Transcript of the qualifying examination of Dmitri Pavlov.

August 25, 2008, 14–17, room 959, Evans Hall.

This is a rough transcript of my qual, which occurred on August 25, 2008. Since I wrote this transcript using my memory, not the actual recording, the words ascribed to the participants do not coincide with the original phrases.

Participants: Dmitri Pavlov, Peter Teichner (advisor), Constantin Teleman (committee chair), Dan-Virgil Voiculescu, Raphael Bousso (Department of Physics).

Teichner: What do you want to start with?

Me: Let's start with topology.

Teichner: How about the cohomology ring of $S^2 \times S^2$.

[I compute group structure by Künneth formula.]

Teichner: What about ring structure?

[I use Poncaré duality. It leaves two variants for ring structure. I choose the wrong one.]

Teichner: How did you obtain this?

[I correct myself and point out the right ring structure.]

Teichner: Yes, and Künneth formula is valid for ring structure. Can you write it down?

[I write down how to obtain this ring structure via Künneth formula.]

Teichner: But can give an example of manifold with the ring structure you originally wrote?

Me: Maybe connected sum of two-dimensional projective space with itself will work.

Teichner: Can you compute the cohomology of this manifold?

Me: One can use Mayer-Vietoris sequence or de Rham cohomology.

Teichner: You can use either.

[I write down Mayer-Vietoris sequence and compute cohomology groups.]

Teichner: But what about the ring structure?

Me: We can use de Rham cohomology. We immediately obtain that $xy = 0$, where x and y are two generators of second cohomology group.

Teichner: What about x^2 and y^2?

Me: By Poincaré duality x^2 and y^2 are generators of H^4. Whether we have $x^2 = y^2$ or $x^2 = -y^2$ depends on orientation.

Teichner: Finally you came to the question of orientation of $\mathbf{C}P^2$. First let's decide whether $\mathbf{C}P^2$ and $\mathbf{C}P^2$ are different manifolds. In other words, can you tell us whether there is an orientation-reversing diffeomorphism of $\mathbf{C}P^2$?

Me: Such a diffeomorphism should induce a negative identity on top cohomology.

Teichner: You already computed ring structure on cohomology of $\mathbf{C}P^2$.

Me: Yes, from this structure one immediately obtains that there is no orientation-reversing diffeomorphism of $\mathbf{C}P^2$.

Me: The manifolds $\mathbf{C}P^2 \# \mathbf{C}P^2$ and $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$ have cohomology rings $x^2 = -y^2 = z$ and $x^2 = y^2 = z$.

Teichner: Therefore these manifolds are not homeomorphic.

Teichner to Teleman: You want to ask some questions on topology?

Teleman: Can you compute the cohomology of $\mathbf{R}P^2 \times \mathbf{R}P^2$.

Me: Yes, we can again use Künneth formula. This time the cohomology has torsion and we have to use the version with Tor functors.

[I write down the Künneth exact sequence. I make a mistake and write $n-1$ instead of $n+1$ in the sum of Tor terms.]

Teleman: Is it $n-1$ or $n+1$ there?

Me: I think it is $n-1$.

Teleman: OK, go on and compute cohomology.

Me: First let's write down the left term of Künneth exact sequence. For this we need cohomology of $\mathbf{C}P^2$.

[I write down *homology* of $\mathbf{R}P^2$.]

Teleman: Are we computing homology or cohomology?

Me: Oops, this is homology.

Teleman: How do you compute cohomology from homology?

Me: Universal coefficient theorem.

[I write down the universal coefficient theorem and apply it to $\mathbf{R}P^2$. I say that $\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z},\mathbf{Z}) = \mathbf{Z}$ but then correct myself.]

[I substitute the cohomology of $\mathbf{R}P^2$ into Künneth exact sequence. The fifth cohomology group turns out to be nonzero.]

Me: Oops, we really should have $n+1$ in the Künneth formula.

Teleman: Signs should change when you pass from homology to cohomology. That's how I remember all these formulas.

Teichner: There are all these fancy topics in the syllabus. Let me choose one. What's Dold-Thom theorem?

[I write down the Dold-Thom theorem. I explains what symmetric product is and say that it converts Moore spaces to Eilenberg-Mac Lane spaces.]

Teichner: Is symmetric product a monoid?

Me: Yes, it is a commutative topological monoid.

Teichner: Is it an abelian group?

Me: No, the symmetric product is a free commutative topological monoid on a topological space, therefore never a group.

Teichner: But Eilenberg-Mac Lane space is a group.

Me: Yes.

Teichner: Can you tell us what will happen if we replace free commutative monoid by free commutative abelian group?

Me: We obtain zero singular chain group. Reduced zero singular chain group.

Teichner: What I meant is connection with Eilenberg-Mac Lane spaces.

Me: One can obtain Eilenberg-Mac Lane spaces by iterating classifying space construction. In this way one immediately obtains that Eilenberg-Mac Lane space is an abelian group.

Teichner: Let's make a five-minute break.

[Break.]

Teleman: Tell us about the relation between divisors and line bundles.

[I define Weil divisor group as the free abelian group generated by hypersurfaces.]

Teleman: Can you define hypersurfaces?

[I write down the definition in local charts.]

Teleman: These are smooth hypersurfaces.

Me: Yes, generally in a local chart one should have a zero set of a holomorphic function.

Teleman: Yes. OK, what about line bundles?

Me: There is a canonical map from the group of divisors to the Picard group of line bundles. For each hypersurface there is a unique line bundle and a global section of this bundle such that its zero set coincides with hypersurface.

Teichner: Is it well-defined? I think the map goes the other way.

Me: Yes, it is well-defined. The map cannot go the other way because there is no canonical way to attach a global holomorphic section to a line bundle.

Teichner: Yes, but is it well-defined?

Me: There is another way to construct this map. Consider short exact sequence of sheaves $0 \to O^* \to K^* \to K^*/O^* \to 0$. The boundary map $H^0(K^*/O^*) \to H^1(O^*)$ is exactly the map we need.

Teleman: Sheaves of what?

Me: Sheaves of abelian groups.

Teleman: With what group structure?

Me: Multiplicative.

Teleman: Why $H^0(K^*/O^*)$ and $H^1(O^*)$ are the groups we need?

[I explain the isomorphism for the Picard group via local trivializations.]

Teleman: What is K^*?

Me: Sheaf of meromorphic functions.

Teleman: But you mentioned only holomorphic functions and their sets of zeroes.

Me: For meromorphic functions one should subtract the divisor of poles from the divisor of zeroes.

Teleman: The map from divisors to line bundles, is it bijective, injective or surjective?

Me: No. The long exact sequence I wrote down before tells us it isn't. The kernel is the group of principal divisors, $H^0(K^*)$. They are precisely the divisors defined by global meromorphic functions.

Teleman: And the cokernel?

Me: The cokernel is $H^1(K^*)$. It can also be nonzero.

Teleman: Do you know when it is zero?

Me: It is zero for projective manifolds.

Teichner: What is Serre duality?

[I write it down as an isomorphism.]

Teleman: What about the pairing?

[I write Serre duality as a nondegenerate pairing.]

Teleman: What is the isomorphism from $H^{n,n}$ to complex numbers?

Me: Integration.

Teleman: I would like to ask about the Hodge conjecture, but it isn't on the syllabus.

Teichner: It is.

Teleman: Then tell us about the Hodge conjecture.

[I write down the Hodge conjecture.]

Teleman: This description suffices as the first approximation. But if we want to make sense of this definition, we need to say something about the cohomology group in your statement.

Me: We can embed sheaf cohomology with complex coefficients into Dolbeault cohomology.

Teleman: Yes, we can do this, but this doesn't bring us any closer. We need something else. It is on your syllabus.

Me: I don't know.

Teleman: Can you tell us what Hodge decomposition is?

[I write down Hodge decomposition.]

Teleman: So now we can identify de Rham cohomology with direct sum of Dolbeault cohomology.

Teleman: Can you tell us what do you mean by the rational 1,1-cohomology?

Me: It is the intersection of Dolbeault cohomology and rational singular cohomology.

Teichner: Let's have another five-minute break.

[Break.]

Voiculescu: What is a von Neumann algebra?

Me: It is a weakly closed $*$-subalgebra of the algebra of bounded operators.

Voiculescu: Can you give another definition?

Me: It is a $*$-subalgebra of the algebra of bounded operators coinciding with its double commutant.

Voiculescu: What I want to do is to ask some questions on commutative von Neumann algebras and then go to type II_1 factors. What can you say about commutative von Neumann algebras?

Me: Every von Neumann algebra is isomorphic to L^∞ of some measurable space.

Voiculescu: In your definition a von Neumann algebra acts on some Hilbert space.

Me: L^∞ acts on L^2 by multiplication.

Voiculescu: What is an isomorphism of von Neumann algebras?

[I say that it is a norm-preserving bijection. By bijection I meant an isomorphism of $*$-algebras, but forgot to say this. Voiculescu and Teichner were uncontent by such wording and eventually made write down all the properties of $*$-algebra isomorphism. Only then I noticed my terminological mistake.]

Voiculescu: So basically you want to say that an isomorphism of von Neumann algebras is a C*-algebra isomorphism.

Me: Yes.

Voiculescu: This is correct, although usually another definition is used and then this statement is derived as a corollary. Do you know another definition of isomorphism?

Me: No, I don't.

Voiculescu: You can replace norm-preservation by ultraweak continuity.

Teichner: Can you say what ultraweak topology is?

[I write down the definition.]

Voiculescu: Can you define what a type II_1 factor is?

Teichner: By the way, what is a factor?

Me: A factor is a von Neumann algebra with a trivial center.

Me: First we define an order on the set of all projections of a factor.

[I write down the definition. I use non-standard notation for two orders on projections.]

Voiculescu: You really want to use another notation for projections.

[I rewrite the definitions using another notation.]

Me: Now a type II_1 factor is a factor whose ordering of projections is isomorphic to $[0, 1]$ and the unit projection is finite.

Voiculescu: What is a finite projection?

Me: A finite projection is a projection that is not isomorphic to any of its subprojections.

Voiculescu: Can you define type II_1 factors in another way?

Me: Yes. A type II_1 factor is a factor with faithful ultraweak continuous normalized trace.

Voiculescu: Yes, this definition is correct, even though you can omit some of the conditions.

Teichner: So you can get rid of faithfulness?

Voiculescu: Yes. You can get rid of faithfulness, you can get rid of ultraweak continuity, you can get rid of almost anything.

Voiculescu: Can you say what is a general condition that guarantees that group algebra of a group is a type II_1 factor?

Me: All conjugacy classes are infinite except for the unit.

[Voiculescu then asked a question on a topic not on a syllabus. I did not know this topic. Voiculescu did not insist, because this topic was not on my syllabus.]

[I leave the room and wait about five minutes. Then Teleman opens the door and congratulates me. Other committee members follow.]