The classification of two-dimensional extended conformal field theories

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Quantum Field Theory and Topological Phases via Homotopy Theory and Operator Algebras
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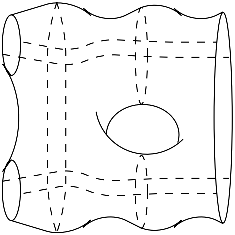
These slides: `https://dmitripavlov.org/cmsa.pdf`

arXiv:2011.01208, arXiv:2111.01095 (joint with Daniel Grady)
+ work in progress



Main theorem: conformal field theory

Theorem

The following categories are equivalent:

- *extended conformal field theories;*
- *Serre-twisted homotopy coherent reps of* $\mathbf{R}^2 \rtimes \widetilde{\mathrm{Conf}}(2)$.

Notation:

- $\widetilde{\mathrm{Conf}}(2)$*: the universal covering of* $\mathrm{Conf}(2)$.
- $\mathrm{Conf}(2)$*:* $z \mapsto \sum_{k \geq 1} a_k z^k$, $a_1 \neq 0$, *convergent with* $R > 0$, *group operation: composition.*
- *Serre-twisted: restricting to* $\mathbf{Z} \subset \widetilde{\mathrm{Conf}}(2) \subset \mathbf{R}^2 \rtimes \widetilde{\mathrm{Conf}}(2)$ *yields powers of Serre automorphisms.*
- *Example: Serre automorphisms are trivial* $\rightsquigarrow$ *homotopy coherent representations of* $\mathbf{R}^2 \rtimes \mathrm{Conf}(2)$.

Variants: Twisted/relative, chiral, $2|1$-Euclidean.

Features of the geometric bordism category

- Locality (Freed, Lawrence): k-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Isotopy (Costello, Hopkins, Lurie): chain complexes to encode BV-BRST
  $\Longrightarrow$ must encode (higher) diffeomorphisms between bordisms
  $\Longrightarrow$ symmetric monoidal (∞, d)-categories
- Geometric (nontopological) structures on bordisms (Segal, Stolz, Teichner): Riemannian/Lorentzian metrics, complex/conformal/symplectic/contact structures, principal G-bundles with connection and isos, higher gauge fields (Kalb–Ramond, Ramond–Ramond)
  $\Longrightarrow$ an $(\infty, 1)$-sheaf of geometric structures
- Smoothness (Stolz, Teichner): values of field theories depend smoothly (or holomorphically, super, ...) on bordisms
  $\Longrightarrow$ $(\infty, 1)$-sheaf of (∞, d)-categories of bordisms

Ingredients of the classification

1. Locality of extended functorial field theories (arXiv:2011.01208) ($\rightsquigarrow$ reduction to simpler geometric structures)
2. Relative geometric cobordism hypothesis (arXiv:2111.01095) (handles of index $\leq k-1 \rightsquigarrow$ handles of index $\leq k$)
3. 1 and 2 $\Rightarrow$ geometric cobordism hypothesis

$$\mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}) \simeq \mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}_d^{fd,\times}),$$

   (topological case: Lurie, 2009)
4. (P.) A computation of the right side for 2-dimensional CFTs (Quillen Theorem A, Thomason's theorem, Riemann mapping theorem).

Other applications of GCH

- (Grady–P.) Invertible geometric FFTs are classified by the geometric Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)
- (Grady–P.) A conjecture of Stolz and Teichner: concordance classes of extended FFTs have a classifying space. (Proof: Locality + the smooth Oka principle (Berwick-Evans–Boavida de Brito–P.).
- (P.) Classification of $2|1$-Euclidean field theories.
- (Grady) Classification of deformation classes of reflection positive invertible geometric FFTs (Conjecture 8.37 in *Reflection positivity and invertible topological phases* by Freed–Hopkins)

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Thank you!

Main theorem 2: $2|1$-Euclidean field theory

Theorem

The following smooth ∞-categories are equivalent:

- *extended* $2|1$*-Euclidean field theories;*
- *Serre-twisted homotopy coherent representations of the Lie supergroup* $\widetilde{\mathrm{Euc}}(2|1)$ *on a 2-dualizable object.*

Notation:

- $\widetilde{\mathrm{Euc}}(2|1)$*: the universal covering of* $\mathrm{Euc}(2|1) = \mathbf{R}^{2|1} \rtimes \mathrm{Spin}(2)$.
- *Serre-twisted: restricting to* $\mathbf{Z} \subset \widetilde{\mathrm{Euc}}(2|1)$ *yields Serre automorphisms.*
- *Serre automorphisms trivial* $\Longrightarrow$ *representations of* $\mathrm{Euc}(2|1)$.

What is functorial field theory?

Want to study integrals of the form

$$\int_{\varphi} \exp(i\hbar^{-1} S(\varphi)) \in \mathbf{C}.$$

- X: spacetime; e.g., $\mathbf{R}^4$
- $\mathcal{F}\colon E \to X$: field bundle; e.g., $\mathbf{R} \times X \to X$
- φ: field: section of $\mathcal{F}\colon E \to X$; e.g., $\varphi \in \mathrm{C}^\infty(X)$ (scalar field)
- $S\colon \Gamma_{\mathcal{F}}(X) \to \mathbf{R}$: action functional.

What kind of manifold is the spacetime X?

- Closed manifold.
- More generally: X is compact with boundary $\partial X = M_0 \sqcup M_1$; write $X\colon M_0 \to M_1$, i.e., X is a bordism from M_0 to M_1.

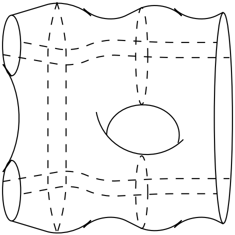
Quantum propagators and Segal gluing

$$\int_\varphi \exp(i\hbar^{-1}S(\varphi)) \in \mathbf{C}, \qquad \varphi \in \Gamma_{\mathcal{F}}(X), \qquad X\colon M_0 \to M_1.$$

- For fixed $\alpha_i = \varphi|_{M_i} \in \Gamma_{\mathcal{F}}(M_i)$, get $K(\alpha_1, \alpha_0) = \int_\varphi \in \mathbf{C}$.
- K is the integral kernel of an operator $F(X)\colon F(M_0) \to F(M_1)$ (propagator).
- Here $F(M_i) = \mathcal{O}(\Gamma_{\mathcal{F}}(M_i))$ (space of states).
- Fubini property (Segal gluing): if $X_1\colon M_0 \to M_1$, $X_2\colon M_1 \to M_2$, then $F(X_2 \sqcup_{M_1} X_1) = F(X_2) \circ F(X_1)$.

$$\int_\varphi \exp(i\hbar^{-1}S(\varphi)) = \int_{\alpha_1}\int_{\varphi_1}\int_{\varphi_2} \exp(i\hbar^{-1}(S(\varphi_1) + S(\varphi_2)))$$

How to compose bordisms



Axioms for quantum propagators in the Schrödinger picture

$\mathcal{F}\colon E \to X$ (field bundle); $F(M) = \mathcal{O}(\Gamma_{\mathcal{F}}(M))$ (space of states)

$$\begin{aligned} F(M \sqcup N) &= \mathcal{O}(\Gamma_{\mathcal{F}}(M \sqcup N)) \cong \mathcal{O}(\Gamma_{\mathcal{F}}(M) \oplus \Gamma_{\mathcal{F}}(N)) \\ &\cong \mathcal{O}(\Gamma_{\mathcal{F}}(M)) \otimes \mathcal{O}_{\mathcal{F}}(\Gamma_{\mathcal{F}}(N)) = F(M) \otimes F(N). \end{aligned}$$

- Segal gluing (Fubini): $F(X_2 \sqcup_{M_1} X_1) = F(X_2) \circ F(X_1)$.
- Monoidality: $F(M \sqcup N) \cong F(M) \otimes F(N)$.
- Segal (following Feynman, Witten): axiomatize Fubini and monoidality as a symmetric monoidal functor (i.e., a functorial field theory)

$$F\colon \mathrm{Bord} \to \mathrm{Vect}.$$

- Bord: objects: $(d-1)$-manifolds M; morphisms: bordisms $X\colon M_0 \to M_1$.
- Vect: objects: vector spaces; morphisms: linear maps.

Geometric structures

Definition

Given $d \geq 0$, the site FEmb_d has

- Objects: submersions $T \to U$ with d-dimensional fibers, where $U \cong \mathbf{R}^n$ is a cartesian manifold;
- Morphisms: commutative squares with $T \to T'$ a fiberwise open embedding over a smooth map $U \to U'$;
- Covering families: open covers on total spaces T.

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Definition (Nijenhuis 1958)

Given $d \geq 0$, a d-dimensional geometric structure is a simplicial presheaf $\mathcal{S}\colon \mathrm{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$.

Example

- $T \to U \mapsto$ the set of fiberwise Riemannian metrics on $T \to U$;
- $(T \to T', U \to U') \mapsto$ the restriction map from T' to T.

Examples of geometric structures

- topological structures (i.e., isotopy-invariant): orientations, spin stuctures, framings, etc. (TQFT as studied by Atiyah, Kontsevich, Reshetikhin, Turaev, Viro, Freed, Lawrence, Quinn, Hopkins, Lurie, . . .);
- fiberwise Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- fiberwise conformal, complex, symplectic, contact, Kähler structures;
- fiberwise foliations, possibly with transversal metrics;
- smooth map to a target manifold M (traditional σ-model);
- smooth map to an orbifold or ∞-sheaf on manifolds;
- fiberwise etale map or an open embedding into a target manifold N;
- fiberwise differential n-forms (possibly closed).

Examples of geometric structures: gauge transformations

Definition

- Send a d-manifold M to (the nerve of) the groupoid $\mathrm{B}_\nabla G(M)$:
  - Objects: principal G-bundles on T with a fiberwise connection on $T \to U$ (gauge fields);
  - Morphisms: connection-preserving isomorphisms (gauge transformations).

Examples of geometric structures: (higher) gauge transformations

- Principal G-bundles with connection on M (gauge fields, e.g., the electromagnetic field);
- Bundle gerbe with connection on M (B-field, Kalb–Ramond field).
- Bundle 2-gerbe with connection on M (supergravity C-field).
- Bundle $(d-1)$-gerbes with connection on M (Deligne cohomology, Cheeger–Simons characters, ordinary differential cohomology, circle d-bundles).
- Geometric tangential structures: geometric Spin^c-structure, String (Waldorf), Fivebrane (Sati–Schreiber–Stasheff), Ninebrane (Sati). (Vanishing of anomaly.)
- differential K-theory (Ramond–Ramond field). Requires ∞-groupoids.

The geometric cobordism hypothesis

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}\colon \mathsf{FEmb}_d^{\mathsf{op}} \to \mathsf{sSet}$.

Constructions:

- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
- A d-dimensional functorial field theory valued in $\mathcal{V}$ with geometric structure $\mathcal{S}$ is a smooth symmetric monoidal (∞, d)-functor $\mathfrak{Bord}_d^{\mathcal{S}} \to \mathcal{V}$.
- The simplicial set of d-dimensional functorial field theories valued in $\mathcal{V}$ with geometric structure $\mathcal{S}$ is the derived mapping simplicial set

$$\mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) = \mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}).$$

Can be refined to a derived internal hom.

The geometric cobordism hypothesis

Conjectures (for topological field theories):

- Freed, Lawrence (1992): $\mathsf{FFT}_{d,\mathcal{V}}$ is an ∞-sheaf.
- Baez–Dolan (1995), Hopkins–Lurie (2008):

$$\mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) \simeq \mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}^{\times}).$$

$\mathcal{V}^{\times}$: fully dualizable objects and invertible morphisms.

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Theorem (Grady–P., The geometric cobordism hypothesis)

Part I (Locality): $\mathfrak{Bord}_d$ is a left adjoint functor:

$$\mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}) \simeq \mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}_d^\times),$$

where $\mathcal{V}_d^\times = \mathsf{FFT}_{d,\mathcal{V}}$, i.e., $\mathcal{V}_d^\times(T \to U) = \mathsf{FFT}_{d,\mathcal{V}}(T \to U)$.

Part II (Framed GCH): The evaluation-at-points map

$$\mathcal{V}_d^\times(\mathbf{R}^d \times U \to U) = \mathsf{FFT}_{d,\mathcal{V}}(\mathbf{R}^d \times U \to U) \to \mathcal{V}^\times(U)$$

is a weak equivalence of simplicial sets functorial in U.

Computing with GCH

- How to compute $\mathcal{V}_d^\times$?
- How to compute $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}_d^\times)$?

Computing with GCH

- How to compute $\mathcal{V}_d^\times$?
  - Simplicial presheaves and sheaf cohomology
  - Integration; differential forms; de Rham theory
  - Need to be done only once per choice of $\mathcal{V}$; precomputed results exist
- How to compute $\mathbf{R}\operatorname{Map}(\mathcal{S}, \mathcal{V}_d^\times)$?
  - Homotopy colimits; Quillen Theorem A; Thomason's theorem
  - Simplicial presheaves and sheaf cohomology
  - Natural operations in differential geometry (Kolář–Michor–Slovák)
  - Homotopy coherent representation theory of (higher) Lie groups

Computing $\mathcal{V}_d^\times$

- Already know $\mathcal{V}_d^\times(\mathbf{R}^d \times U \to U) \simeq \mathcal{V}^\times(U)$, functorial in $U \in \mathsf{Cart}$.
- What are the structure maps for functoriality in FEmb_d?
- Step 1: Guess a map $\mathcal{W} \to \mathcal{V}_d^\times$.
- Step 2: For every U, prove $\mathcal{W}(\mathbf{R}^d \times U \to U) \to \mathcal{V}_d^\times(\mathbf{R}^d \times U \to U) \to \mathcal{V}^\times(U)$ is a weak equivalence.

Example ($\mathcal{V} = \mathrm{B}^d\mathrm{U}(1)$; prequantum FFTs)

- Step 1a: $\mathcal{W}(\mathbf{R}^d \times U \to U) = U\Gamma(\Omega^d_U(\mathbf{R}^d \times U) \leftarrow \cdots \leftarrow \Omega^1_U(\mathbf{R}^d \times U) \leftarrow \mathrm{C}^\infty(\mathbf{R}^d \times U, \mathrm{U}(1)))$.
- Step 1b: $\mathcal{W} \to \mathcal{V}_d^\times$: $\omega \mapsto (B \mapsto \exp(\frac{i}{\hbar}\int_B \omega))$.
- Step 2: Poincaré lemma: $\mathcal{W}(\mathbf{R}^d \times U \to U) \xrightarrow{\sim} \mathrm{B}^d\mathrm{C}^\infty(U, \mathrm{U}(1))$

How to compute $\mathbf{R}\operatorname{Map}(\mathcal{S}, \mathcal{W})$?

Two main options:

- Use the theory of natural operations, working on the site FEmb_d.
  Examples: differential characteristic classes yield prequantum field theories.
- Use an adjunction to switch to a different category: $\operatorname{Fun}(\mathsf{Cart}^{\mathsf{op}}, \mathsf{sSet}^{\mathrm{O}(d)})$.
  Examples: classification of conformal or Euclidean field theories.

Categories of geometric structures

Proposition

The functors q^* and ι^* are right Quillen equivalences.

$$\begin{array}{ccccc} \mathcal{S}h(\mathrm{FEmb}_d) & \xleftarrow{\rho^*} & \mathcal{S}h(\mathfrak{FEmb}_d) & \xrightarrow{\iota^*} & \mathcal{S}h(\mathrm{Cart})^{\mathrm{O}(d)} \\ {\scriptstyle q^*}\downarrow & & {\scriptstyle q^*}\downarrow & \nearrow{\scriptstyle \iota^*} & \\ \mathcal{S}h(\mathrm{FEmbCart}_d) & \xleftarrow{\rho^*} & \mathcal{S}h(\mathfrak{FEmbCart}_d). & & \end{array}$$

- $\mathcal{S}h(C)$: simplicial presheaves on C, Čech-local model structure
- $\mathfrak{FEmb}_d$: like FEmb_d, but enriched in spaces
- $\mathrm{FEmbCart}_d$: full subcategory of FEmb_d on $\mathrm{D}_U := (\mathbf{R}^d \times U \to U)$
- $\mathfrak{FEmbCart}_d$: equivalent to $\mathrm{Cart} \times \mathrm{BO}(d)$ by C^∞ Kister–Mazur

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\mathcal{S}h(\mathsf{FEmbCart}_d) & \xleftarrow{\rho^*} & \mathcal{S}h(\mathfrak{FEmbCart}_d). & &
\end{array}$$

The functor $\rho_!$ adds "rank d homotopies / isotopies" to a geometric structure.

d-dimensional holonomy is invariant under rank d homotopies.

$d = 1$: Kobayashi, Barrett, Caetano–Picken

$d > 1$: Bunke–Turner–Willerton, Picken, Mackaay–Picken

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\end{array}
$$

Recipe to compute $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, \rho^*\mathcal{V}_d^\times)$.

- Use q^* to move to $\mathrm{FEmbCart}_d$ / $\mathfrak{FEmbCart}_d$. (Suppressed from the notation.)
- $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, \rho^*\mathcal{V}_d^\times) \simeq \mathbf{R}\,\mathrm{Map}(\rho_!\mathcal{S}, \mathcal{V}_d^\times)$.
- Compute $\rho_!\mathcal{S}$.
- $\mathbf{R}\,\mathrm{Map}(\rho_!\mathcal{S}, \mathcal{V}_d^\times) \simeq \mathbf{R}\,\mathrm{Map}(\iota^*\rho_!\mathcal{S}, \iota^*\mathcal{V}_d^\times)$. ($\mathrm{C}^\infty$ Kister–Mazur)

How to compute $\rho_! \mathcal{S}$?

Notation:

- $\mathsf{FEmbCart}_d$: Objects $\mathsf{D}_U = (\mathbf{R}^d \times U \to U)$, morphisms: fiberwise open embeddings.
- $\mathfrak{FEmbCart}_d$: Objects $\mathfrak{D}_U$, space of morphisms.
- $\rho\colon \mathsf{FEmbCart}_d \to \mathfrak{FEmbCart}_d$: inclusion.
- $\rho_!\colon \mathcal{S}h(\mathsf{FEmbCart}_d) \to \mathcal{S}h(\mathfrak{FEmbCart}_d)$: left Kan extension.

Computation:

- $\rho_! \mathcal{S} = \rho_! \operatorname{hocolim}_{\mathsf{D}_U \to \mathcal{S}} Y(\mathsf{D}_U) = \operatorname{hocolim}_{\mathsf{D}_U \to \mathcal{S}} Y(\mathfrak{D}_U)$.
- Evaluate on $\mathfrak{D}_W$:

$$(\rho_! \mathcal{S})(\mathfrak{D}_W) = \operatorname*{hocolim}_{\mathsf{D}_U \to \mathcal{S}} \mathfrak{FEmbCart}_d(\mathfrak{D}_W, \mathfrak{D}_U).$$

- $\mathfrak{FEmbCart}_d(\mathfrak{D}_W, \mathfrak{D}_U)$ is 1-truncated. Ob: $\varphi\colon \mathsf{D}_W \to \mathsf{D}_U$. Mor $\gamma\colon \varphi \to \varphi'$: isotopy classes of isotopies from φ to φ' (form a $\mathbf{Z}$-torsor).

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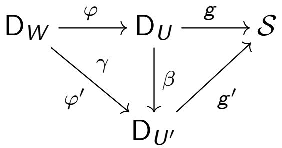
- Thomason's theorem: hocolim computed as the Grothendieck construction F. Ob: $\mathsf{D}_W \xrightarrow{\varphi} \mathsf{D}_U \xrightarrow{g} \mathcal{S}$. Mor $(\varphi, g) \to (\varphi', g')$: $\beta \colon \mathsf{D}_U \to \mathsf{D}_{U'}$: $g = g'\beta$, $\gamma \colon \beta\varphi \to \varphi'$.



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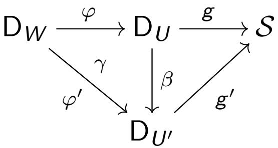

- $\mathrm{BC}^\infty(W, \mathbf{R}^2 \rtimes \widetilde{\mathrm{Conf}}(2))$. Ob: germ of D_W around 0. Mor: displacement + automorphism of a germ.

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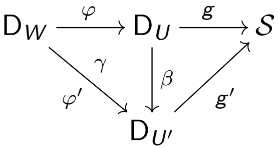

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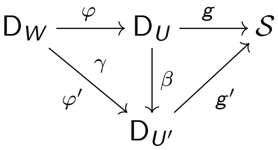
- Projection functor $\pi\colon F \to \mathrm{BC}^\infty(W, \mathbf{R}^2 \rtimes \widetilde{\mathrm{Conf}}(2))$.
  - $(\varphi, g) \mapsto$ germ of D_W around 0.
  - $(\beta, \gamma) \mapsto B\colon W \to \mathbf{R}^2 \rtimes \widetilde{\mathrm{Conf}}(2)$

How to compute $\rho_! \mathcal{S}$?

- Grothendieck construction F:



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  - $(\varphi')^{-1}\gamma$ is an isotopy class of isotopies $(\varphi')^{-1}\beta\varphi \to \mathrm{id}_{D_W}$.
  - $W \to \mathbf{R}^2$: the displacement of the origin.
  - $W \to \widetilde{\mathrm{Conf}}(2)$: the germ of embedding + winding number.

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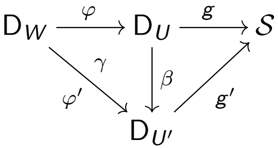

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  - $W \to \widetilde{\mathsf{Conf}}(2)$: the germ of embedding + winding number.
- Quillen's Theorem A: $*/\pi$ is a directed poset $\implies$ weakly contractible nerve

How to compute $\rho_! \mathcal{S}$?

- $\mathrm{BC}^\infty(W, \mathbf{R}^2 \rtimes \widetilde{\mathsf{Conf}}(2))$. Ob: germ of D_W around 0. Mor: displacement + automorphism of a germ.
- Projection functor $\pi\colon F \to \mathrm{BC}^\infty(W, \mathbf{R}^2 \rtimes \widetilde{\mathsf{Conf}}(2))$.
  - $(\varphi, g) \mapsto$ germ of D_W around 0.
  - $(\beta, \gamma) \mapsto B\colon W \to \mathbf{R}^2 \rtimes \widetilde{\mathsf{Conf}}(2)$
  - $(\varphi')^{-1}\gamma$ is an isotopy class of isotopies $(\varphi')^{-1}\beta\varphi \to \mathrm{id}_{\mathsf{D}_W}$.
  - $W \to \mathbf{R}^2$: the displacement of the origin.
  - $W \to \widetilde{\mathsf{Conf}}(2)$: the germ of embedding + winding number.
- Quillen's Theorem A: $*/\pi$ is a directed poset $\implies$ weakly contractible nerve
- Theorem: $(\rho_!\mathcal{S})(\mathfrak{D}_W) \simeq \mathrm{BC}^\infty(W, \mathbf{R}^2 \rtimes \widetilde{\mathsf{Conf}}(2))$.

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- Theorem: $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}_d^\times) \simeq \mathbf{R}\,\mathrm{Map}(\mathrm{B}(\mathbf{R}^2 \rtimes \widetilde{\mathrm{Conf}}(2)), \iota^* \mathcal{V}_d^\times)$.

Applications (current)

- Consequence of the GCH: smooth invertible FFTs are classified by the smooth Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)
- The Stolz–Teichner conjecture: concordance classes of extended FFTs have a classifying space. (Proof: Locality + the smooth Oka principle (Berwick-Evans–Boavida de Brito–P.).
- Construction of power operations on the level of FFTs (extending Barthel–Berwick-Evans–Stapleton).
- (Grady) The Freed–Hopkins conjecture (Conjecture 8.37 in *Reflection positivity and invertible topological phases*)

Applications (ongoing)

- Construction of prequantum FFTs from geometric/topological data. Differential characteristic classes as FFTs. (cf. Berthomieu 2008; Bunke–Schick 2010; Bunke 2010).
- Atiyah–Singer index invariants (index, η-invariant, determinant line, index gerbe) as a fully extended FFT (cf. Bunke 2002; Hopkins–Singer 2002; Bunke–Schick 2007).
- Quantization of functorial field theories. Examples: 2d Yang–Mills.

Example: the prequantum Chern–Simons theory (1)

Input data:

- G: a Lie group;
- $\mathcal{S} = \mathrm{B}_\nabla G$ (fiberwise principal G-bundles with connection);
- $\mathcal{V} = \mathrm{B}^3\mathrm{U}(1)$ (a single k-morphism for $k < 3$; 3-morphisms are $\mathrm{U}(1)$ as a Lie group).

Output data: a fully extended 3-dimensional G-gauged FFT:

$$\mathfrak{Bord}_3^{\mathrm{B}_\nabla G} \to \mathrm{B}^3\mathrm{U}(1).$$

- Closed 3-manifold $M \mapsto$ the Chern–Simons action of M;
- Closed 2-manifold $B \mapsto$ the prequantum line bundle of B;
- Closed 1-manifold $C \mapsto$ the Wess–Zumino–Witten gerbe (B-field) of C (Carey–Johnson–Murray–Stevenson–Wang);
- Point $\mapsto$ the Chern–Simons 2-gerbe (Waldorf).

Example: the prequantum Chern–Simons theory (2)

Step 1 Compute $\mathcal{V}_3^\times = (\mathrm{B}^3\mathrm{U}(1))_3^\times$.

Step 1a W is the fiberwise Deligne complex of $T \to U$:

$$W(T \to U) = \Omega^3 \leftarrow \Omega^2 \leftarrow \Omega^1 \leftarrow \mathrm{C}^\infty(T, \mathrm{U}(1)).$$

Step 1b $W \to \mathcal{V}_3^\times$: a fiberwise 3-form ω on $T \to U$
$\mapsto$ framed FFT: 3-bordism $B \mapsto \exp(\int_B \omega)$.

Step 1c The composition

$$W(T \to U) \to \mathcal{V}_3^\times(T \to U) \to \mathcal{V}^\times(U) = \mathrm{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(T, \mathrm{U}(1))$$

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Step 2 Construct a point in

$$\mathbf{R}\,\mathrm{Map}(\mathrm{B}_\nabla G, W)$$
$$= \mathbf{R}\,\mathrm{Map}(\Omega^1(-, \mathfrak{g})//\mathrm{C}^\infty(-, G), \mathrm{B}^3\mathrm{C}^\infty_{\mathrm{fconst}}(-, \mathrm{U}(1))).$$

(Brylinski–McLaughlin 1996, Fiorenza–Sati–Schreiber 2013)

Step 2′ Even better: can compute the whole space $\mathbf{R}\,\mathrm{Map}(\mathrm{B}_\nabla G, W)$.

Example: the prequantum Chern–Simons theory (2)

Step 1 Result: $\mathcal{V}_3^\times = (\mathsf{B}^3\mathrm{U}(1))_3^\times = \mathsf{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(-, \mathrm{U}(1))$.

Step 2 Construct a point in

$$\begin{aligned} &\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, W) \\ &= \mathbf{R}\,\mathrm{Map}(\Omega^1(-, \mathfrak{g})//\mathrm{C}^\infty(-, G), \mathsf{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(-, \mathrm{U}(1))). \end{aligned}$$

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Quantization of functorial field theories

X: the prequantum geometric structure
Y: the quantum geometric structure (e.g., a point)

$$
\begin{array}{ccc}
\mathsf{FFT}_{d,\mathcal{V}}(X) & \xrightarrow[\simeq]{\mathsf{GCH}} & \mathbf{R}\,\mathrm{Map}(X, \mathcal{V}_d^\times) \\
\Big\downarrow{\scriptstyle\int} & & \Big\downarrow{\scriptstyle Q} \\
\mathsf{FFT}_{d,\mathcal{V}}(Y) & \xrightarrow[\mathsf{GCH}]{\simeq} & \mathbf{R}\,\mathrm{Map}(Y, \mathcal{V}_d^\times)
\end{array}
$$

$d = 1$: recover the Spin^c geometric quantization when X is a smooth manifold, $Y = \mathsf{Riem}_{1|1}$, $\mathcal{V} =$ Fredholm complexes.