B}^n\mathfrak{C}A$. For Grb_n^∇ the homotopy colimit of the resulting simplicial diagram can be computed as the totalization of the corresponding bicomplex of normalized chains, whose term in bidegree (k,l) is $\Omega^{n-k}(\mathbf{\Delta}^l, g)$ if $k < n$ and $\mathrm{C}^\infty(\mathbf{\Delta}^l, A)$ if $k = n$. As shown in 1.6, for a fixed $k < n$ the corresponding complex $\Omega^{n-k}(\mathbf{\Delta}^\bullet, g)$ is exact and for $k = n$ we get a complex $\mathrm{C}^\infty(\mathbf{\Delta}^\bullet, A)$ that presents $\mathfrak{C}A$. Thus the totalization is the chain complex that presents $\mathrm{B}^n\mathfrak{C}A$. In other words, smooth bundle gerbes with or without connection have the same classifying space, as expected. ∎

Remark 4.6. It is easy to construct an explicit chain homotopy equivalence from the above totalization to $A[n]$. Nonzero terms live in bidegrees $(n-k,k)$, where for $k > 0$ we have maps $\Omega^k(\mathbf{\Delta}^k, g) \to A$ give by integration over the compact part of $\mathbf{\Delta}^k$ and applying the exponential map $g \to A$, whereas for $k = 0$ we simply take the identity map. The constructed map is nothing but the *higher holonomy map.* It can then be extended to a chain homotopy in a familiar fashion.

5 Total stacks of bundles and gerbes

In this section we explore examples that construct a smooth refinement of the notion of the total space of a principal bundle or a bundle $(n-1)$-gerbe. The statements of this section were suggested to the third author by Urs Schreiber.

Given a Lie group G, consider the universal principal G-bundle with connection $\mathrm{TBun}_G^\nabla \to \mathrm{Bun}_G^\nabla$. Freed and Hopkins [CWFAHT] denote this morphism by $E_\nabla G \to B_\nabla G$. Recall that TBun_G^∇ is weakly equivalent to the sheaf of 1-forms valued in the Lie algebra of G. Furthermore, the value of Bun_G^∇ on any geometric simplex can be computed as the groupoid of trivializable principal G-bundles with connection defined in Remark 3.6 because any principal G-bundle on $\mathbf{\Delta}^n$ is trivial.

Likewise, given an abelian Lie group A, we can define the universal bundle $(n-1)$-gerbe with connection $\mathrm{TGrb}_n^\nabla A \to \mathrm{Grb}_n^\nabla A$, whose base $\mathrm{Grb}_n^\nabla A$ is the stackification of the prestack of A-valued Deligne complexes (i.e., n-truncated differential forms with values in the Lie algebra of A, with 0-forms replaced by smooth A-valued functions) and whose total stack $\mathrm{TGrb}_n^\nabla A$ is the stack of differential n-forms valued in the Lie algebra of A, which one should think of as *trivialized bundle $(n-1)$-gerbes with band A.*

Even more generally, one could consider a Lie ∞-group G and define the universal principal G-bundle with connection. We recover the above examples when G is an ordinary Lie group or $G = \mathrm{B}^{n-1}A$ respectively, where B denotes delooping in stacks of E_∞-spaces.

Proposition 5.1. Consider a smooth manifold X and an arbitrary morphism $X \to \mathrm{Bun}_G^\nabla$, which classifies a principal G-bundle with connection over X. Consider the homotopy pullback diagram

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{TBun}_G^\nabla \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Bun}_G^\nabla. \end{array}$$

The stack T is the *total stack* of the principal G-bundle with connection classified by the map $X \to \mathrm{Bun}_G^\nabla$. The concordance space functor sends this square to the homotopy pullback square

$$\begin{array}{ccc} \mathfrak{C}T & \longrightarrow & \mathrm{E}\mathfrak{C}G \\ \downarrow & & \downarrow \\ \mathfrak{C}X & \longrightarrow & \mathrm{B}\mathfrak{C}G. \end{array}$$

Here $\mathfrak{C}X$ is simply the underlying homotopy type of X, $\mathrm{E}\mathfrak{C}G \to \mathrm{B}\mathfrak{C}G$ is the usual universal principal G-bundle in the sense of homotopy theory, the map $\mathfrak{C}X \to \mathrm{B}\mathfrak{C}G$ classifies the underlying topological bundle, and $\mathfrak{C}T$ is the total space of that bundle, understood as a homotopy type in the traditional sense. The above remains true if G is replaced by $\mathrm{B}^{n-1}A$ for an abelian Lie group A, Bun^∇ and TBun^∇ are replaced by Grb_n^∇ and TGrb_n^∇, and principal G-bundles are replaced by bundle $(n-1)$-gerbes with band A.

Proof. We claim that the morphism $\mathcal{R}\mathrm{TBun}_G^\nabla \to \mathcal{R}\mathrm{Bun}_G^\nabla$ (here $\mathcal{R}F = F^{\mathbf{\Delta}^\bullet}$ is the concordance resolution, see) is a homotopical Kan fibration, meaning that it satisfies the homotopical analog of the horn filling condition: given a square diagram of simplicial objects in simplicial sets (in fact, groupoids) of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathrm{TBun}_G^\nabla(\mathbf{\Delta}^\bullet) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathrm{Bun}_G^\nabla(\mathbf{\Delta}^\bullet) \end{tikzcd}$$

together with a levelwise homotopy between the two compositions, we can find a diagonal arrow such that both triangles commute up to a levelwise homotopy and the composition of these homotopies is itself homotopic to the original homotopy. This boils down to saying that a trivialized principal G-bundle with connection over $\mathbf{\Lambda}^n_k$ together with a principal G-bundle with connection over $\mathbf{\Delta}^n$ that extends it to the whole simplex admits an extension of the trivialization to the whole simplex. If we use the presentations of $\mathrm{TBun}_G^\nabla(\mathbf{\Delta}^\bullet)$ and $\mathrm{Bun}_G^\nabla(\mathbf{\Delta}^\bullet)$ identified above, this amounts to saying that the zero 1-form over $\mathbf{\Delta}^n_k$ can be extended to the zero 1-form over $\mathbf{\Delta}^n$, which is tautologically true. For the statement about bundle $(n-1)$-gerbes it suffices to replace 1-forms valued in the Lie algebra of G with n-forms valued in the Lie algebra of A. ∎

6 Liftings, Hirsch—Mazur, and Kirby—Siebenmann

6.1. ***Basic theory***

Consider a morphism $p\colon E \to B$ of stacks. Given a morphism $f\colon X \to B$ of stacks, we are interested in the *stack of liftings of* f, which is defined as the homotopy pullback of $\mathrm{Hom}(X,E) \to \mathrm{Hom}(X,B) \leftarrow \mathrm{pt}$, where the right map is induced by f.

To obtain a formula for the concordance stack of the stack of liftings of f, we impose an additional condition on $E \to B$: we require it to be a fiber ∞-bundle with a typical fiber V. This means that there is an effective ∞-epimorphism $U \to B$ such that the base change of $E \to B$ with respect to $U \to B$ is equivalent to the projection $U \times V \to U$. For more details, see Definition 4.1 in Nikolaus, Schreiber, and Stevenson [Principal].

Recall that although the concordification functor preserves finite homotopy products, it does not necessarily preserve homotopy pullbacks. The following proposition offers a partial remedy for this.

Proposition 6.2. The concordification functor preserves homotopy pullbacks of stacks, where one of the maps is an effective ∞-epimorphism.

Proof. It suffices to show that the concordance resolution of a hypercover $H \to X$ is a homotopy Kan fibration. This follows immediately from the fact that any morphism $\mathbf{\Lambda}^n_k \to H$ factors through some representable M. ▮

Corollary 6.3. The concordification of a V-fiber ∞-bundle is a $\mathcal{C}V$-fiber ∞-bundle. In particular, the concordance space has a typical fiber $\mathfrak{C}V$.

Corollary 6.4. The concordification of a G-principal ∞-bundle is a $\mathcal{C}G$-principal ∞-bundle. In particular, the concordance space has a typical fiber $\mathfrak{C}G$.

Corollary 6.5. Given a G-principal ∞-bundle $E \to B$ and a morphism $f\colon X \to B$, the concordification of the stack of liftings of f to E can be computed as the homotopy pullback of $\mathrm{Hom}(X,\mathcal{C}E) \to \mathrm{Hom}(X,\mathcal{C}B) \leftarrow \mathrm{pt}$. In particular, the concordance space of liftings is either empty or a torsor over $\mathrm{Hom}(\mathfrak{C}X,\mathfrak{C}G)$.

6.6. ***Smooth tangential structures***

In this section the word "manifold" can mean smooth, PL, or topological manifold, and $\mathrm{Aut}(n)$ denotes the structure group of an n-dimensional tangent bundle, i.e., $\mathrm{GL}(n)$, $\mathrm{PL}(n)$, and $\mathrm{TOP}(n)$ respectively. The tangent bundle of an n-manifold M is modulated by a map $M \to \mathrm{B}\,\mathrm{Aut}(n)$. Recall that an n-dimensional *structure group* is a morphism $\mathrm{B}G \to \mathrm{B}\,\mathrm{Aut}(n)$, where G is an ∞-group.

Proposition 6.7. Given a structure group $G \to \mathrm{Aut}(n)$, the concordance space of tangential G-structures on M is either empty or a torsor over $\mathrm{Map}(\mathfrak{C}M,\mathfrak{C}H)$, where $H \to \mathrm{B}G \to \mathrm{B}\,\mathrm{Aut}(n)$ is a fiber sequence.

Corollary 6.8. For any manifold M the set of orientations of M is a torsor over $\mathrm{Map}(M,\mathbf{Z}/2)$, the groupoid of spin (respectively spin^c) structures on M is a torsor over $\mathrm{Map}(M,\mathrm{B}\mathbf{Z}/2)$ (respectively $\mathrm{Map}(M,\mathrm{BU}(1))$), the 2-groupoid of string structures on M is a torsor over $\mathrm{Map}(M,\mathrm{B}^2\mathrm{U}(1))$.

6.9. ***Hirsch—Mazur theory***

Proposition 6.10. For any smoothable PL-manifold M the concordance space of smoothings of M is a torsor over $\mathrm{Map}(M,\mathrm{PL}/\mathrm{O})$.

6.11. ***Kirby—Siebenmann theory***

Proposition 6.12. For any topological manifold M that admits a PL-structure the concordance space of PL-structures of M is a torsor over $\mathrm{Map}(M,\mathrm{TOP}/\mathrm{PL}) = \mathrm{Map}(M,\mathrm{K}(\mathbf{Z}/2,3))$.

7 Field theories and factorization algebras

7.1. ***Groupoids of stacks***

Fix some target ∞-category V and consider the prestack St_V that sends a manifold X to the ∞-groupoid of V-valued stacks on X with 1-morphisms being equivalences of stacks. The prestack St_V satisfies descent and therefore is a stack; a recent exposition of this fact can be found in our work with Gwilliam [Fact].

Proposition 7.2. The canonical map $\mathcal{C}\mathrm{St}_V(X) \to \mathrm{Map}(\mathfrak{C}X, \mathfrak{C}\mathrm{St}_V)$ is an equivalence. In particular, concordance classes of V-valued stacks over X are in bijection with $[X, \mathfrak{C}\mathrm{St}_V]$.

fft 7.3. ***Field theories***

Functorial field theories (not necessarily topological) in the sense of Atiyah and Segal are functors from some category of $(n-1)$-dimensional manifolds and n-dimensional bordisms between them to the category of vector spaces and linear maps.

Freed proposed to consider *extended* field theories, which are functors from the n-category whose k-morphisms are k-dimensional manifolds with corners, considered as bordisms, to the appropriately generalized target category, e.g., bundle $(n-1)$-gerbes or E_{n-1}-algebras. Bordisms can also be equipped with additional geometric structures, such as a smooth map to some target manifold X.

As observed by Stolz and Teichner in [Elliptic], the advantage of considering extended field theories is the following result, shown in its full generality (with arbitrary geometric structures, like Riemannian metrics or spin structures) in the upcoming paper [ExtLoc].

Theorem 7.4. The (covariant) functor $X \mapsto \mathrm{Bord}_n(X)$ from smooth manifolds to symmetric monoidal (∞, n)-categories satisfies the homotopy codescent condition, i.e., is a costack.

Corollary 7.5. Given some target symmetric monoidal (∞, n)-category T, the prestack $\mathrm{FT}_n = (X \mapsto \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n(X), T))$ satisfies the homotopy descent condition, i.e., is a stack. Here $\mathrm{Fun}^{\otimes}$ denote the ∞-groupoids of symmetric monoidal functors.

Applying Theorem 0.2, we immediately obtain the following result.

Proposition 7.6. Concordances classes of field theories over X (i.e., objects in $\mathrm{FT}_n(X)$) are representable by the space $\mathfrak{C}\mathrm{FT}_n$.

One immediate application of this result involves the Stolz—Teichner program [SUSYFT], which conjectures a bijection between concordance classes of $2|1$-dimensional Euclidean field thoeries over X and $TMF^0(X)$, the topological modular forms spectrum of Hopkins and Miller [TMF]. The above proposition allows us to conclude that such concordance classes are representable, which, of course, is the first step toward identifying them with the value of some cohomology theory.

7.7. ***The Madsen—Tillmann spectra***

Consider the stack Bord_n that sends a smooth manifold S to the simplicial set $\mathrm{Bord}_n(S)$ whose m-simplices are smooth manifolds M equipped with a smooth map $M \to \mathbf{\Delta}^m \times S$ whose composition with the projection $\mathbf{\Delta}^m \times S \to S$ is a submersion whose fibers are smooth maps $M_s \to \mathbf{\Delta}^m$ of rank at most n that are transversal to each face of $\mathbf{\Delta}^m$. The simplicial maps are given by pullbacks along $\mathbf{\Delta}^{m'} \to \mathbf{\Delta}^m$, which exist by transversality. The prestack structure is given by pullbacks along $S' \to S$, which exist because the relevant maps are submersions. Strictly speaking (pun intended), we only get a Grothendieck fibration in sets over $\Delta \times \mathrm{Man}$, which one can rectify to an honest functor in a standard fashion. The homotopy descent condition is satisfied, so the prestack Bord_n is a stack.

Disjoint union of manifolds turns Bord_n into a stack of E_∞-spaces. (As usual, the strictness issues can be addressed by organizing the above construction into a Γ-space, for example.) This E_∞-space is group-like, the inverse of a manifold being supplied by the same manifold with the opposite orientation. The forgetful functor from connective spectra to spaces creates homotopy limits, so we in fact get a stack of connective spectra.

More generally, one considers manifolds with a tangential θ-structure for some fixed $\theta\colon B \to \mathrm{BO}(n)$. As established by Galatius, Madsen, Tillmann, and Weiss in [CobCat] for the $(n-1, n)$-dimensional slice and later extended by Lurie in [TFT] to all dimensions, the concordance space of Bord_n can be computed as the Madsen—Tillmann spectrum $\mathrm{MT}(\theta)$, defined as the Thom spectrum of the additive inverse of the pullback of the universal n-bundle on $\mathrm{BO}(n)$ to B.

Proposition 7.8. The space $\mathfrak{C}\mathrm{Bord}_\theta$ is weakly equivalent to the Madsen—Tillmann spectrum $\Sigma^n \mathrm{MT}(\theta)$.

Corollary 7.9. For any smooth manifold X the mapping space $\mathrm{Map}(X, \mathrm{MT}(\theta))$ is weakly equivalent to the concordance space of bordisms over X.

Remark 7.10. For the first n levels there is no difference between concordances of bordisms and bordisms themselves because any concordance can be converted to a bordism. Starting from the level n the rank condition kicks in, which prevents the stack Bord^n from being concordance-invariant. Put differently, if we fix some closed n-manifold X and take the loop space $\Omega^n(\mathrm{Bord}_n(S), *)$ at the point $*$ given by X, we recover the classifying space of the discrete group $\mathrm{C}^\infty(S, \mathrm{Diff}(X))$. The concordification functor recovers the underlying homotopy type of $\mathrm{Diff}(X)$.

7.11. ***Factorization algebras***

Recall that the ∞-category of factorization algebras on a site S valued in a closed symmetric monoidal presentable ∞-category T is defined as the ∞-category of T-valued strong symmetric monoidal costacks on S in the Weiss topology. A costack on S is a covariant functor $S \to T$ such that the induced functor $S^{\mathrm{op}} \to T^{\mathrm{op}}$ is a stack. The *Weiss topology* induced by a given Grothendieck topology C is defined by taking all covering families τ such that τ^k is a covering family in C for all $k \geq 0$. Here τ^k is the family obtained by taking the k-fold cartesian power of every element in τ. Without the loss of generality we can assume that our ∞-site has finite homotopy products and coproducts. The site S is equipped with the monoidal structure given by the disjoint union, and it is in this sense that the costack must be strong monoidal.

Consider now the prestack that sends a manifold X to the ∞-category of factorization algebras on the site of manifolds equipped with a map to X. This prestack is a stack, hence we obtain the following result:

Proposition 7.12. The concordance space $\mathcal{C}\mathrm{FA}(X)$ of factorization algebras on the site of manifolds equipped with a map to X is naturally weakly equivalent to $\mathrm{Map}(X, \mathfrak{C}\mathrm{FA})$. In particular, concordance classes of factorization algebras over X are precisely homotopy classes of maps from X to $\mathfrak{C}\mathrm{FA}$.

Computing the spaces $\mathfrak{C}\mathrm{FA}$ for some choice of the target T is an interesting open problem.

Remark 7.13. One can also organize factorization algebras in a stack in a very different fashion. Suppose we have a stack T of closed symmetric monoidal presentable ∞-categories. The prestack that sends a manifold X to the ∞-category of factorization algebras with values in $T(X)$ is a stack. One can think of these as smooth families of factorization algebras parametrized by X. The concordance space over X is again representable, though the classifying space might be different from the one identified above.

8 Thom spectra and bordism

Having already explored various geometric models for ordinary cohomology and K-theory, we now proceed to another classical cohomology theory, namely cobordism. Our machinery requires sheaves on smooth manifolds as an input, and one way to ensure that the relevant pullbacks always exist is to use the machinery of derived smooth manifolds of Lurie and Spivak. We refer to Spivak [DSM] and Borisov and Noel [SADDM] for the relevant facts about derived smooth manifolds. We denote by DMan the simplicial category of derived smooth manifolds as constructed in Theorem 1 of [SADDM].

Definition 8.1. The stack Bord on the site of smooth manifolds sends a smooth manifold S to the full subcategory of DMan/S consisting of proper maps $X \to S$ of virtual dimension 0.

Proposition 8.2. We have $\pi_k \mathbf{C}\mathrm{Bord} \cong \pi_k \mathrm{MO}$.

Proof. According to Proposition 14.1, $\pi_k \mathbf{C}\mathrm{Bord}$ can be computed as pointed concordance classes of Bord over $\mathbf{S}^k$. Being pointed in this context amounts to the fiber over the basepoint of $\mathbf{S}^k$ being empty, and removing the basepoint turns $\mathbf{S}^k$ into $\mathbf{R}^k$. The map into $\mathbf{R}^k$ is discarded after taking the concordance classes, so what remains is simply equivalence classes of proper derived smooth manifolds of dimension k module the equivalence relation of concordance, which in this case is nothing but bordism. According to Theorem 2.6 in Spivak [DSM] the derived bordism groups coincide with the ordinary bordism groups, which completes the proof. ∎

Proposition 8.3. We have $\mathfrak{C}\mathrm{Bord} \simeq \mathrm{MO}$.

Proof sketch. We establish a zigzag of weak equivalences with the Quinn model of Thom spectra, see the paper of Laures and McClure [MultQ] for the relevant definitions. The intermediate step in the zigzag is a simplicial subset of $\mathfrak{C}\mathrm{Bord}$ consisting of those simplices of $\mathfrak{C}\mathrm{Bord}$ that correspond to ordinary manifolds and the map to $\boldsymbol{\Delta}^n$ is transversal at all faces of $\boldsymbol{\Delta}^n$. The map to the Quinn spectrum is then given by discarding the map $\boldsymbol{\Delta}^n$ while preserving the stratified structure of fibers over various faces of $\boldsymbol{\Delta}^n$, which constitutes an n-ad. It remains to observe that the induced map on homotopy groups is an isomorphism, which is established by explicitly computing the homotopy groups as in the previous proposition. ∎

9 Representability of concordified stacks of spaces

A *space* for us is an object of the relative category sSet of simplicial sets equipped with the usual (Kan) weak equivalences. A *simplicial space* is then a bisimplicial set, an object of the relative category ssSet equipped with the levelwise weak equivalences. Our choice of terminology emphasizes that bisimplicial sets have a preferred simplicial direction. Sets embed into spaces as discrete spaces, therefore simplicial sets embed into simplicial spaces via the functor $\iota\colon \mathrm{sSet} \to \mathrm{ssSet}$.

The notation $\mathrm{Map}(X,Y)$ always denotes the *mapping space* given by the simplicial set obtained by taking the enriched hom from X to Y, where X and Y are objects in a simplicially enriched category, e.g., simplicial sets, simplicial spaces, or simplicial presheaves. The *derived mapping space* functor RMap for simplicial sets is the functor $\mathrm{RMap}\colon \mathrm{sSet}^{\mathrm{op}} \times \mathrm{sSet} \to \mathrm{sSet}$ defined as $\mathrm{RMap}(X,Y) = \mathrm{Map}(X, \mathrm{Ex}^\infty Y)$. We use the same notation for simplicial presheaves and simplicial spaces, with appropriate replacements for Ex^∞. The *derived internal hom* functor RHom (which we only use for simplicial spaces) is the functor $\mathrm{RHom}\colon \mathrm{ssSet}^{\mathrm{op}} \times \mathrm{ssSet} \to \mathrm{ssSet}$ defined as $\mathrm{RHom}(X,Y) = \mathrm{Hom}(X, RY)$, where R is a Reedy fibrant replacement functor.

Recall the adjoint triple $\lambda \dashv \Re \dashv \rho$, where the functor $\Re = \mathrm{diag}\colon \mathrm{ssSet} \to \mathrm{sSet}$ takes diagonal of a simplicial space, which computes its realization (homotopy colimit). Concretely, λ is the unique cocontinuous functor $\mathrm{sSet} \to \mathrm{ssSet}$ that maps $\Delta^n \to \Delta^{n,n}$ and $(\rho X)_{m,n} = \mathrm{Hom}(\Delta^m \times \Delta^n, X)$. A *realization equivalence* is a morphism of simplicial spaces whose realization is a weak equivalence of spaces. Equipping simplicial sets with Kan weak equivalences and simplicial spaces with realization equivalences, we upgrade the above adjoint triple to an adjoint triple of relative categories so that all three functors create (i.e., preserve and reflect) weak equivalences. For the adjunction $\lambda \dashv \Re$ the unit map $\mathrm{id} \to \Re\lambda$ is a weak equivalence and the counit map $\lambda\Re \to \mathrm{id}$ is a realization equivalence. For the adjunction $\Re \dashv \rho$ the unit map $\mathrm{id} \to \rho\Re$ is a realization equivalence and the counit map $\Re\rho \to \mathrm{id}$ is a weak equivalence. Thus both adjunctions are homotopy equivalences of relative categories and one can freely pass between spaces and simplicial spaces using any of these functors. We emphasize that there is no need to derive these functors because they preserve weak equivalences.

The standard cosimplicial object $\bullet\colon \Delta \to \Delta$ is given by the identity functor and can be substituted in any place where a simplex is required, e.g., $F(\mathbf{\Delta}^\bullet) = (k \in \Delta^{\mathrm{op}} \mapsto F(\mathbf{\Delta}^k))$ and $\mathbf{\Delta}^\bullet \times \mathbf{\Delta}^\bullet = (k \in \Delta^{\mathrm{op}} \mapsto \mathbf{\Delta}^k \times \mathbf{\Delta}^k)$. The blank $-$ is likewise used to denote the standard precostack $-\colon \mathrm{Man} \to \mathrm{Man}$ given by the identity functor and can be substituted in any place where a manifold or a prestack is required, e.g., $F(X \times -) = (S \in \mathrm{Man}^{\mathrm{op}} \mapsto F(X \times S))$.

Definition 9.1. A *prestack* is a functor $\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$, i.e., a simplicial presheaf on the site of manifolds. A *stack* is a prestack F that satisfies the *descent condition* for open covers of manifolds: for any open cover U of a manifold X the canonical map $F(X) \to \mathrm{holim}_{i \in \Delta/I} F(U_{i_0} \cap \cdots \cap U_{i_n})$ is a weak equivalence. A *weak equivalence* of prestacks is a map $F \to G$ whose components $F(X) \to G(X)$ are weak equivalences for any manifold X.

Definition 9.2. The *concordance prestack* $\mathcal{C}F$ of a prestack F is the prestack $X \mapsto \Re F(\mathbf{\Delta}^\bullet \times X)$. The *concordance space* $\mathfrak{C}F$ of a prestack F is the space $\mathcal{C}F(\mathrm{pt}) = \Re F(\mathbf{\Delta}^\bullet)$.

conccomp **Definition 9.3.** The *concordance comparison map* of a prestack F is a morphism of prestacks

$$\mathcal{C}F(-) \to \mathrm{RMap}(\mathfrak{C}-, \mathfrak{C}F)$$

that at a manifold X is given by the map of spaces

$$\Re F(\mathbf{\Delta}^\bullet \times X) \to \mathrm{Map}(\Re\,\mathrm{Map}(\mathbf{\Delta}^\bullet, X), \mathrm{Ex}^\infty \Re F(\mathbf{\Delta}^\bullet))$$

whose adjoint is

$$\Re F(\mathbf{\Delta}^\bullet \times X) \times \Re\,\mathrm{Map}(\mathbf{\Delta}^\bullet, X) \to \Re F(\mathbf{\Delta}^\bullet \times \mathbf{\Delta}^\bullet) \to \Re F(\mathbf{\Delta}^\bullet) \to \mathrm{Ex}^\infty \Re F(\mathbf{\Delta}^\bullet) = \mathrm{Ex}^\infty \mathfrak{C}F,$$

where the second map is induced by the diagonal map $\mathbf{\Delta}^\bullet \to \mathbf{\Delta}^\bullet \times \mathbf{\Delta}^\bullet$. Used in I.1.

Definition 9.4. The *prestack geometric realization functor* $S \mapsto \|S\|_p$ sends a simplicial space S to the prestack given by the weighted (homotopy) colimit $S \otimes_\Delta \mathbf{\Delta}^\bullet$. The *stack geometric realization functor*

$S \mapsto \|S\|$ sends S to the stackification of $\|S\|_p$. We use the same notation when S is a simplicial set, which we first replace with the simplicial space ιS.

Remark 9.5. The functor $S \mapsto \|S\|_p$ from simplicial spaces to prestacks is left adjoint to the functor $F \mapsto F(\boldsymbol{\Delta}^\bullet)$ from prestacks to simplicial spaces. This adjunction is a homotopical adjunction (both functors preserve weak equivalences).

Remark 9.6. Due to the Reedy-cosimplicial injective-presheaf cofibrancy of the cosimplicial diagram $\boldsymbol{\Delta}^\bullet$ valued in prestacks, the weighted homotopy colimit $S \otimes_\Delta \boldsymbol{\Delta}^\bullet$ can be computed as the ordinary weighted colimit. In particular, the resulting prestack $\|S\|_p$ takes values in sets (i.e., discrete spaces) and its value on a manifold X can be described concretely as the set of maps $X \to u(S)$ that factor as a smooth map $U \to \boldsymbol{\Delta}^n$ followed by the inclusion $\boldsymbol{\Delta}^n \to \|S\|_p$. Here $u(S)$ denotes the set of points of $\|S\|_p$, equivalently, $\|S\|$. The stackification of a prestack of sets can be computed as the ordinary sheafification. In particular, the value of $\|S\|$ on a manifold X is the set of maps $X \to u(S)$ that locally in X factor as a smooth map $U \to \boldsymbol{\Delta}^n$ followed by the inclusion $\boldsymbol{\Delta}^n \to \|S\|$. It is instructive to look at the example of maps $\boldsymbol{\Delta}^1 \to \|\Lambda^2_1\|$. The target $\|\Lambda^2_1\|$ looks like a cross (recall that all simplices are extended) and a map $\boldsymbol{\Delta}^1 \to \|\Lambda^2_1\|$ must locally by a smooth map to one of the constituent lines; in particular, whenever it hits the intersection point, it must remain there for a while.

Definition 9.7. A *triangulation* T of a manifold X is a morphism $T \to \mathfrak{C}X$ of simplicial sets such that for any nondegenerate simplex $\Delta^n \to T$ the induced map $\boldsymbol{\Delta}^n \to X$ is an immersion and the map from the ordinary geometric realization of T (as a topological space) to the underlying topological space of X is a homeomorphism of topological spaces.

Remark 9.8. Any smooth manifold admits a triangulation. The bulk of the argument is supplied by Theorem 10.6 in Munkres [EDT]. Whitney's extension theorem then extends each simplex in the triangulation to an extended simplex, inductively on the dimension of the simplex. Finally, the underlying simplicial complex of the triangulation must be turned into a simplicial set by choosing an arbitrary linear order on the vertices of the triangulation.

Remark 9.9. The full strength of the smooth triangulation theorem is not necessary for our main result. In fact, the only fact we need is that every smooth manifold is smoothly homotopy equivalent to a smoothly triangulable manifold, which is quite easy to prove, see Lemma 12.3.

smoothconccomp **Theorem 9.10.** The concordance comparison map of a stack F is a weak equivalence of prestacks. Used in 12.2*.

Proof. By Lemma 9.11, the concordance comparison map of F evaluated at a manifold X is weakly equivalent to the realization of the morphism of simplicial spaces $\mathrm{RHom}(\lambda T, F(\boldsymbol{\Delta}^\bullet)) \to \mathrm{RHom}(\lambda T, \rho\Re F(\boldsymbol{\Delta}^\bullet))$ for a triangulation T of X. By Lemma 9.12, the latter morphism is a realization equivalence. ∎

The intuition behind the following lemma is that in the concordance comparison map the manifold on which we evaluate can be replaced with the combinatorial data of its triangulation.

rewrite **Lemma 9.11.** The concordance comparison map of a stack F at a manifold X triangulated by T is weakly equivalent to the realization of the morphism of simplicial spaces

$$\mathrm{RHom}(\lambda T, F(\mathbf{\Delta}^\bullet)) \to \mathrm{RHom}(\lambda T, \rho\Re F(\mathbf{\Delta}^\bullet))$$

induced by the counit map $F(\mathbf{\Delta}^\bullet) \to \rho\Re F(\mathbf{\Delta}^\bullet)$. Used in 9.10*.

Proof. We have the following commutative diagram of spaces:

$$\begin{tikzcd}
\mathcal{C}F(X) \arrow[r] \arrow[d, "="] & \mathrm{RMap}(\mathfrak{C}X, \mathfrak{C}F) \arrow[d, "="] \\
\mathfrak{C}\,\mathrm{Hom}(X, F) \arrow[r] \arrow[d, "(1)"', "\sim"] & \mathrm{RMap}(\Re_k \mathrm{Map}(\mathbf{\Delta}^k, X), \Re_k F(\mathbf{\Delta}^k)) \arrow[d, "(2)"', "\sim"] \\
\mathfrak{C}\,\mathrm{Hom}(\|T\|, F) \arrow[r] \arrow[d, "(3)"', "\cong"] & \mathrm{RMap}(T, \Re_k F(\mathbf{\Delta}^k)) \\
\Re_k \mathrm{Map}(\|T\| \times \mathbf{\Delta}^k, F) \arrow[d, "(5)"', "\sim"] & \mathrm{RMap}(\Re\lambda T, \Re_k F(\mathbf{\Delta}^k)) \arrow[u, "(4)", "\sim"'] \\
\Re_k \mathrm{Map}(\|T \times \Delta^k\|, F) \arrow[d, "(7)"', "\cong"] & \mathrm{RMap}(\lambda T, \rho\Re_k F(\mathbf{\Delta}^k)) \arrow[u, "(6)", "\cong"'] \\
\Re_k \mathrm{Map}(\|T \times \Delta^k\|_p, F) \arrow[d, "(9)"', "\cong"] & \mathrm{RMap}(\iota T, \rho\Re_k F(\mathbf{\Delta}^k)) \arrow[u, "(8)", "\sim"'] \\
\Re_k \mathrm{Map}(\iota T \times \iota\Delta^k, F(\mathbf{\Delta}^\bullet)) \arrow[r, "(11)", "\cong"'] & \Re\,\mathrm{Hom}(\iota T, F(\mathbf{\Delta}^\bullet)). \arrow[u, "(10)"]
\end{tikzcd}$$

If X is a representable prestack, the canonical maps $\mathrm{Map}(X, -) \to \mathrm{RMap}(X, -)$ and $\mathrm{Hom}(X, -) \to \mathrm{RHom}(X, -)$ are weak equivalences. We also have $\mathrm{Map}(X, F) = F(X)$ and $\mathrm{Hom}(X, F) = F(X \times -)$. A weak equivalence $F \to F'$ of prestacks induces a weak equivalence of morphisms of simplicial spaces used in the statement. Thus we can fibrantly replace F in the injective model structure on simplicial presheaves. Under this assumption, the underived Map and Hom above actually compute their derived versions: all simplicial presheaves are injectively cofibrant and the target F is injectively fibrant. Furthermore, the injective fibrancy of F implies the Reedy fibrancy of the simplicial space $F(\mathbf{\Delta}^\bullet)$ because the latter amounts to the right lifting property of F with respect to the map $\|\partial\Delta^n\|_p \to \|\Delta^n\|_p$, which is an injective cofibration.

The map (1) is induced by the morphism of stacks $\|T\| \to X$, which is a concordance equivalence of stacks by Lemma 9.13. The functor $\mathrm{Hom}(-, F)$ preserves this concordance equivalence and the functor $\mathfrak{C}$ sends concordance equivalences of stacks to weak equivalences of spaces. The map (2) is induced by the weak equivalence of spaces $T \to \Re_k \mathrm{Map}(\mathbf{\Delta}^k, X)$. The map (3) expands the definition of the functor $\mathfrak{C}$. The map (4) is induced by the weak equivalence of spaces $T \to \Re\lambda T$. The map (5) is induced by the map $\|T \times \Delta^k\| \to \|T\| \times \|\Delta^k\| = \|T\| \times \mathbf{\Delta}^k$. Its two components are induced by the projections $T \times \Delta^k \to T$ and $T \times \Delta^k \to \Delta^k$ and the map itself is a concordance equivalence by Lemma 9.13. The map (5) is the evaluation at the point of the same map with Map replaced by Hom. Thus suffices to show that the latter map is a weak equivalence of prestacks. The functor $\mathrm{Hom}(-, F)$ preserves concordance equivalences and $\Re$ sends the resulting simplicial diagram of concordance equivalences to a concordance equivalence between concordance-invariant prestacks, hence also a weak equivalence of prestacks. The source $\Re_k \mathrm{Hom}(\|T\| \times \mathbf{\Delta}^k, F)$ and target $\Re_k \mathrm{Hom}(\|T \times \Delta^k\|, F)$ are concordance-invariant by direct inspection. The map (6) is induced by the adjunction $\Re \dashv \rho$. The map (7) is induced by the map $\|T \times \Delta^k\|_p \to \|T \times \Delta^k\|$, which gets mapped to a weak equivalence by $\mathrm{Map}(-, F)$ because F is a stack. The map (8) is induced by the weak equivalence of simplicial spaces $\lambda T \to \iota T$. The map (9) is induced by the adjunction $\|-\|_p \dashv -(\mathbf{\Delta}^\bullet)$ between simplicial spaces and stacks. The map (11) is the formula for computing the internal hom of simplicial spaces in terms of mapping spaces. The map (10) is the map in the statement. ▮

commute **Lemma 9.12.** If F is a stack and T is a triangulation of a manifold X, the canonical map

$$\operatorname{RHom}(\iota T, F(\boldsymbol{\Delta}^\bullet)) \to \operatorname{RHom}(\iota T, \rho\Re F(\boldsymbol{\Delta}^\bullet))$$

induced by the counit map $F(\boldsymbol{\Delta}^\bullet) \to \rho\Re F(\boldsymbol{\Delta}^\bullet)$ is a realization equivalence. Used in 9.10*, 13.4, 13.4.

Proof. We apply the sphere filling criterion 11.1 to the above morphism of simplicial spaces and use the adjunction to move ιT to the left. The relevant diagram reads

$$\begin{tikzcd} \iota(\operatorname{Sd}^i \partial\Delta^n \times T) \arrow[r] \arrow[d] & F(\boldsymbol{\Delta}^\bullet) \arrow[d] \\ \iota(\operatorname{Sd}^i \Delta^n \times T) \arrow[r] \arrow[ur, dotted] & \rho\Re F(\boldsymbol{\Delta}^\bullet). \end{tikzcd}$$

Starting with the above commutative square, we must increase i as necessary and construct the diagonal arrow so that the upper triangle commutes and the two morphisms in the lower triangle are connected by a homotopy relative boundary.

Using Lemma 11.2 we construct a subdivision $L \to \operatorname{Sd}^i \Delta^n \times T$ together with a morphism $L \to F(\boldsymbol{\Delta}^\bullet)$ with the properties explained there. The adjoint of the latter morphism is $\|L\| \to F$. Using Lemma 9.13 we construct a morphism $\|\operatorname{Sd}^i \Delta^n \times T\| \to \|L\|$ that is a part of the concordance equivalence, as explained there. Composing, we get a morphism $\|\operatorname{Sd}^i \Delta^n \times T\| \to F$. Taking its adjoint, we get $\operatorname{Sd}^i \Delta^n \times T \to F(\boldsymbol{\Delta}^\bullet)$, and restricting along $\operatorname{Sd}^i \partial\Delta^n \times T \to \operatorname{Sd}^i \Delta^n \times T$ we get another commutative square of the same form as above, with a specified diagonal arrow that makes the upper triangle commute strictly and the lower triangle up to a levelwise homotopy. This square is homotopic (in the sense of Lemma 11.1) to the original square, using the concordance constructed in Lemma 9.13, which completes the proof. ▮

conctriang **Lemma 9.13.** Suppose X is a manifold with boundary, triangulated by T. Then there is a concordance equivalence between X and $\|T\|$, where the map $\|T\| \to X$ in this concordance equivalence is supplied by the triangulation T. Used in 9.11*, 9.11*, 9.12*, 9.12*.

Proof. Abusing the notation, we identify any simplex s in T with its image in X, which is a closed compact subset of X. The *open star* U_s of a simplex s in T consists of the union of the interiors of all simplices in X that contain s. The *closed star* W_s is defined in the same way but without interiors. The *link* L_s is defined as $W_s - U_s$. Construct an open cover U of X by taking the open star of each vertex in T. The elements of U are indexed by vertices of T. The intersection of a finite subfamily of U is empty if the corresponding vertices do not form a simplex in T. Otherwise it is the open star of the corresponding simplex in T. In particular, U is a good open cover of T: finite intersections of elements of U are either empty or cartesian.

Choose a partition of unity h subordinate to U. Thus for any index i the function $h_i\colon X \to [0,1]$ is a smooth function with support in U_i. We construct a morphism $g\colon X \to \|T\|$ that informally (on the point-set level) is given by the formula $x \mapsto \sum_i h_i(x) \cdot i$, where i is interpreted as a vertex in T (hence in $\|S\|$) and the sum is interpreted as an affine combination in the simplex determined by the vertices i for which $h_i(x) \neq 0$; the definition of a partition of unity guarantees that this combination is affine (the sum of coefficients is 1).

To make the above informal description of the map $g\colon X \to \|T\|$ precise, we first construct another open cover V of X and then define g locally on V. The elements of V are indexed by the simplices of T. For such a simplex s we take the vertices in the link of s; there are only finitely many such vertices by definition of a triangulation. The union of the supports of h_i for such vertices i is a closed set B such that $B \cap s = \emptyset$. By normality of X the closed subsets B and s can be separated by open subsets $B' \supset B$ and $s' \supset s$. Now set $V_s := s' \cap U_s$. Thus $V'_s := \bigcup_{t \subset s} V_t$ for all faces t of s is an open neighborhood of s.

A crucial property of the open sets V_s constructed above is that the only vertices i for which $h_i|_{V_s}$ is not the zero function are precisely the vertices of s. Thus $\sum_{i \in s} h_i = 1$ on V_s and we get a map $(h_i)_{i \in s}\colon V_s \to \boldsymbol{\Delta}^s$. Postcomposing it with the map $\boldsymbol{\Delta}^s \to \|T\|$ that corresponds to the simplex s, we get a map $V_s \to \|T\|$.

Consider two such maps $V_s \to \|T\|$ and $V_t \to \|T\|$. The intersection $V_s \cap V_t$ is empty if so is $s \cap t$. Otherwise $s \cap t$ is a simplex and $V_s \cap V_t$ is an open subset of X such that h_i vanishes on it unless $i \in s \cap t$. Thus $V_s \to \|T\|$ and $V_t \to \|T\|$ factor through $\boldsymbol{\Delta}^{s\cap t} \to \boldsymbol{\Delta}^s$ and $\boldsymbol{\Delta}^{s\cap t} \to \boldsymbol{\Delta}^t$ when restricted to $V_s \cap V_t$. Therefore their postcompositions with $\boldsymbol{\Delta}^s \to \|T\|$ and $\boldsymbol{\Delta}^t \to \|T\|$ are the same because both coincide with the map $V_{s\cap t} \to \|T\|$ constructed for the simplex $s \cap t$.

By construction, the map $X \to \|T\|$ sends any (compact) simplex of T to the corresponding simplex of $\|T\|$. Thus the composition $\|T\| \to X \to \|T\|$ preserves the simplices of $\|T\|$, hence we can construct a smooth concordance $\mathbf{\Delta}^1 \times \|T\| \to \|T\|$ from this composition to the identity map on $\|T\|$ using linear interpolation on each simplex.

The composition $X \to \|T\| \to X$ preserves the simplices of T. We would like to construct a concordance $\mathbf{\Delta}^1 \times X \to X$ from this composition to the identity map on X. We start with the same trick with linear interpolation and construct a *continuous* concordance, which is *not* smooth on simplices of codimension 1 and higher. By the Whitney approximation theorem (see, for example, Theorem 2.2.6 in Hirsch [DiffTop]), two maps $X \to X$ are smoothly concordant if and only if they are continuously concordant. ▮

10 Simplicial spaces

The following result can be interpreted as saying that any map to $\mathrm{Ex}^\infty T$ can be presented as a map to T from some barycentric subdivision of the source. Individual simplices of the source might have to be subdivided arbitrarily many times, so we cannot simply apply the functor Sd^i for some $i \ge 0$.

liftexi **Proposition 10.1.** For any simplicial sets K and T, and any map $K \to \mathrm{Ex}^\infty T$ there is weak equivalence $K' \to K$, a map $K' \to T$, and a homotopy $\Delta^1 \times K' \to \mathrm{Ex}^\infty T$ that connects the compositions $K' \to K \to \mathrm{Ex}^\infty T$ and $K' \to T \to \mathrm{Ex}^\infty T$. (As revealed in the proof, K' is obtained from K by barycentrically subdividing its simplices.) Used in 10.2*, 10.2*.

Proof. We present $\emptyset \to K$ as a transfinite composition of the diagram $A_0 \to A_1 \to \cdots$, where $A_{n-1} \to A_n$ for any $n \ge 1$ is a cobase change of $\coprod \partial\Delta^n \to \coprod \Delta^n$, where the coproducts are taken over all nondegenerate n-simplices of K. The map $K' \to K$ and the homotopy $\Delta^1 \times K' \to \mathrm{Ex}^\infty T$ will have an identical presentation in terms of transfinite compositions, coproducts, and cobase changes, so we concentrate our attention on a single nondegenerate simplex $\Delta^n \to K \to \mathrm{Ex}^\infty T$.

Any map $\Delta^n \to \mathrm{Ex}^\infty T$ factors as $\Delta^n \to \mathrm{Ex}^i T \to \mathrm{Ex}^\infty T$, and the first map is adjoint to $\mathrm{Sd}^i \Delta^n \to T$. We take i to be as small as possible here. The attaching map $\mathrm{Sd}^i \partial\Delta^n \to K'$ is constructed by induction on the skeleton of $\partial\Delta^n$ using iterations of last vertex maps to reduce the value of i if necessary (here we use the fact that i is as small as possible so that all simplices in the skeleton have the same or smaller value of i). The last vertex map $\mathrm{Sd}^i(\partial\Delta^n \to \Delta^n) \to (\partial\Delta^n \to \Delta^n)$ yields the weak equivalence $K' \to K$. ▮

The following result allows one to present any map to $\mathrm{Ex}^\infty \Re W$ as a map to W from a barycentrically subdivided source, to which we apply the functor λ to turn it into a simplicial space.

Proposition 10.2. For any simplicial set K, any simplicial space W, and any map $K \to \mathrm{Ex}^\infty \Re W$ there is a weak equivalence $K' \to K$ and a map $\lambda K' \to W$ such that the compositions $K' \to \Re\lambda K' \to \Re W \to \mathrm{Ex}^\infty \Re W$ and $K' \to K \to \mathrm{Ex}^\infty \Re W$ are connected by a homotopy $\Delta^1 \times K' \to \mathrm{Ex}^\infty \Re W$.

Proof. Using Proposition 10.1, we construct $K' \to K$, a map $K' \to \Re W$, and a homotopy $\Delta^1 \times K' \to \mathrm{Ex}^\infty \Re W$ that connects the compositions $K' \to K \to \mathrm{Ex}^\infty \Re W$ and $K' \to \Re W \to \mathrm{Ex}^\infty \Re W$. Using the adjunction $\lambda \dashv \Re$, we get a map $\lambda K' \to W$ whose image under $\Re$ is the original map $K' \to \Re W$. The relevant homotopy is supplied by Proposition 10.1. ▮

Proposition 10.3. A morphism of simplicial spaces $W \to X$ is a realization equivalence if and only if for any $n \ge 0$, $i \ge 0$, and a commutative square

$$\begin{tikzcd} \lambda\,\mathrm{Sd}^i \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ \lambda\,\mathrm{Sd}^i \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

one can increase i and find a diagonal arrow such that the upper triangle commutes strictly and the bottom triangle commutes up to a homotopy $\lambda\,\mathrm{Sd}^i(\Delta^1 \times \Delta^n) \to Y$.

Proof. The morphism of simplicial spaces $W \to X$ is a realization equivalence if and only if $\mathrm{Ex}^\infty \Re W \to \mathrm{Ex}^\infty \Re X$ is a weak equivalence of fibrant simplicial sets, which is equivalent to requiring that for any commutative square

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Ex}^\infty \Re X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathrm{Ex}^\infty \Re Y \end{tikzcd}$$

one can find a diagonal arrow such that the upper triangle commutes strictly and the lower triangle commutes up to a homotopy $\Delta^1 \times \Delta^n \to \mathrm{Ex}^\infty \Re Y$. The simplicial sets $\partial\Delta^n$ and Δ^n are compact and therefore maps from them factor through some finite stage Ex^i (this is when we might need to increase i, since the newly constructed maps might factor through a higher stage). The adjunction $\lambda \dashv \Re$ completes the proof. ∎

weakeq 11 Weak equivalences of simplicial spaces

In this section we establish two criteria for detecting weak equivalences of simplicial spaces, which generalize the classical results about simplicial sets and topological spaces. We start by proving a homotopy coherent version of this result and then leverage it to obtain a statement about simplicial spaces.

The following proposition generalizes a classical criterion in simplicial homotopy theory for weak equivalences, see Theorem 5.2 in Dugger and Isaksen [WESP]. We briefly recall the statement for simplicial sets first: a morphism $f\colon X \to Y$ of fibrant simplicial sets is a weak equivalence if and only if for any commutative square

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

we can find a diagonal arrow $\Delta^n \to X$ such that the upper triangle commutes strictly and the two compositions in the lower triangle are connected by a simplicial homotopy $\Delta^1 \times \Delta^n \to Y$.

sweakeqcrit **Sphere filling criterion 11.1.** A morphism $f\colon X \to Y$ of Reedy fibrant simplicial spaces is a realization equivalence if and only if for each commutative square

$$\begin{tikzcd} \iota\,\mathrm{Sd}^i\,\partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ \iota\,\mathrm{Sd}^i\,\Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

we can increase i so that there is a diagonal morphism $\iota\,\mathrm{Sd}^i\,\Delta^n \to X$ such that the upper triangle commutes and the two maps $\iota\,\mathrm{Sd}^i\,\Delta^n \to Y$ coming from the lower triangle are connected by a homotopy $\iota\,\mathrm{Sd}^i(\Delta^1 \times \Delta^n) \to Y$ whose restriction to $\iota\,\mathrm{Sd}^i(\Delta^1 \times \partial\Delta^n)$ factors through the projection to $\iota\,\mathrm{Sd}^i\,\partial\Delta^n$. Used in 9.12*, 9.12*.

Proof. ∎

For a quasicategorical proof of the following lemma see Corollary 6.2 and Remark 6.3 in Mazel-Gee [MIC]. Recall that an object A is compact if $\mathrm{Hom}(A,-)$ preserves filtered colimits. A simplicial set is compact if and only if it has finitely many nondegenerate simplices.

lift **Lifting lemma 11.2.** Given a Reedy fibrant simplicial space W, a cofibration $K \to L$ of compact simplicial sets, a morphism of simplicial spaces $\iota K \to W$ that presents some map $K \to \Re W$, and a morphism $L \to \Re W$ together with a homotopy $h\colon \Delta^1 \times K \to \Re W$ from $K \to \Re W$ to $K \to L \to \Re W$, we can find $i \geq 0$ and a morphism $\iota\,\mathrm{Sd}^i\,L \to W$ such that $\iota\,\mathrm{Sd}^i\,K \to \iota\,\mathrm{Sd}^i\,L \to W$ equals $\iota\,\mathrm{Sd}^i\,K \to \iota K \to W$, and a simplicial homotopy p from $\mathrm{Sd}^i\,L \to \Re W$ to $\mathrm{Sd}^i\,L \to L \to \Re W$ whose whiskering with $\mathrm{Sd}^i\,K \to \mathrm{Sd}^i\,L$ is h whiskered with $\mathrm{Sd}^i\,K \to K$:

$$\begin{tikzcd} \iota\,\mathrm{Sd}^i\,K \arrow[r] \arrow[d] & \iota K \arrow[d] & \mathrm{Sd}^i\,K \arrow[r] \arrow[d] & K \arrow[d] \arrow[dr] & \\ \iota\,\mathrm{Sd}^i\,L \arrow[r] & W, & \mathrm{Sd}^i\,L \arrow[r] & L \arrow[r] & \Re W. \end{tikzcd}$$

In particular, for $K = \emptyset$ any morphism $L \to \Re W$ can be lifted to a morphism $\iota\,\mathrm{Sd}^i\,L \to W$, meaning that the latter presents a morphism homotopic to the composition $\mathrm{Sd}^i\,L \to L \to \Re W$. Used in 9.12*.

Proof. We start by modifying the input data so that h is trivial. The pushout product of the acyclic cofibration $1\colon \Delta^0 \to \Delta^1$ and the cofibration $K \to L$ is the acyclic cofibration $L \sqcup_K \Delta^1 \times K \to \Delta^1 \times L$. The data of the map $L \to \Re W$ and the homotopy $h\colon \Delta^1 \times K \to \Re W$ assemble into a map $L \sqcup_K \Delta^1 \times K \to \Re W$, which by the fibrancy of $\Re W$ admits an extension to $\Delta^1 \times L$. The restriction of the resulting map $\Delta^1 \times L \to \Re W$ to $0 \times L$ is the desired new map $L \to \Re W$. Since h is an identity now, the whiskering of the homotopy p will also be an identity, i.e., the homotopy p will be relative $\mathrm{Sd}^i K$. The above diagrams can now be interpreted as strictly commutative diagrams, except for p.

A morphism $L \to \Re W = \mathrm{Ex}^\infty \operatorname{diag} W$ factors through $\mathrm{Ex}^i \operatorname{diag} W$ for some $i \geq 0$ by compactness of L. The resulting morphism $L \to \mathrm{Ex}^i \operatorname{diag} W$ is adjoint to $\mathrm{Sd}^i L \to \operatorname{diag} W$. The latter morphism in its turn is adjoint to $\kappa \,\mathrm{Sd}^i L \to W$, where $\kappa \dashv \operatorname{diag}$. This is almost the morphism we need, except that its domain is $\kappa \,\mathrm{Sd}^i L$, not $\iota \,\mathrm{Sd}^i L$. We remark that the restriction of this morphism to $\kappa \,\mathrm{Sd}^i K$ equals the composition $\kappa \,\mathrm{Sd}^i K \to \iota \,\mathrm{Sd}^K \to \iota K \to W$ by construction.

We now construct two Reedy acyclic cofibrations (i.e., componentwise acyclic cofibrations) of simplicial spaces, with domains $\kappa \,\mathrm{Sd}^i L$ and $\iota \,\mathrm{Sd}^i L$. Both functors κ and ι can be defined using left Kan extensions for functors $n \mapsto \Delta^{n,n}$ and $n \mapsto \Delta^{n,0}$. Thus it suffices to find a zigzag of natural Reedy acyclic cofibrations $\Delta^{n,n} \to A_n \leftarrow \Delta^{n,0}$. Take A_n to be the external bisimplicial product of Δ^n and the simplicial join $\Delta^0 \star \Delta^n$ of Δ^0 and Δ^n. The natural acyclic cofibrations of spaces $\Delta^{n,0} \to A_n$ and $\Delta^{n,n} \to A_n$ cover the first and second component of the join, respectively. Thus we have a zigzag of natural transformations $\kappa \to A \leftarrow \iota$ and also $A \to \iota$, all of which are weak equivalences.

The map $\kappa \,\mathrm{Sd}^i L \to W$ and the map $q\colon A(\mathrm{Sd}^i K) \to \iota \,\mathrm{Sd}^i K \to \iota K \to W$ assemble together into a map $\kappa \,\mathrm{Sd}^i L \sqcup_{\kappa \,\mathrm{Sd}^i K} A(\mathrm{Sd}^i K) \to W$. The inclusion $\kappa \,\mathrm{Sd}^i L \sqcup_{\kappa \,\mathrm{Sd}^i K} A(\mathrm{Sd}^i K) \to A(\mathrm{Sd}^i L)$ is a Reedy acyclic cofibration because it is a cobase change of the Reedy acyclic cofibration $\kappa \,\mathrm{Sd}^i K \to A(\mathrm{Sd}^i K)$. Since the simplicial space W is Reedy fibrant, we can extend the map q along this inclusion, obtaining a map $A(\mathrm{Sd}^i L) \to W$. The map $A(\mathrm{Sd}^i L) \to W$ can then be restricted along the Reedy acyclic cofibration $\iota \,\mathrm{Sd}^i L \to A(\mathrm{Sd}^i L)$, giving us the desired morphism $\iota \,\mathrm{Sd}^i L \to W$, whose restriction to $\iota \,\mathrm{Sd}^i K$ coincides with the composition $\iota \,\mathrm{Sd}^i K \to \iota K \to W$ by construction. The induced map $\operatorname{diag} A(\mathrm{Sd}^i L) \to \Re W$ implements a homotopy (of nonstandard shape) from $\mathrm{Sd}^i L = \operatorname{diag} \iota \,\mathrm{Sd}^i L \to \Re W$ to $\operatorname{diag} \kappa \,\mathrm{Sd}^i L \to \Re W$, which is relative $\mathrm{Sd}^i K$ by construction. This homotopy can be composed with the constant homotopy (of the same nonstandard shape $\operatorname{diag} A(\mathrm{Sd}^i L) \to \mathrm{Sd}^i L \to \Re W$) from $\operatorname{diag} \kappa \,\mathrm{Sd}^i L \to \Re W$ to $\mathrm{Sd}^i L = \operatorname{diag} \iota \,\mathrm{Sd}^i L \to \Re W$. The resulting composition is a homotopy of the form $A(\mathrm{Sd}^i L) \sqcup_{\kappa \,\mathrm{Sd}^i L} A(\mathrm{Sd}^i L) \to \Re W$ relative $\mathrm{Sd}^i K$ (by construction), with the desired source and target. The only remaining problem is that the constructed homotopy uses a nonstandard cylinder object instead of $\Delta^1 \times \mathrm{Sd}^i L$. Since $\Re W$ is fibrant, we use a trick with extension and restriction along acyclic cofibrations once again to turn this nonstandard relative homotopy into a standard homotopy $\Delta^1 \times \mathrm{Sd}^i L \to \Re W$ relative $\mathrm{Sd}^i K$. ▮

sources 12 Sources other than smooth manifolds

We now generalize the main theorem for stacks of spaces on the site of smooth manifolds to nontriangulable topological manifolds. This is achieved by the following lemma.

toptrian **Lemma 12.1.** Any topological manifold is homotopy equivalent to a triangulable topological manifold.

Used in 12.2*.

Proof. We reduce to the case of connected (hence second countable) manifolds X. Choose a proper map $X \to \mathbf{R}^N$. By Topological General Position Lemma 1 in Dancis [GenPos] for sufficiently high N every proper map $X \to \mathbf{R}^N$ can be properly homotoped to a proper locally flat embedding. By Theorem 2.1 in Edwards [TRN] this embedding admits a topological regular neighborhood. The interior of this neighborhood is an open subset of $\mathbf{R}^N$ that is homotopy equivalent to the original manifold. As explained in the proof of Lemma 12.3, any open subset of $\mathbf{R}^N$ admits a smooth triangulation. ▮

Corollary 12.2. On the site of topological manifolds, the concordance comparison map of a stack F is a weak equivalence of prestacks.

Proof. For triangulable topological manifolds see the proof of Theorem 9.10; dropping smoothness allows one to discard some parts of the proof. A nontriangulable topological manifold is homotopy equivalent to a triangulable topological manifold by Lemma 12.1. The resulting natural square of concordance comparison maps has weak equivalences for three out of four maps: the concordance comparison map for the triangulable manifold is a weak equivalence by the above argument, whereas the two maps induced by the homotopy equivalence are themselves homotopy equivalences (recall that mapping a homotopy equivalence into $\mathcal{C}F$ produces a homotopy equivalence because $\mathcal{C}F$ is concordance-invariant). Hence, the concordance comparison map for any topological map is a weak equivalence. ▮

The following lemma is not necessary for the main theorem, however, it allows us to prove it without referencing the rather subtle and delicate proof of the existence of smooth triangulations of smooth manifolds.

smoothtrian **Lemma 12.3.** Any smooth manifold is smooth homotopy equivalent to a smoothly triangulable smooth manifold. Used in 9.9, 12.1*.

Proof. We can immediately reduce to the case of connected (hence second countable) manifolds. By the Whitney theorem any smooth manifold admits a proper smooth embedding into some $\mathbf{R}^N$. The interior of a tubular neighborhood of this embedding is an open submanifold of $\mathbf{R}^N$ that is smoothly homotopy equivalent to the original manfold. A smooth triangulation of an open subset U of $\mathbf{R}^N$ can be constructed by picking any triangulation of $\mathbf{R}^N$, taking its simplices that are contained in U, then barycentrically subdividing the remaining simplices, and repeating this process indefinitely. ▮

targets 13 Targets other than spaces

We now use the main theorem for stacks of spaces as a launchpad to obtain a generalization that allows for much more general class of targets than spaces. Below the word "∞-category" is used in an invariant fashion and can mean any of the existing equivalent models for ∞-categories such as Quillen's model categories, Joyal's quasicategories, Segal categories, relative categories etc. All constructions used below, such as mapping spaces and (co)limits are by definition derived. We stick to model categories for the sake of being definite, but nothing in our presentation is specific to the case of model categories.

An object G of an ∞-category T is *compact projective* if its corepresentable functor

$$\mathrm{Map}(G,-)\colon T \to \mathrm{sSet}$$

preserves homotopy sifted colimits. An ∞-category is *algebraic* if it is homotopy locally presentable and the corepresentable functors of compact projective objects detect equivalences: a morphism $f\colon A \to B$ in T is an equivalence if (and only if) the morphism $\mathrm{Hom}(G,f)\colon \mathrm{Hom}(G,A) \to \mathrm{Hom}(G,B)$ is an equivalence of spaces for all compact projective objects G. For details, see Lurie, §5.5.8 in [HTT] and §7.1.4 in [HA], and Rosický [HoVar]. Homotopy sifted colimits commute with finite homotopy products in any algebraic category.

Algebraic categories are closed under the following constructions. (1) An overcategory of an algebraic category is again algebraic. (2) If a right adjoint functor is conservative, preserves sifted colimits, and its codomain is algebraic, then its domain is also algebraic. (Proposition 7.1.4.12(4) in Lurie [HA].) (3) Algebras over a monad that preserves sifted colimits in an algebraic category are again algebraic. (4) Algebras over an multisorted algebraic theory in an algebraic category are again algebraic. (5) Algebras over an operad in a closed symmetric monoidal algebraic category are again algebraic. (6) Presheaves valued in an algebraic category are again algebraic.

Examples of algebraic ∞-categories include spaces (Example 5.5.8.24 in Lurie [HA]), connective spectra (Corollary 7.1.4.13 in Lurie [HA]), module spectra over a connective ring spectrum (Corollary 7.1.4.15 in Lurie [HA]), E_n-spaces, group-like E_n-spaces, E_n-semirings, E_n-rings, connective E_n-ring spectra, connective E_n-algebra spectra over a connective E_{n+1}-ring spectrum (Corollary 7.1.4.17 in Lurie [HA]), categories and E_n-monoidal categories (possibly enriched in a closed symmetric monoidal algebraic category) with a fixed set of objects (and functors that are identity on objects).

maintheorem **Theorem 13.1.** The concordification functor $\mathcal{C}$ preserves stacks valued in any algebraic ∞-category T. Used in I.4, I.8.

Proof. Given an object G in T and a T-valued prestack F denote by $\mathrm{Map}(G,F)$ the space-valued prestack obtained by applying $\mathrm{Map}(G,-)$ componentwise. The functor $\mathrm{Map}(G,-)$ preserves homotopy limits, hence it preserves stacks because the latter are specified using a homotopy limit over Δ. The functor $\mathrm{Map}(G,-)$ also preserves concordance-invariant homotopy (pre)sheaves and in fact it commutes with the concordification functor $\mathcal{C}$ because the latter can be computed using a homotopy colimit over Δ^{op}, which is homotopy sifted and therefore is preserved by the corepresentable functor of a compact projective object. We can now apply the main theorem for the case of spaces and deduce that $\mathcal{C}\,\mathrm{Map}(G,F)$ is a stack of spaces for any T-valued stack F. The descent condition for $\mathcal{C}F$ requires that the restriction map $\mathcal{C}F(X) \to \mathrm{holim}\,\mathcal{C}F(U^\bullet)$ is an equivalence for all covers U of X. Equivalences in T are detected by the functors $\mathrm{Map}(G,-)$ for all compact projective objects G. The latter functors commute with $\mathcal{C}$ and the homotopy limit over Δ^{op}, so the resulting maps are equivalences because $\mathcal{C}\,\mathrm{Map}(G,F)$ is a stack for any compact projective object G. ▮

mainthmrep **Corollary 13.2.** For any stack F on the site of smooth manifolds valued in an algebraic ∞-category T its concordification $\mathcal{C}F$ is representable by $\mathfrak{C}F = (\mathcal{C}F)(\mathrm{pt})$, i.e., the canonical natural transformation

$(\mathcal{C}F)(X) \to \mathrm{Map}(\mathfrak{C}X, \mathfrak{C}F)$ is a weak equivalence, where Map denotes the canonical powering of T over spaces. ▌Used in 14.1*.

unboundeddescent **Remark 13.3.** The fact that T is an algebraic ∞-category is crucial for the main theorem. Two important examples of nonalgebraic ∞-categories are nonconnective spectra and unbounded chain complexes of abelian groups. Indeed, it is easy to see that infinite additivity already fails for the case of unbounded chain complexes. In this case homotopy colimits of simplicial diagrams can be computed by using the Dold—Kan correspondence to pass to a connective chain complex of unbounded chain complexes, i.e., a double complex that is connective in one direction and unbounded in another. One then takes the direct sums along each diagonal, which yields the desired homotopy colimit. If the chain complexes were connective, then each sum would be finite and therefore could also be computed as the product, and finite products commute with infinite products. In the unbounded case infinite direct sums do not commute with infinite products, and it is easy to exploit this fact to construct an explicit counterexample of a stack F whose concordification is not infinitely additive, e.g., set $F(S)$ to be the unbounded chain complex with the zero differential that has $\Omega^k(S)$ in degree $-k$. Used in I.0*, I.4, A.0*.

Remark 13.4. In the case of stable ∞-categories one can still show that the concordance comparison map $\mathcal{C}F(X) \to \mathrm{Map}(\mathfrak{C}X, \mathfrak{C}F)$ is a weak equivalence for any manifold X that admits a finite triangulation T, e.g., a compact manifold. In this case the functor $\mathrm{Hom}(T, -)$ in Lemma 9.12 can be rewritten as a finite homotopy limit, and in a stable ∞-category finite homotopy limits commute with arbitrary small colimits, hence Lemma 9.12 holds in this setting, and the rest of the proof is the same. This is essentially Proposition 7.6 in Bunke, Nikolaus, and Völkl [DiffSpec], which shows that the concordification functor preserves stacks with respect to the Grothendieck topology of *finite* open covers.

14 Homotopy groups and higher fundamental groupoids of concordance spaces

In this section we apply the main theorem to establish a relation between geometric and categorical homotopy groups of F, the latter being defined as the homotopy groups of $\mathfrak{C}F$ and the former are obtained by taking pointed concordance classes of F over smooth spheres. This gives us a powerful tool that allows to perform computations with $\mathfrak{C}F$ by working with sections of F.

hogroups **Proposition 14.1.** For any linear presheaf F and any choice of a basepoint $* \in \mathfrak{C}F$ (equivalently, $* \in F(\mathrm{pt})$) we have a natural equivalence $\mathcal{C}F(\mathfrak{S}^n_*) \to (\mathfrak{C}F)^{\mathcal{S}^n_*}$ (presented by a zigzag of natural weak equivalences), in particular there is an isomorphism $F[\mathfrak{S}^n_*] \to \pi_n(\mathfrak{C}F)$ between pointed concordance classes of F over the smooth n-sphere and elements in the nth homotopy group of $\mathfrak{C}F$. Used in 1.3*, 8.2*, 14.5.

Proof. The comparison map $\mathcal{C}F(\mathfrak{S}^n) \to \mathrm{Map}(\mathcal{S}^n, \mathfrak{C}F)$ is a weak equivalence by the main theorem 13.2, or rather its linear part. We also have $\mathcal{C}F(\mathrm{pt}) = \mathfrak{C}F$ and there are two canonical maps $\mathcal{C}F(\mathfrak{S}^n) \to \mathcal{C}F(\mathrm{pt})$ and $\mathrm{Map}(\mathcal{S}^n, \mathfrak{C}F) \to \mathrm{Map}(\mathrm{pt}, \mathfrak{C}F) = \mathfrak{C}F$ induced by the inclusions $\mathrm{pt} \to \mathfrak{S}^n$ and $\mathrm{pt} \to \mathcal{S}^n$. These maps can be organized in a commutative diagram

$$\begin{array}{ccc} \mathcal{C}F(\mathfrak{S}^n) & \longrightarrow & \mathrm{Map}(\mathcal{S}^n, \mathfrak{C}F) \\ \downarrow & & \downarrow \\ \mathcal{C}F(\mathrm{pt}) & \longrightarrow & \mathrm{Map}(\mathrm{pt}, \mathfrak{C}F), \end{array}$$

where all objects and maps are pointed, by using the given basepoint $*$. The horizontal maps are weak equivalences, hence so is their homotopy fiber $\mathcal{C}F(\mathfrak{S}^n_*) \to \mathrm{Map}(\mathcal{S}^n_*, \mathfrak{C}F)$.

The following proposition is a generalization of the above construction.

Proposition 14.2. For any simplicial set S and a stack F (or a linear prestack if S is compact), the canonical natural map $\Re((\mathcal{R}F)^S) \to (\mathfrak{C}F)^{\Re S}$ is a weak equivalence.

Recall that the n-*coskeleton* functor of a simplicial set X for $n \geq 0$ computes the nth level of the Postnikov tower X, which can be denoted as $\pi_{<n}X$ (the fundamental $(n-1)$-groupoid of X), meaning that homotopy groups in degrees less than n are preserved and the other homotopy groups are killed. The n-coskeleton of X can also be computed as $k \in \Delta^{\mathrm{op}} \mapsto \mathrm{Hom}(\mathrm{skel}_n \Delta^k, X)$, where skel_n is the n-*skeleton* functor, the left adjoint of cosk_n. The latter definition makes sense for simplicial objects, and we use it below.

Proposition 14.3. Given a cofibration $K \to L$ of simplicial sets, a stack F, and a morphism $K \to \mathcal{R}F$ that presents a point in $(\mathcal{R}F)^K$, hence in $(\mathfrak{C}F)^{\Re K}$, the homotopy pullback $(\mathfrak{C}F)^{\Re L} \to (\mathfrak{C}F)^{\Re K} \leftarrow \mathrm{pt}$ can be computed as the realization of the homotopy pullback of $(\mathcal{R}F)^L \to (\mathcal{R}F)^K \leftarrow \mathrm{pt}$.

Remark 14.4. Already for $n = 0$ one can see that the essence of the proposition lies in the fact that a single concordance is sufficient to relate any pair of objects that lie in the same connected component of $\mathfrak{C}F$, even though a priori one might need chains of concordances of arbitrary length.

Remark 14.5. Substituting first $\emptyset \to \mathrm{S}^{n-1}$ and then $\mathrm{S}^{n-1} \to (\Delta^1 \times \mathrm{S}^{n-1}/\Delta^1 \times *)$ we obtain a variant of 14.1, where instead of $F[\mathbf{S}^{n-1}_*]$ we have $F[\mathbf{\Delta}^{n-1}_{\partial\mathbf{\Delta}^{n-1}}]$, i.e., concordance classes of sections of F over $\mathbf{\Delta}^{n-1}$ whose restriction to $\partial\mathbf{\Delta}^{n-1}$ is pulled back via $\partial\mathbf{\Delta}^{n-1} \to \mathbf{\Delta}^0$.

Proof. We have a canonical map $\Re \operatorname{cosk}_n G \to \operatorname{cosk}_n \Re G$ (the left cosk_n is a simplicial space, whereas the right cosk_n is a space), whose adjoint map $\operatorname{cosk}_n G \to \operatorname{const} \operatorname{cosk}_n \Re G$. ∎

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