CHAPTER 34

Non-Commutative L^p-Spaces

Gilles Pisier*

Équipe d'Analyse, Université Paris VI, Case 186, F-75252 Paris Cedex 05, France
Texas A&M University, College Station, TX 77843, USA
E-mail: gip@ccr.jussieu.fr

Quanhua Xu

Laboratoire de Mathématiques, Université de Franche-Comté, UFR des Sciences et Techniques,
16, Route de Gray, 25030 Besançon Cedex, France
E-mail: qx@math.univ-fcomte.fr

Contents

*Partially supported by NSF and Texas Advanced Research Program 010366-163.

HANDBOOK OF THE GEOMETRY OF BANACH SPACES, VOL. 2
Edited by William B. Johnson and Joram Lindenstrauss

Introduction

This survey is devoted to the theory of non-commutative L^p-spaces. This theory (in the tracial case) was laid out in the early 50's by Segal [181] and Dixmier [47] (see also [110, 183]). Since then the theory has been extensively studied, extended and applied, and by now the strong parallelism between non-commutative and classical Lebesgue integration is well-known.

We will see that on the one hand, non-commutative L^p-spaces share many properties with the usual L^p-spaces (to which we will refer as *commutative* L^p-spaces), and on the other, they are very different from the latter. They provide interesting (often "pathological") examples which cannot exist among the usual function or sequence spaces. They are also used as fundamental tools in some other directions of mathematics (such as operator algebra theory, non-commutative geometry and non-commutative probability), as well as in mathematical physics.

Some tools in the study of the usual commutative L^p-spaces still work in the non-commutative setting. However, most of the time, new techniques must be invented. To illustrate the difficulties one may encounter when studying non-commutative L^p-spaces, we mention here three well-known facts. Let H be a complex Hilbert space, and let $B(H)$ denote the algebra of all bounded operators on H. The first fact states that the usual triangle inequality for the modulus of complex numbers is no longer valid for the modulus of operators, namely, in general, we do not have $|x+y| \leqslant |x| + |y|$ for $x, y \in B(H)$, where $|x| = (x^*x)^{1/2}$ is the modulus of x. However, there is a useful substitute, obtained in [1], which reads as follows. For any $x, y \in B(H)$ there are two isometries u and v in $B(H)$ such that

$$|x+y| \leqslant u|x|u^* + v|y|v^*.$$

The second fact is about operator monotone functions. Let α be a positive real number. In general, the condition that $0 \leqslant x \leqslant y$ $(x, y \in B(H))$ does not imply $x^\alpha \leqslant y^\alpha$. This implication holds only in the case of $\alpha \leqslant 1$. The last fact concerns the convexity of the map $x \mapsto x^\alpha$ on the positive part $B(H)_+$ of $B(H)$. For $\alpha < 1$ this map is concave (actually, the function $(x, y) \to x^\alpha \otimes y^{1-\alpha}$ is concave on $B(H)_+ \times B(H)_+$, [112]), but for $\alpha \geqslant 1$ convexity holds only if $1 \leqslant \alpha \leqslant 2$. The reader can find more results of this nature in [17].

Some even worse phenomena may happen. It is well known that composed with the usual trace Tr on $B(H)$, all the preceding maps have the usual desired properties. For instance, the function $x \mapsto \mathrm{Tr}(x^\alpha)$ becomes convex for all $\alpha \geqslant 1$, as one can expect. Now consider the function

$$(x_1, \dots, x_n) \mapsto \mathrm{Tr}\big[\big(x_1^\alpha + \dots + x_n^\alpha\big)^{1/\alpha}\big]$$

on $B(H)_+^n$. In the commutative setting, the convexity of this function for all $\alpha \geqslant 1$ and $n \geqslant 1$ is extremely useful in many situations. Again, in the non-commutative case, this convexity is not guaranteed, at least for $\alpha > 2$ (cf. [36]; see also [10] for some related results).

Despite the difficulty caused by the lack of these elementary properties, we feel the theory has now matured enough for us to be able to present the reader with a rather satisfactory picture. Of course much remains to be done, as shown by the many open problems which we will encounter.

We now briefly describe the organization of this survey. After a preliminary section, we discuss the interpolation of non-commutative L^p-spaces (associated with a trace) in Section 2. This is one of the oldest subjects in the field. The main result there allows to reduce all interpolation problems on non-commutative L^p-spaces to the corresponding ones on commutative L^p-spaces.

Section 3 can be still considered as a preliminary one. There we introduce the non-commutative L^p-spaces associated with a state or weight. This section also contains two useful results. The first one says that the non-commutative L^p-spaces over the hyperfinite II_1 factor are the smallest ones among all those over von Neumann algebras not of type I. The second one is Haagerup's approximation theorem.

In the short Section 4 we discuss very briefly some similarities and differences between the commutative L^p-spaces and their non-commutative counterparts. One remarkable result in the early stage of the non-commutative L^p-space theory is the Gordon–Lewis theorem on local unconditional structure of the Schatten classes. This (negative) result shows that compared with the usual function spaces, the Schatten classes (and so the general non-commutative L^p-spaces) are, in a certain sense, "very non-commutative".

Section 5 discusses the uniform convexities and smoothness, and the related type and cotype properties. Although the problem on the uniform (real) convexity of the non-commutative L^p-spaces goes back to the 50's, the best constant for the modulus of convexity was found only at the beginning of the 90's. Two uniform complex convexities (the uniform PL-convexity and Hardy convexity) are also discussed in this section.

The central object in Section 6 is the non-commutative Khintchine inequalities, of paramount importance in this theory. Like in the commutative case, they are the key to a large part of non-commutative analysis, including of course the type and cotype properties of non-commutative L^p-spaces, and closely linked to the non-commutative Grothendieck theorem.

Section 7 presents some very recent results on non-commutative martingale inequalities. In view of its close relations with quantum (= non-commutative) probability, this direction, which is still at an early stage of development, is likely to get more attention in the near future.

Section 8 deals with the non-commutative Hardy spaces. We present there some non-commutative analogues of the classical theorems on the Hardy spaces in the unit disc, such as the boundedness of the Hilbert transformation, Szegö and Riesz factorizations.

The first result in Section 9 is Peller's characterization of the membership of a Hankel operator in a Schatten class. This result is related to Schur multipliers. The rest of this section gives an outline of the recent works by Harcharras on Schur multipliers and non-commutative $\Lambda(p)$-sets.

The last section concerns the embedding and isomorphism of non-commutative L^p-spaces. Almost all results given there were obtained just in the last few years. This is still a very active direction.

We end this introductory section by pointing out that we will freely use standard notation and notions from operator algebra theory, for which we refer to [48,104,178,184,185,190].

1. Preliminaries

In this section we give some necessary preliminaries on non-commutative L^p-spaces associated with a trace. This requires that the underlying von Neumann algebra be semifinite (see below the definition). In Section 3, we will consider the non-tracial case.

$\mathcal{M}$ will always denote a von Neumann algebra, and $\mathcal{M}_+$ its positive part. We recall that a *trace* on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying

(i) $\tau(x+y) = \tau(x) + \tau(y)$, $\forall x, y \in \mathcal{M}_+$;
(ii) $\tau(\lambda x) = \lambda\tau(x)$, $\forall \lambda \in [0, \infty)$, $x \in \mathcal{M}_+$;
(iii) $\tau(u^*u) = \tau(uu^*)$, $\forall u \in \mathcal{M}$.

τ is said to be *normal* if $\sup_\alpha \tau(x_\alpha) = \tau(\sup_\alpha x_\alpha)$ for any bounded increasing net (x_α) in $\mathcal{M}_+$, *semifinite* if for any non-zero $x \in \mathcal{M}_+$ there is a non-zero $y \in \mathcal{M}_+$ such that $y \leqslant x$ and $\tau(y) < \infty$, and *faithful* if $\tau(x) = 0$ implies $x = 0$. If $\tau(1) < \infty$ (1 denoting the identity of $\mathcal{M}$), τ is said to be *finite*. If τ is finite, we will assume almost systematically that τ is normalized, that is, $\tau(1) = 1$. We often think of τ as a non-commutative ($=$quantum) probability.

A von Neumann algebra $\mathcal{M}$ is called semifinite if it admits a normal semifinite faithful (abbreviated as *n.s.f.*) trace τ, which we assume in the remainder of this section. Then let $\mathcal{S}_+$ be the set of all $x \in \mathcal{M}_+$ such that $\tau(\operatorname{supp} x) < \infty$, where $\operatorname{supp} x$ denotes the support of x (defined as the least projection p in $\mathcal{M}$ such that $px = x$ or equivalently $xp = x$). Let $\mathcal{S}$ be the linear span of $\mathcal{S}_+$. It is easy to check that $\mathcal{S}$ is a $*$-subalgebra of $\mathcal{M}$ which is w^*-dense in $\mathcal{M}$, moreover for any $0 < p < \infty$, $x \in \mathcal{S}$ implies $|x|^p \in \mathcal{S}_+$ (and so $\tau(|x|^p) < \infty$), where $|x| = (x^*x)^{1/2}$ is the modulus of x. Now we define

$$\|x\|_p = \left[\tau\left(|x|^p\right)\right]^{1/p}, \quad x \in \mathcal{S}.$$

One can show that $\|\cdot\|_p$ is a norm on $\mathcal{S}$ if $1 \leqslant p < \infty$, and a quasi-norm (more precisely, a p-norm) if $0 < p < 1$. The completion of $(\mathcal{S}, \|\cdot\|_p)$ is denoted by $L^p(\mathcal{M}, \tau)$. This is the non-commutative L^p-space associated with $(\mathcal{M}, \tau)$. For convenience, we set $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm. The trace τ can be extended to a linear functional on $\mathcal{S}$, which will be still denoted by τ. Then

$$\left|\tau(x)\right| \leqslant \|x\|_1, \quad \forall x \in \mathcal{S}.$$

Thus τ extends to a continuous functional on $L^1(\mathcal{M}, \tau)$.

The elements in $L^p(\mathcal{M}, \tau)$ can be viewed as closed densely defined operators on H (H being the Hilbert space on which $\mathcal{M}$ acts). We recall this briefly. A closed densely defined operator x on H is said to be *affiliated* with $\mathcal{M}$ if $xu = ux$ for any unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. An affiliated operator x is said to be τ-*measurable* or simply *measurable* if

$\tau(e_\lambda(|x|)) < \infty$ for some $\lambda > 0$, where $e_\lambda(|x|)$ denotes the spectral resolution of $|x|$ (corresponding to the indicator function of (λ, ∞)). For any measurable operator x we define the *generalized singular numbers* by

$$\mu_t(x) = \inf\{\lambda > 0\colon\ \tau(e_\lambda(|x|)) \leqslant t\}, \quad t > 0.$$

It will be convenient to denote simply by $\mu(x)$ the function $t \to \mu_t(x)$.

Note that $\mu(x)$ is a non-increasing function on $(0, \infty)$. This notion is the generalization of the usual singular numbers for compact operators on a Hilbert space (see [72]). It was first introduced in a Bourbaki seminar note by Grothendieck [77]. It was studied in details in [132,62] and [64].

Let $L^0(\mathcal{M}, \tau)$ denote the space of all measurable operators in $\mathcal{M}$. Then $L^0(\mathcal{M}, \tau)$ is a $*$-algebra, which can be made into a topological $*$-algebra as follows. Let

$$V(\varepsilon, \delta) = \{x \in L^0(\mathcal{M}, \tau)\colon\ \mu_\varepsilon(x) \leqslant \delta\}.$$

Then $\{V(\varepsilon, \delta)\colon\ \varepsilon, \delta > 0\}$ is a system of neighbourhoods at 0 for which $L^0(\mathcal{M}, \tau)$ becomes a metrizable topological $*$-algebra. The convergence with respect to this topology is called the *convergence in measure*. Then $\mathcal{M}$ is dense in $L^0(\mathcal{M}, \tau)$. We refer to [131] and [191] for more information.

The trace τ is extended to a positive tracial functional on the positive part $L^0_+(\mathcal{M}, \tau)$ of $L^0(\mathcal{M}, \tau)$, still denoted by τ, satisfying

$$\tau(x) = \int_0^\infty \mu_t(x)\,\mathrm{d}t, \quad x \in L^0_+(\mathcal{M}, \tau).$$

Then for $0 < p < \infty$,

$$L^p(\mathcal{M}, \tau) = \{x \in L^0(\mathcal{M}, \tau)\colon\ \tau(|x|^p) < \infty\} \quad \text{and} \quad \|x\|_p = \left(\tau(|x|^p)\right)^{1/p}.$$

Also note that $x \in L^p(\mathcal{M}, \tau)$ iff $\mu(x) \in L^p(0, \infty)$, and $\|x\|_p = \|\mu(x)\|_{L^p(0,\infty)}$. Recall that $\mu(x) = \mu(x^*) = \mu(|x|)$; so $x \in L^p(\mathcal{M}, \tau)$ iff $x^* \in L^p(\mathcal{M}, \tau)$, and we have $\|x\|_p = \|x^*\|_p$.

The usual Hölder inequality extends to the non-commutative setting. Let $0 < r, p, q \leqslant \infty$ be such that $1/r = 1/p + 1/q$. Then

$$x \in L^p(\mathcal{M}, \tau),\ y \in L^q(\mathcal{M}, \tau) \Longrightarrow xy \in L^r(\mathcal{M}, \tau) \text{ and } \|xy\|_r \leqslant \|x\|_p \|y\|_q.$$

In particular, if $r = 1$,

$$|\tau(xy)| \leqslant \|xy\|_1 \leqslant \|x\|_p \|y\|_q, \quad x \in L^p(\mathcal{M}, \tau), y \in L^q(\mathcal{M}, \tau).$$

This defines a natural duality between $L^p(\mathcal{M}, \tau)$ and $L^q(\mathcal{M}, \tau)$: $\langle x, y\rangle = \tau(xy)$. Then for any $1 \leqslant p < \infty$ we have

$$\left(L^p(\mathcal{M}, \tau)\right)^* = L^q(\mathcal{M}, \tau) \quad \text{(isometrically)}.$$

Thus, $L^1(\mathcal{M},\tau)$ is the predual of $\mathcal{M}$, and $L^p(\mathcal{M},\tau)$ is reflexive for $1<p<\infty$. Note that the classical theorem of Day on the dual of L^p for $0<p<1$ was extended to the non-commutative setting by Saito [176]: the dual of $L^p(\mathcal{M},\tau)$, $0<p<1$, is trivial iff $\mathcal{M}$ has no minimal projection.

REMARK. [114] contains a different construction of non-commutative L^p-spaces via a non-commutative upper integral.

Although we will concentrate on non-commutative L^p-spaces in this survey, the more general so-called "symmetric operator spaces" are worth mentioning: let E be a rearrangement invariant (in short *r.i.*) function space on $(0,\infty)$, the symmetric operator space associated with $(\mathcal{M},\tau)$ and E is defined by

$$E(\mathcal{M},\tau)=\{x\in L^0(\mathcal{M},\tau)\colon\ \mu(x)\in E\}\quad\text{and}\quad \|x\|_{E(\mathcal{M},\tau)}=\|\mu(x)\|_E.$$

In particular, if $E=L^p(0,\infty)$, we recover $L^p(\mathcal{M},\tau)$. These symmetric operator spaces have been extensively studied, see, e.g., [51,53,54,133,134] and [203] for more information.

We end this section by some examples.

(i) *Commutative L^p-spaces.* Let $\mathcal{M}$ be an Abelian von Neumann algebra. Then $\mathcal{M}=L^\infty(\Omega,\mu)$ for a measure space (Ω,μ), integration with respect to the measure μ gives us an *n.s.f.* trace, and $L^p(\mathcal{M},\tau)$ is just the commutative L^p-space $L^p(\Omega,\mu)$.

(ii) *Schatten classes.* Let $\mathcal{M}=B(H)$, the algebra of all bounded operators on H, and $\tau=\mathrm{Tr}$, the usual trace on $B(H)$. Then the associated L^p-space $L^p(\mathcal{M},\tau)$ is the Schatten class $S^p(H)$. If H is separable and $\dim H=\infty$ (resp. $\dim H=n$), we denote $S^p(H)$ by S^p (resp. S^p_n). Note that in our notation $S^\infty(H)$ is not the ideal of all compact operators on H but $B(H)$ itself. [72,128] and [182] contain elementary properties of $S^p(H)$.

(iii) *The hyperfinite II_1 factor.* Let M_n denote the full algebra of all complex $n\times n$ matrices, equipped with the normalized trace σ_n. Let

$$(R,\tau)=\bigotimes_{n\geqslant 1}(\mathcal{A}_n,\tau_n),\qquad (\mathcal{A}_n,\tau_n)=(M_2,\sigma_2),\ n\in\mathbb{N},$$

be the von Neumann algebra tensor product. Then R is the hyperfinite II_1 factor and τ is the (unique) normalized trace on R. There is another useful description of R. Let $(\varepsilon_n)_{n\geqslant 1}$ be a sequence of self-adjoint unitaries on a Hilbert space, satisfying the following canonical anticommutation relations

$$\varepsilon_i\varepsilon_j+\varepsilon_j\varepsilon_i=2\delta_{ij},\quad i,j\in\mathbb{N}. \tag{CAR}$$

Let R_0 be the C^*-algebra generated by the ε_i's. Then R_0 admits a unique faithful tracial state, denoted by τ, which is defined as follows. For any finite subset $A=\{i_1,\dots,i_n\}\subset\mathbb{N}$ with $i_1<\cdots<i_n$ we put $w_A=\varepsilon_{i_1}\cdots\varepsilon_{i_n}$, and $w_\emptyset=1$. Then the trace τ is uniquely determined by its action on the w_A's: $\tau(w_A)=1$ (resp. $=0$) if $A=\emptyset$ (resp. $\neq\emptyset$). Consider R_0 as a C^*-algebra acting on $L^2(\tau)$ by left multiplication. Then the von Neumann algebra

generated by R_0 in $B(L^2(\tau))$ can be (isomorphically) regarded as the hyperfinite II_1 factor R. Note that the family of all linear combinations of the w_A's are w^*-dense in R and dense in $L^p(R)$ for all $0 < p < \infty$; also note that $\{w_A\colon A \subset \mathbb{N}\}$ is an orthonormal basis of $L^2(R)$ $(= L^2(\tau))$. Finally, we mention that the von Neumann subalgebra generated by $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ is isomorphic to M_{2^n}, and then the restriction of τ to this subalgebra is just the normalized trace of M_{2^n}. We refer to [25] and [158] for more information.

(iv) *Group algebras.* Consider a discrete group Γ. Let $vN(\Gamma) \subset B(\ell_2(\Gamma))$ be the associated von Neumann algebra generated by the left translations. Let τ_Γ be the canonical trace on $vN(\Gamma)$, defined as follows: $\tau_\Gamma(x) = \langle x(\delta_e), \delta_e\rangle$ for any $x \in vN(\Gamma)$, where $(\delta_g)_{g\in\Gamma}$ denotes the canonical basis of $l_2(\Gamma)$, and where e is the identity of Γ. This is a normal faithful normalized finite trace on $vN(\Gamma)$. A particularly interesting case is when $\Gamma = \mathbb{F}_n$, the free group on n generators. We refer to [67] and [196] for more on this theme.

2. Interpolation

This section is devoted to the interpolation of non-commutative L^p-spaces. It is well known that the non-commutative L^p-spaces associated with a semifinite von Neumann algebra form an interpolation scale with respect to both the real and complex interpolation methods (see (2.1) and (2.2) below). This result not only is useful in applications but also can be taken as a starting point to define non-commutative L^p-spaces associated to a von Neumann algebra of type III (which admits no *n.s.f.* trace). This is indeed the viewpoint taken by Kosaki [106] (see also [192]). We will discuss this point in the next section. Here we restrict ourselves only to semifinite von Neumann algebras. Thus throughout this section, $\mathcal{M}$ will always denote a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. We refer to [15] for all notions and notation from interpolation theory used below. Let $1 \leqslant p_0, p_1 \leqslant \infty$ and $0 < \theta < 1$. It is well known that

$$L^p(\mathcal{M},\tau) = \big(L^{p_0}(\mathcal{M},\tau), L^{p_1}(\mathcal{M},\tau)\big)_\theta \quad \text{(with equal norms)}, \tag{2.1}$$

$$L^p(\mathcal{M},\tau) = \big(L^{p_0}(\mathcal{M},\tau), L^{p_1}(\mathcal{M},\tau)\big)_{\theta,p} \quad \text{(with equivalent norms)}, \tag{2.2}$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$, and where $(\cdot,\cdot)_\theta$, $(\cdot,\cdot)_{\theta,p}$ denote respectively the complex and real interpolation methods. It is not easy to retrace the origin of these interpolation results. Some weaker or particular forms go back to the 50's (cf., e.g., [47,110,172]). The results in the full generality were achieved by Ovchinikov [133,134] (see also [135] for the real interpolation, and [141] in the case of Schatten classes).

(2.1) and (2.2) easily follow from the following result. Recall that $\mu(x)$ denotes the generalized singular number of x (see Section 1) and that a map $T : X \to Y$ is called contractive (or a contraction) if $\|T\| \leqslant 1$.

THEOREM 2.1. *For any fixed $x \in L^1(\mathcal{M},\tau) + L^\infty(\mathcal{M},\tau)$ there are linear maps T and S (which may depend on x) satisfying the following properties:*

(i) *$T : L^1(\mathcal{M},\tau) + L^\infty(\mathcal{M},\tau) \to L^1(0,\infty) + L^\infty(0,\infty)$, T is contractive from $L^p(\mathcal{M},\tau)$ to $L^p(0,\infty)$ for $p = 1$ and $p = \infty$, and $Tx = \mu(x)$;*

(ii) $S: L^1(0,\infty) + L^\infty(0,\infty) \to L^1(\mathcal{M},\tau) + L^\infty(\mathcal{M},\tau)$, *$S$ is contractive from $L^p(0,\infty)$ to $L^p(\mathcal{M},\tau)$ for $p=1$ and $p=\infty$, and $S\mu(x)=x$.*

Although not explicitly stated, Theorem 2.1 is implicit in the literature. It is essentially contained in [4] for Schatten classes, and in some different (weaker) form in [53] for the general case. We will include a proof at the end of the section.

REMARK. In interpolation language, Theorem 2.1 implies that the pair $(L^1(\mathcal{M},\tau), L^\infty(\mathcal{M},\tau))$ is a (contractive) *partial retract* of $(L^1(0,\infty), L^\infty(0,\infty))$. We should emphasize the usefulness of such a result: it reduces all interpolation problems on $(L^1(\mathcal{M},\tau), L^\infty(\mathcal{M},\tau))$ to those on $(L^1(0,\infty), L^\infty(0,\infty))$. Recall that $(L^1(0,\infty), L^\infty(0,\infty))$ is one of the best understood pairs in interpolation theory. We now illustrate this by some examples. More applications can be found in [4,53,54] and [133,134].

First let us show how to get (2.1) and (2.2) from their commutative counterparts.

PROOF OF (2.1) AND (2.2). Let $x \in L^p(\mathcal{M},\tau)$ (noting that $L^p(\mathcal{M},\tau) \subset L^1(\mathcal{M},\tau) + L^\infty(\mathcal{M},\tau)$). Let S be the map associated to x given by Theorem 2.1. Then by interpolation

$$S: \big(L^1(0,\infty), L^\infty(0,\infty)\big)_\theta \to \big(L^1(\mathcal{M},\tau), L^\infty(\mathcal{M},\tau)\big)_\theta$$

is a contraction. However, it is classical that

$$\big(L^1(0,\infty), L^\infty(0,\infty)\big)_\theta = L^p(0,\infty) \quad \text{(with equal norms)}.$$

Thus we deduce

$$\|x\|_\theta = \big\|S\mu(x)\big\|_\theta \leqslant \big\|\mu(x)\big\|_\theta = \|x\|_p;$$

whence

$$L^p(\mathcal{M},\tau) \subset \big(L^1(\mathcal{M},\tau), L^\infty(\mathcal{M},\tau)\big)_\theta, \quad \text{a contractive inclusion.}$$

The inverse inclusion is proved similarly by means of the map T. Therefore, we have shown (2.1). In the same way, we get (2.2). $\square$

The above argument works in a more general setting as well.

COROLLARY 2.2. *Let $\mathcal{F}$ be an interpolation functor. Then*

$$\mathcal{F}\big(L^1(\mathcal{M},\tau), L^\infty(\mathcal{M},\tau)\big) = \mathcal{F}\big(L^1(0,\infty), L^\infty(0,\infty)\big)(\mathcal{M},\tau).$$

More generally, for any r.i. function spaces E_0, E_1 on $(0,\infty)$

$$\mathcal{F}\big(E_0(\mathcal{M},\tau), E_1(\mathcal{M},\tau)\big) = \mathcal{F}(E_0,E_1)(\mathcal{M},\tau).$$

This corollary is contained in [53] (and in [4,141] for Schatten classes).

REMARK. As a consequence of Corollary 2.2, we have that for any *r.i.* function space E the map T (resp. S) in Theorem 2.1 is contractive from $E(\mathcal{M},\tau)$ (resp. E) to E (resp. $E(\mathcal{M},\tau)$). In particular, T and S are contractions between the L^p-spaces in consideration for all $1 \leqslant p \leqslant \infty$.

The following particular case of Corollary 2.2 is worth being mentioned explicitly. Here K_t denotes the usual K-functional from interpolation theory.

COROLLARY 2.3. *Let $1 \leqslant p_0, p_1 \leqslant \infty$. Then for any $x \in L^{p_0}(\mathcal{M},\tau) + L^{p_1}(\mathcal{M},\tau)$ and any $t > 0$*

$$K_t\big(x; L^{p_0}(\mathcal{M},\tau), L^{p_1}(\mathcal{M},\tau)\big) = K_t\big(\mu(x); L^{p_0}(0,\infty), L^{p_1}(0,\infty)\big).$$

In particular,

$$K_t\big(x; L^1(\mathcal{M},\tau), L^\infty(\mathcal{M},\tau)\big) = \int_0^t \mu_s(x)\,\mathrm{d}s.$$

REMARKS. (i) Using a factorization argument, one can easily extend Corollary 2.3 to the case of quasi-Banach spaces, so that p_0, p_1 are now allowed to be in $(0,\infty]$. Then the equality there has to be replaced by an equivalence with relevant constants depending only on p_0, p_1 (see also [135]).

(ii) As a consequence of the preceding remark, the indices p_0, p_1 in (2.2) can vary in $(0,\infty]$.

(iii) (2.1) also extends to the quasi-Banach space case (cf. [201]).

PROOF OF THEOREM 2.1. Fix an $x \in L^1(\mathcal{M},\tau) + L^\infty(\mathcal{M},\tau)$. We may assume $x \geqslant 0$. Indeed, by polar decomposition, it is easy to reduce the proof to this case.

First we suppose x is an *elementary operator*, i.e., of the form

$$x = \sum_{k=1}^n a_k e_k,$$

where for all $1 \leqslant k \leqslant n$, $a_k \in (0,\infty)$, and where the e_k's are disjoint projections with $\tau(e_k) \in (0,\infty)$. Then we define

$$Py = \sum_{k=1}^n \frac{\tau(ye_k)}{\tau(e_k)} e_k, \quad y \in L^1(\mathcal{M},\tau) + L^\infty(\mathcal{M},\tau).$$

Note that P is the orthogonal projection of $L^2(\mathcal{M},\tau)$ onto its subspace generated by $\{e_1,\dots,e_n\}$. In particular, P is selfadjoint. Let $y \in L^\infty(\mathcal{M},\tau)$. Then

$$\|Py\|_\infty \leqslant \sup_{1\leqslant k\leqslant n} \frac{|\tau(ye_k)|}{\tau(e_k)} \leqslant \sup_{1\leqslant k\leqslant n} \frac{\|y\|_\infty \tau(e_k)}{\tau(e_k)} = \|y\|_\infty.$$

Therefore, P is a contraction on $L^\infty(\mathcal{M},\tau)$. By duality, P is a contraction on $L^1(\mathcal{M},\tau)$ as well.

Let $\mathcal{N}$ be the subalgebra generated by $\{e_1,\dots,e_n\}$ (the identity of $\mathcal{N}$ being $e=e_1+\dots+e_n$). Then $\mathcal{N}$ is isomorphic to l_∞^n and τ induces a weighted counting measure ν on l_∞^n, namely, $\nu(\{k\})=\tau(e_k)$ for all $1\leqslant k\leqslant n$. It is clear that for any p

$$L^p(\mathcal{N},\tau|_{\mathcal{N}})=l_p^n(\nu)\quad\text{isometrically.}$$

Thus we can identify $(\mathcal{N},\tau|_{\mathcal{N}})$ with (l_∞^n,ν). With this identification, $\mu(x)$ is exactly the usual non-increasing rearrangement of x with respect to the measure ν. On the other hand, it is classical (and easy to prove in our special case) that there are linear maps R and Q satisfying (cf. [33])

(i) $R: l_1^n(\nu)+l_\infty^n(\nu)\to L^1(0,\infty)+L^\infty(0,\infty)$, R is contractive from $l_p^n(\nu)$ to $L^p(0,\infty)$ for $p=1,\infty$, and $Rx=\mu(x)$;

(ii) $Q: L^1(0,\infty)+L^\infty(0,\infty)\to l_1^n(\nu)+l_\infty^n(\nu)$, Q is contractive from $L^p(0,\infty)$ to $l_p^n(\nu)$ for $p=1,\infty$, and $Q\mu(x)=x$.

Then we set $T=RP$ and $S=iQ$, where i is the natural inclusion of $l_1^n(\nu)+l_\infty^n(\nu)$ $(=L^1(\mathcal{N},\tau|_{\mathcal{N}})+L^\infty(\mathcal{N},\tau|_{\mathcal{N}}))$ into $L^1(\mathcal{M},\tau)+L^\infty(\mathcal{M},\tau)$. One easily checks that T and S satisfy all requirements of Theorem 2.1. Therefore, Theorem 2.1 is proved for elementary operators.

Before passing to general (positive) operators, we note that T and S constructed above are positive in the sense that $y\geqslant 0$ (resp. $f\geqslant 0$) implies $Ty\geqslant 0$ (resp. $Sf\geqslant 0$). Now for a positive $x\in L^1(\mathcal{M},\tau)+L^\infty(\mathcal{M},\tau)$, using the spectral decomposition of x, we may choose an increasing sequence $\{x_n\}$ of elementary positive operators such that $x_n\leqslant x$ for all $n\geqslant 1$, $\lim_{n\to\infty}\mu_t(x_n)=\mu_t(x)$ for all $t>0$ and $\lim_{n\to\infty}x_n=x$ in the topology $\sigma(L^1(\mathcal{M},\tau)+L^\infty(\mathcal{M},\tau), L^1(\mathcal{M},\tau)\cap L^\infty(\mathcal{M},\tau))$. See [64], pp. 277–278. By the first part of the proof, for each n there are T_n and S_n associated with x_n as in Theorem 2.1. Thus (T_n) is a bounded sequence in $B(L^\infty(\mathcal{M},\tau),L^\infty(0,\infty))$. Since $B(L^\infty(\mathcal{M},\tau),L^\infty(0,\infty))$ is a dual space with predual $L^\infty(\mathcal{M},\tau)\otimes^\wedge L^1(0,\infty)$, passing to a subsequence if necessary, we may assume that T_n converges to T' in $B(L^\infty(\mathcal{M},\tau),L^\infty(0,\infty))$ with respect to the w^*-topology. Thus T' is a contraction from $L^\infty(\mathcal{M},\tau)$ to $L^\infty(0,\infty)$. To show that T' also defines a contraction from $L^1(\mathcal{M},\tau)$ to $L^1(0,\infty)$ let $y\in L^1(\mathcal{M},\tau)\cap L^\infty(\mathcal{M},\tau)$ and $f\in L^1(0,\infty)\cap L^\infty(0,\infty)$. Then

$$\int_0^\infty T'(y)f=\lim_{n\to\infty}\int_0^\infty T_n(y)f;$$

whence

$$\left|\int_0^\infty T'(y)f\right|\leqslant\limsup_{n\to\infty}\left\|T_n(y)\right\|_1\|f\|_\infty\leqslant\|y\|_1\|f\|_\infty,$$

which implies that $T'y\in L^1(0,\infty)$ and $\|T'y\|_1\leqslant\|y\|_1$. Hence T' extends to a contraction from $L^1(\mathcal{M},\tau)$ into $L^1(0,\infty)$.

On the other hand, by the positivity of T_n, we have $\mu(x_n) = T_n x_n \leqslant T_n x$. Taking limits, we find $\mu(x) \leqslant T'x$. Hence $\mu(x) = wT'x$ for some $w \in L^\infty(0,\infty)$ with $\|w\|_\infty \leqslant 1$. Then one sees that the map T defined by $Ty = wT'y$ has the property (i) of Theorem 2.1.

Similarly, from the sequence (S_n) we get the desired map S. □

REMARK 2.4. The result of interpolation applied to a compatible pair (X_0, X_1) of Banach spaces depends in general very much on the way in which we view this pair as compatible. There is however an elementary "invariance" property which we will invoke in the sequel, as follows: let (X_0, X_1) be a compatible pair of Banach spaces. Now let (Y_0, Y_1) be another compatible pair of Banach spaces and let $u_0 : Y_0 \to X_0$ and $u_1 : Y_1 \to X_1$ be isometric isomorphisms, which coincide on $Y_0 \cap Y_1$ (in that case it is customary in interpolation theory to think of u_0 and u_1 as the "same" map!). Equivalently, we have an isometric isomorphism $u : Y_0 + Y_1 \to X_0 + X_1$ such that the restrictions $u_0 = u_{|Y_0}$ and $u_1 = u_{|Y_1}$ are isometric isomorphisms respectively from Y_0 to X_0 and from Y_1 to X_1. Then u defines an isometric isomorphism from $(Y_0, Y_1)_\theta$ to $(X_0, X_1)_\theta$, so that

$$(Y_0, Y_1)_\theta \simeq (X_0, X_1)_\theta \quad (0 < \theta < 1).$$

This follows from the interpolation property applied separately to u and its inverse. If we assume that the pairs are made compatible with respect to continuous injections $J : X_0 \to X_1$ and $j : Y_0 \to Y_1$. Then to say that u_0 and u_1 are the "same" map means that $u_1 j = J u_0$. In particular, if $X_0 = Y_0$ and if u_0 is the identity on $X_0 = Y_0$, then this reduces to $u_1 j = J$.

3. General von Neumann algebras, including type III

The construction of non-commutative L^p-spaces based on *n.s.f.* traces outlined in Section 1 does not apply to von Neumann algebras of type III, which do not admit *n.s.f.* traces. However, it is known that any von Neumann algebra has an *n.s.f.* weight (a weight is simply an additive and positively homogeneous functional on the positive cone with values in $[0,\infty]$). This section is devoted to the non-commutative L^p-spaces associated with a von Neumann algebra equipped with an *n.s.f.* weight. There are several ways to construct the latter spaces (cf., e.g., [3,78,87,106,113,192]). We will present two of them. The first one is to reduce von Neumann algebras of type III to semifinite von Neumann algebras with the help of crossed products, as proposed by Haagerup [78]. The second way is via the complex interpolation; so it can be considered as a continuation of the results established in the previous section for the semifinite case. This was developed by Kosaki [106] and Terp [192] (see also [88,89] for related results).

We begin with the construction via interpolation. Let $\mathcal{M}$ be a von Neumann algebra. We know that $\mathcal{M}$ is a dual space with a unique predual, denoted by $\mathcal{M}_*$. We define, as usual, $L^1(\mathcal{M}) = \mathcal{M}_*$ and $L^\infty(\mathcal{M}) = \mathcal{M}$. Now we are confronted with the problem of defining $L^p(\mathcal{M})$ for any $1 < p < \infty$. For simplicity and clarity we will consider only the case where $\mathcal{M}$ is σ-finite, as in [106]. The reader is referred to [192] for the general case. Fix a distinguished normal faithful state φ on $\mathcal{M}$. Then we embed $\mathcal{M}$ into $\mathcal{M}_*$ by the following left injection

$$j:\mathcal{M}\to\mathcal{M}_*,\quad j(x)=x\varphi \quad (\text{here } x\varphi(y)=\varphi(yx)\ \forall y\in\mathcal{M}).$$

It is clear that j is a contractive injection with dense range. Thus we obtain a "compatibility" for the pair $(\mathcal{M},\mathcal{M}_*)$ with respect to which we may consider interpolation spaces between $\mathcal{M}$ and $\mathcal{M}_*$. Now let $1<p<\infty$. Following Kosaki, we introduce the corresponding non-commutative L^p-space as

$$L^p(\mathcal{M},\varphi)=(\mathcal{M},\mathcal{M}_*)_{1/p}.$$

To show the so-defined non-commutative L^p-spaces possess all properties one can expect, one should first note the important fact that $L^2(\mathcal{M},\varphi)$ is a Hilbert space, more precisely, $L^2(\mathcal{M},\varphi)=H_\varphi$, where H_φ is the Hilbert space in the GNS construction induced by φ (obtained after completion of $\mathcal{M}$ equipped with the inner product $\langle x,y\rangle=\varphi(y^*x)$). The proof of this fact given in [106] uses the modular theory. Here, we would like to point out that it directly follows from a general result in interpolation theory, that we describe as follows.

Let X be a complex Banach space. Let $\overline{X}$ denote the conjugate space of X, i.e., $\overline{X}$ is just X itself but equipped with the conjugate complex multiplication: $\lambda\cdot x=\bar{\lambda}x$ for any $\lambda\in\mathbb{C}$ and $x\in X$. For $x\in X$, $\bar{x}$ denotes the element x considered as an element in $\overline{X}$. Given a linear map $v:X\to Y$, we denote by $\bar{v}:\overline{X}\to\overline{Y}$ the same map acting on the "conjugates". Now suppose that there is a bounded linear map $J:X^*\to\overline{X}$ which is injective and of dense range. This allows us to consider $(X^*,\overline{X})$ as a compatible pair of Banach spaces. Suppose further that J is positive, i.e., $\xi(J(\xi))\geqslant 0$ for any $\xi\in X^*$. Then $\langle\xi,\eta\rangle=\xi(J(\eta))$ defines a scalar product on X^*. (Note: for $\xi\in X^*$ and $x\in X$ we write $\xi(\bar{x})=\xi(x)$.) Let H be the completion of X^* with respect to the above scalar product. Note that H contains X^* as a dense linear subspace. Thus we can define a bounded linear injection $v:H\to\overline{X}$ by simply setting (on an element of X^*) $v(\xi)=J(\xi)$, and extending by density to the whole of H. Identifying $\overline{H^*}$ with H as well as $(\overline{X})^*$ with $\overline{X^*}$, and denoting by ${}^t v:(\overline{X})^*\to H^*=\overline{H}$ the adjoint of v (in the Banach space sense) we see that $J=v\,{}^t\bar{v}$. These facts are well known (and easy to check). The general theorem referred to above is the following

THEOREM 3.1. *With the above assumptions, $(X^*,\overline{X})_{1/2}=H$ with equal norms.*

REMARK. This is well-known ([116]) with the additional assumption that X is reflexive. The general form as above was observed in [153], p. 26 (see also [197] and [42] for related results).

COROLLARY 3.2. *$L^2(\mathcal{M},\varphi)=H_\varphi$ with equal norms.*

PROOF. We let $X=\mathcal{M}_*$, $X^*=\mathcal{M}$. Recall that the involution on $\mathcal{M}_*$ is defined by $\psi^*(x)=\overline{\psi(x^*)}$ ($\psi\in\mathcal{M}_*$, $x\in\mathcal{M}$). Let $J:\mathcal{M}\to\overline{\mathcal{M}_*}$ be the map taking x to $\overline{j(x)^*}=\overline{\varphi x^*}$ and let $u_1:\mathcal{M}_*\to\overline{\mathcal{M}_*}$ be the (linear) isometry taking ψ to $\overline{\psi^*}$. We have $\langle\xi,\eta\rangle=$

$\xi(J(\eta)) = \xi(\overline{\varphi\eta^*}) = \xi(\varphi\eta^*) = \varphi(\eta^*\xi)$, thus we find $H = H_\varphi$ with equal norms and we have $J = u_1 j$, so that the result follows by invoking Remark 2.4 (here u_0 is simply the identity of $\mathcal{M}$). □

Using this corollary and the reiteration theorem, we see that the dual space of $L^p(\mathcal{M}, \varphi)$ is (isometrically) equal to $L^{p'}(\mathcal{M}, \varphi)$ for any $1 < p < \infty$ $(1/p + 1/p' = 1)$. The duality is induced by the scalar product of H_φ, that is, $\langle x, y\rangle = \varphi(y^*x)$, $x, y \in \mathcal{M}$. Corollary 3.2 also yields the Clarkson inequalities in $L^p(\mathcal{M}, \varphi)$ for any $1 < p < \infty$ (see Theorem 5.1 below). Thus, $L^p(\mathcal{M}, \varphi)$ $(1 < p < \infty)$ is uniformly convex. We will see more precise results on this in Section 5.

REMARK. Instead of the left injection considered previously, one could equally take the right injection of $\mathcal{M}$ into $\mathcal{M}_*$, i.e., $x \mapsto \varphi x$ (here $\varphi x(y) = \varphi(xy)$ $\forall y \in \mathcal{M}$). Then the resulting interpolation spaces are isometric to those obtained previously.

In view of the results in the last section, one is naturally led to consider the real interpolation as well. Set, for $1 < p < \infty$

$$L^{p,p}(\mathcal{M}, \varphi) = (\mathcal{M}, \mathcal{M}_*)_{1/p,p}.$$

The problem now is whether $L^{p,p}(\mathcal{M}, \varphi)$ and $L^p(\mathcal{M}, \varphi)$ are isomorphic. For the special case of $p = 2$, the answer is affirmative. Indeed, Theorem 3.1 admits a counterpart for the real interpolation as well (see [116] in the case of reflexive spaces; [205], p. 519 for the general case; see also [42] for more related results). Thus $L^{2,2}(\mathcal{M}, \varphi) = L^2(\mathcal{M}, \varphi)$ with equivalent norms. However, this is no longer true for all other values of p, as shown by the following example, due to Junge and the second named author.

EXAMPLE 3.3. Let φ be the state of $B(l^2)$ given by a diagonal operator D of trace 1 and whose diagonal entries are all positive. Then, for any $1 < p \neq 2 < \infty$, the two spaces $(B(l^2), B(l^2)_*)_{1/p}$ and $(B(l^2), B(l^2)_*)_{1/p,\,p}$ do not coincide. Indeed, let R (resp. R_*) be the subspace of $B(l^2)$ (resp. $B(l^2)_*$) consisting of matrices whose all rows but the first are zero. It is clear that $R = l^2$ and $R_* = l^2(d)$ isometrically, where $d = (d_n)_n$ is the sequence of the diagonal entries of D, and where $l^2(d)$ is the weighted l^2-space with the norm

$$\|x\|_{l^2(d)} = \left(\sum_n |x_n d_n|^2\right)^{1/2}.$$

On the other hand, let $P : B(l^2) \to R$ be the natural projection. P is contractive on $B(l^2)$. It is easy to check that under the left injection associated with φ, P is also a contractive projection from $B(l^2)_*$ onto R_*. Now assume that for some $1 < p < \infty$ the two interpolation spaces $(B(l^2), B(l^2)_*)_{1/p}$ and $(B(l^2), B(l^2)_*)_{1/p,\,p}$ have equivalent norms. Then we deduce that $(R, R_*)_{1/p}$ and $(R, R_*)_{1/p,\,p}$ have equivalent norms too. However, it is well-known that the first space is still a weighted l^2-space (and hence a Hilbert space), while the second one is isomorphic to a Hilbert space only when $p = 2$. Thus we have proved our assertion.

This construction by interpolation has several disadvantages: there is no natural notion of positive cone, no reasonably handy bimodule action by multiplication of $\mathcal{M}$ on $L^p(\mathcal{M}, \varphi)$, and finally the case $p < 1$ is excluded. However, these difficulties disappear in Haagerup's construction, to which we now turn. Our main reference for Haagerup's L^p-spaces is [191]. Let $\mathcal{M}$ be a von Neumann algebra equipped with a distinguished *n.s.f* weight φ. Let $\sigma_t = \sigma_t^{\varphi}$, $t \in \mathbb{R}$, denote the one parameter modular automorphism group of $\mathbb{R}$ on $\mathcal{M}$ associated with φ. We consider the crossed product $\mathcal{R} = \mathcal{M} \rtimes_{\sigma} \mathbb{R}$. Recall briefly the definition of $\mathcal{R}$. If $\mathcal{M}$ acts on a Hilbert space H, $\mathcal{R}$ is a von Neumann algebra acting on $L^2(\mathbb{R}, H)$, generated by the operators $\pi(x), x \in \mathcal{M}$, and the operators $\lambda(s), s \in \mathbb{R}$, defined by the following conditions: for any $\xi \in L^2(\mathbb{R}, H)$ and $t \in \mathbb{R}$

$$\pi(x)(\xi)(t) = \sigma_{-t}(x)\xi(t) \quad \text{and} \quad \lambda(s)(\xi)(t) = \xi(t-s).$$

Note that π is a normal faithful representation of $\mathcal{M}$ on $L^2(\mathbb{R}, H)$. Thus we may identify $\mathcal{M}$ with $\pi(\mathcal{M})$. Then the modular automorphism group $\{\sigma_t\}_{t\in\mathbb{R}}$ is given by

$$\sigma_t(x) = \lambda(t)x\lambda(t)^*, \quad x \in \mathcal{M},\ t \in \mathbb{R}.$$

There is a dual action $\{\hat{\sigma}_t\}_{t\in\mathbb{R}}$ of $\mathbb{R}$ on $\mathcal{R}$. This is a one parameter automorphism group of $\mathbb{R}$ on $\mathcal{R}$, implemented by the unitary representation $\{W(t)\}_{t\in\mathbb{R}}$ of $\mathbb{R}$ on $L^2(\mathbb{R}, H)$:

$$\hat{\sigma}_t(x) = W(t)xW(t)^*, \quad t \in \mathbb{R},\ x \in \mathcal{R},$$

where

$$W(t)(\xi)(s) = \mathrm{e}^{-its}\xi(s), \quad \xi \in L^2(\mathbb{R}, H),\ t, s \in \mathbb{R}.$$

Note that the dual action $\hat{\sigma}_t$ is also uniquely determined by the following conditions

$$\hat{\sigma}_t(x) = x \quad \text{and} \quad \hat{\sigma}_t\big(\lambda(s)\big) = \mathrm{e}^{-ist}\lambda(s), \quad \forall x \in \mathcal{M},\ s, t \in \mathbb{R}.$$

Thus $\mathcal{M}$ is invariant under $\{\hat{\sigma}_t\}_{t\in\mathbb{R}}$. In fact, $\mathcal{M}$ is exactly the space of the fixed points of $\{\hat{\sigma}_t\}_{t\in\mathbb{R}}$, namely,

$$\mathcal{M} = \big\{x \in \mathcal{R}\colon\ \hat{\sigma}_t(x) = x, \forall t \in \mathbb{R}\big\}.$$

Recall that the crossed product $\mathcal{R}$ is semifinite. Let τ be its *n.s.f.* trace satisfying

$$\tau \circ \hat{\sigma}_t = \mathrm{e}^{-t}\tau, \quad \forall\, t \in \mathbb{R}.$$

Also recall that any *n.s.f.* weight ψ on $\mathcal{M}$ induces a dual *n.s.f.* weight $\tilde{\psi}$ on $\mathcal{R}$. Then $\tilde{\psi}$ admits a Radon–Nikodym derivative with respect to τ. In particular, the dual weight $\tilde{\varphi}$ of our distinguished weight φ has a Radon–Nikodym derivative D with respect to τ. Then

$$\tilde{\varphi}(x) = \tau(Dx), \quad x \in \mathcal{R}_+.$$

Recall that D is an invertible positive selfadjoint operator on $L^2(\mathbb{R}, H)$, affiliated with $\mathcal{R}$, and that the regular representation $\lambda(t)$ above is given by

$$\lambda(t) = D^{\mathrm{i}t}, \quad t \in \mathbb{R}.$$

Now we define the Haagerup non-commutative L^p-spaces $(0 < p \leqslant \infty)$ by

$$\Lambda^p(\mathcal{M}, \varphi) = \{x \in L^0(\mathcal{R}, \tau) \colon \hat{\sigma}_t(x) = \mathrm{e}^{-t/p}x, \ \forall t \in \mathbb{R}\}.$$

(Recall that $L^0(\mathcal{R}, \tau)$ denotes the topological $*$-algebra of all operators on $L^2(\mathbb{R}, H)$ measurable with respect to $(\mathcal{R}, \tau)$.) It is clear that $\Lambda^p(\mathcal{M}, \varphi)$ is a vector subspace of $L^0(\mathcal{R}, \tau)$, invariant under the $*$-operation. The algebraic structure of $\Lambda^p(\mathcal{M}, \varphi)$ is inherited from that of $L^0(\mathcal{R}, \tau)$. Let $x \in \Lambda^p(\mathcal{M}, \varphi)$ and $x = u|x|$ its polar decomposition. Then $u \in \mathcal{M}$ and $|x| \in \Lambda^p(\mathcal{M}, \varphi)$. Recall that

$$\Lambda^\infty(\mathcal{M}, \varphi) = \mathcal{M} \quad \text{and} \quad \Lambda^1(\mathcal{M}, \varphi) = \mathcal{M}_*.$$

The latter equality is understood as follows. As mentioned previously, for any $\omega \in \mathcal{M}_*^+$, the dual weight $\tilde{\omega}$ has a Radon–Nikodym derivative, denoted by h_ω, with respect to τ:

$$\tilde{\omega}(x) = \tau(h_\omega x), \quad x \in \mathcal{R}_+.$$

Then

$$h_\omega \in L^0(\mathcal{R}, \tau) \quad \text{and} \quad \hat{\sigma}_t(h_\omega) = \mathrm{e}^{-t}h_\omega, \quad \forall t \in \mathbb{R}.$$

Thus $h_\omega \in \Lambda^1(\mathcal{M}, \varphi)_+$. This correspondence between $\mathcal{M}_*^+$ and $\Lambda^1(\mathcal{M}, \varphi)_+$ extends to a bijection between $\mathcal{M}_*$ and $\Lambda^1(\mathcal{M}, \varphi)$. Then for any $\omega \in \mathcal{M}_*$, if $\omega = u|\omega|$ is its polar decomposition, the corresponding $h_\omega \in \Lambda^1(\mathcal{M}, \varphi)$ admits the polar decomposition

$$h_\omega = u|h_\omega| = uh_{|\omega|}.$$

Thus we can define a norm on $\Lambda^1(\mathcal{M}, \varphi)$ by

$$\|h_\omega\|_1 = |\omega|(1) = \|\omega\|_{\mathcal{M}_*}, \quad \omega \in \mathcal{M}_*.$$

In this way, $\Lambda^1(\mathcal{M}, \varphi) = \mathcal{M}_*$ isometrically. Now let $0 < p < \infty$. Since $x \in \Lambda^p(\mathcal{M}, \varphi)$ iff $|x|^p \in \Lambda^1(\mathcal{M}, \varphi)$, we define

$$\|x\|_p = \bigl\||x|^p\bigr\|_1^{1/p}, \quad x \in \Lambda^p(\mathcal{M}, \varphi).$$

Then if $1 \leqslant p < \infty$, $\|\cdot\|_p$ is a norm (cf. [78] and [191]), and if $0 < p < 1$, $\|\cdot\|_p$ is a p-norm (cf. [108]). Equipped with $\|\cdot\|_p$, $\Lambda^p(\mathcal{M}, \varphi)$ becomes a Banach space or a quasi-Banach space, according to $1 \leqslant p < \infty$ or $0 < p < 1$. Clearly,

$$\|x\|_p = \|x^*\|_p = \bigl\||x|\bigr\|_p, \quad x \in \Lambda^p(\mathcal{M}, \varphi).$$

REMARKS. (i) Using [191] Lemma II.5, one easily checks that $\Lambda^p(\mathcal{M},\varphi)$ is isometric to a subspace of the non-commutative weak L^p-space $L^{p,\infty}(\mathcal{R},\tau)$. Also note that in $\Lambda^p(\mathcal{M},\varphi)$ the topology defined by $\|\cdot\|_p$ coincides with the topology induced by that of $L^0(\mathcal{R},\tau)$ (cf. [191]).

(ii) One weak point of the Haagerup non-commutative L^p-spaces is the fact that for any $p\neq q$ the intersection of $\Lambda^p(\mathcal{M},\varphi)$ and $\Lambda^q(\mathcal{M},\varphi)$ is trivial. In particular, these spaces do not form an interpolation scale. This causes some difficulties in applications (especially when interpolation is used).

As usual, for $1\leqslant p<\infty$ the dual space of $\Lambda^p(\mathcal{M},\varphi)$ is $\Lambda^{p'}(\mathcal{M},\varphi)$, $1/p+1/p'=1$. To describe this duality, we need to introduce a distinguished linear functional on $\Lambda^1(\mathcal{M},\varphi)$, called *trace* and denoted by tr, which is defined by

$$\mathrm{tr}(x)=\omega_x(1),\quad x\in\Lambda^1(\mathcal{M},\varphi),$$

where $\omega_x\in\mathcal{M}_*$ is the unique normal functional associated with x by the above identification between $\mathcal{M}_*$ and $\Lambda^1(\mathcal{M},\varphi)$. Then tr is a continuous functional on $\Lambda^1(\mathcal{M},\varphi)$ satisfying

$$\bigl|\mathrm{tr}(x)\bigr|\leqslant\mathrm{tr}\bigl(|x|\bigr)=\|x\|_1,\quad x\in\Lambda^1(\mathcal{M},\varphi).$$

The usual Hölder inequality also holds for these non-commutative L^p-spaces. Let $0<p,q,r\leqslant\infty$ such that $1/r=1/p+1/q$. Then

$$\begin{aligned}&x\in\Lambda^p(\mathcal{M},\varphi)\text{ and }y\in\Lambda^q(\mathcal{M},\varphi)\\&\quad\Longrightarrow xy\in\Lambda^r(\mathcal{M},\varphi)\text{ and }\|xy\|_r\leqslant\|x\|_p\|y\|_q.\end{aligned}$$

In particular, for any $1\leqslant p\leqslant\infty$ we have

$$\bigl|\mathrm{tr}(xy)\bigr|\leqslant\|xy\|_1\leqslant\|x\|_p\|y\|_{p'},\quad x\in\Lambda^p(\mathcal{M},\varphi),\ y\in\Lambda^{p'}(\mathcal{M},\varphi).$$

Thus, $(x,y)\mapsto\mathrm{tr}(xy)$ defines a duality between $\Lambda^p(\mathcal{M},\varphi)$ and $\Lambda^{p'}(\mathcal{M},\varphi)$, with respect to which

$$\bigl(\Lambda^p(\mathcal{M},\varphi)\bigr)^*=\Lambda^{p'}(\mathcal{M},\varphi)\quad\text{isometrically},\ 1\leqslant p<\infty.$$

This functional tr on $\Lambda^1(\mathcal{M},\varphi)$ plays the role of a trace. Indeed, it satisfies the following *tracial* property

$$\mathrm{tr}(xy)=\mathrm{tr}(yx),\quad x\in\Lambda^p(\mathcal{M},\varphi),\ y\in\Lambda^{p'}(\mathcal{M},\varphi).$$

The reader is referred to [191] for more information.

THEOREM 3.4. *Let $\mathcal{M}$ be a von Neumann algebra.*

(i) *Let $0 < p < \infty$. If τ is an n.s.f. trace on $\mathcal{M}$, then $L^p(\mathcal{M}, \tau)$ (the non-commutative L^p-space described in Section 1) is isometric to $\Lambda^p(\mathcal{M}, \varphi)$.*

(ii) *Let $0 < p < \infty$. Then $\Lambda^p(\mathcal{M}, \varphi)$ is independent of φ, i.e., if φ and ψ are two n.s.f. weights on $\mathcal{M}$, then $\Lambda^p(\mathcal{M}, \varphi)$ and $\Lambda^p(\mathcal{M}, \psi)$ are isometric.*

(iii) *Let φ be a normal faithful state on $\mathcal{M}$ and $1 < p < \infty$. Then $L^p(\mathcal{M}, \varphi)$ and $\Lambda^p(\mathcal{M}, \varphi)$ are isometric.*

The first two parts of Theorem 3.4 are due to Haagerup [78] (see also [191]), and the third one to Kosaki [106]. As can be expected, the proof of Theorem 3.4 heavily depends on the modular theory.

The preceding statement allows a considerable simplification of the notation, as follows:

CONVENTION. From now on, given a von Neumann algebra $\mathcal{M}$, $L^p(\mathcal{M})$ will denote any one of the non-commutative L^p-spaces associated with $\mathcal{M}$ appearing in Theorem 3.4. (The latter shows that these spaces are all "the same".) However, if $\mathcal{M}$ is semifinite, we will always assume that $L^p(\mathcal{M})$ is the L^p-space constructed from an *n.s.f.* trace as in Section 1.

The following basic result is very useful to reduce the failure of certain properties of L^p-spaces to the special case of the hyperfinite factor $L^p(R)$. Recall that R denotes the hyperfinite II_1 factor (see Section 1).

THEOREM 3.5. *Let $\mathcal{M}$ be a von Neumann algebra not of type* I. *Then for any $0 < p \leqslant \infty$ (resp. $1 \leqslant p \leqslant \infty$) $L^p(R)$ is isometric to a (resp. 1-complemented) subspace of $L^p(\mathcal{M})$.*

The proof of Theorem 3.5 combines several more or less well-known facts. The key point is that if $\mathcal{M}$ is not of type I, then R is isomorphic, as von Neumann algebra, to a w^*-closed $*$-subalgebra of $\mathcal{M}$ which is the range of a normal conditional expectation on $\mathcal{M}$. The reader is referred to [122] for more details and precise references.

We end this section with Haagerup's approximation theorem of an $L^p(\mathcal{M})$ associated with an algebra $\mathcal{M}$ of type III by those associated with semifinite von Neumann algebras (cf. [79]).

THEOREM 3.6. *Let $\mathcal{M}$ be a von Neumann algebra equipped with an n.s.f. weight φ. Let $\Lambda^p(\mathcal{M}, \varphi)$ be the associated Haagerup L^p-space ($0 < p < \infty$). Then there are a Banach space X (a p-Banach space if $0 < p < 1$), a directed family $\{(\mathcal{M}_i, \tau_i)\}_{i \in I}$ of finite von Neumann algebras $\mathcal{M}_i$ (with normal faithful finite traces τ_i), and a family $\{j_i\}_{i \in I}$ of isometric embeddings $j_i : L^p(\mathcal{M}_i, \tau_i) \to X$ such that*

(i) *$j_i(L^p(\mathcal{M}_i, \tau_i)) \subset j_{i'}(L^p(\mathcal{M}_{i'}, \tau_{i'}))$ for all $i, i' \in I$ with $i \leqslant i'$;*

(ii) *$\bigcup_{i \in I} j_i(L^p(\mathcal{M}_i, \tau_i))$ is dense in X;*

(iii) *$\Lambda^p(\mathcal{M}, \varphi)$ is isometric to a (complemented for $1 \leqslant p < \infty$) subspace of X.*

4. From classic L^p to non-commutative L^p: similarities and differences

A good part of the early theory consisted in extending commutative results over to the non-commutative case; this usually required specific new methods, but without too many surprises. For instance, we have already seen that a non-commutative L^p-space $L^p(\mathcal{M})$ is reflexive for any $1 < p < \infty$. Moreover, just like in the commutative case it is easy to check that $L^1(\mathcal{M})$ has the RNP iff $\mathcal{M}$ is atomic. Indeed, if $\mathcal{M}$ is not atomic, $L^1(\mathcal{M})$ contains a 1-complemented subspace isometric to $L^1(0, 1)$, hence fails the RNP. On the other hand, $L^1(\mathcal{M})$ is weakly sequentially complete for any $\mathcal{M}$. Moreover, there are characterizations of weakly compact subsets in $L^1(\mathcal{M})$, analogous to those in the commutative setting (cf. [190, III.5] and the references given there; see also [140] for more recent results). Moreover, we will see later in Section 5 that any non-commutative L^p-space $(0 < p \leqslant 1)$ has the analytic RNP.

However, the differentiability of the norms of non-commutative L^p-spaces has not been well understood yet. This problem was considered only for the Schatten classes in [194]. It was announced there (with a sketch of proof) that the norm of S^p had the same differentiability as that of l^p $(1 < p < \infty)$. It seems unclear how to extend this to the general case (or, at least, to the semifinite case). In a different direction, the papers [55,56] are devoted to the problem of characterizing the symmetric spaces of measurable operators for which the absolute-value mapping $x \to |x|$ is Lipschitz continuous. In the case of non-commutative L^1-spaces, Kosaki proves in [108] the following useful inequality: for any φ and ψ in such a space, we have

$$\big\| |\varphi| - |\psi| \big\|_1 \leqslant \sqrt{2}\big(\|\varphi + \psi\|_1 \|\varphi - \psi\|_1\big)^{1/2}.$$

The passage from the Schatten classes to von Neumann algebras with semifinite traces, i.e., from the discrete to the continuous case, can sometimes be quite substantial. See, for instance, Brown's extension of Weyl's classical inequalities: $(\sum |\lambda_n(T)|^p)^{1/p} \leqslant \|T\|_{S^p}$ (here $\lambda_n(T)$ are the eigenvalues of T repeated according to multiplicity). Brown [26] had to invent a new kind of spectral measure (now called Brown's measure) to extend this, together with Lidskii's trace theorem, to the semifinite case.

The study of non-commutative L^p-spaces, or more generally, of symmetric operator spaces, goes mainly in two closely related directions: lift topological or geometrical properties from the commutative setting to the non-commutative one, and reduce problems in the non-commutative case to those in the commutative one. We have already seen several examples in both directions.

To discuss more illustrations, it is better to place ourselves in the context of symmetric operator spaces. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with an *n.s.f.* trace τ, and let E be an *r.i.* function space on $(0, \infty)$. One naturally expects that properties of $E(\mathcal{M}, \tau)$ should be reflected by those of E. Works already done in this direction are too numerous to enumerate. Here we content ourselves with only three examples. The first one concerns the (uniform) Kadets–Klee properties. The lifting of these properties from E to $E(\mathcal{M}, \tau)$ has been extensively studied (cf., e.g., [6,37,38,41,50,57]). The second example is about the reduction of weakly compact subsets in $E(\mathcal{M}, \tau)$ to those in E. This was achieved in [52] and [58] (see also the references there for previous works on this problem). Finally, the geometry of the unit ball of $E(\mathcal{M},\tau)$ was studied in [5] and [40]. More examples and references of this kind will be given in appropriate places in the subsequent sections (see also [49]).

Despite the strong analogy between the commutative and non-commutative settings, non-commutative L^p-spaces behave, in some aspects, very differently from their commutative counterparts.

One of the most spectacular differences concerns unconditional bases or "unconditional structures". Already in [111], it was proved that S^1 cannot be embedded into any space with an unconditional basis, in sharp contrast with ℓ^1. But the big surprise came when Gordon and Lewis [74] proved that the Schatten class S^p fails to have any unconditional basis when $p \neq 2$ in sharp contrast with ℓ^p or L^p. More generally they proved that S^p fails "local unconditional structure" in their sense (abbreviated as *GL-l.u.st.*; see [94] for the precise definition). This was the first example of a reflexive Banach space which was not isomorphic to any complemented subspace of a Banach lattice. More precisely, let $lu(X)$ denote the *GL-l.u.st.* constant of a Banach space X ($lu(X)$ is equal to the norm of factorization through a Banach lattice of the identity of X^{**}). The following theorem was proved by Gordon and Lewis [74] using a criterion (necessary but not sufficient) for the *GL-l.u.st.* of a space X: any 1-absolutely summing operator on X must factor through L^1 (this is now called the *GL*-property). More precisely, they obtained the first part of the next statement (the second part comes from [143], see also [180] and [146], 8.d for related results):

THEOREM 4.1. *There is a constant $C > 0$ such that for any $1 \leqslant p \leqslant \infty$ and any $n \geqslant 1$*

$$Cn^{|1/p-1/2|} \leqslant lu\big(S_n^p\big) \leqslant n^{|1/p-1/2|}.$$

Consequently, S^p does not have the GL-l.u.st. for $p \neq 2$. More generally, let X be any Banach lattice of finite cotype (resp. of type > 1), then there is a constant $C > 0$ such that, if E is any n^2-dimensional subspace (resp. subspace of a quotient) of X, we have $d(S_n^p, E) \geqslant C n^{|1/p-1/2|}$.

Combining Theorem 3.5 and Theorem 4.1, we immediately obtain

COROLLARY 4.2. *A non-commutative $L^p(\mathcal{M})$, $1 \leqslant p < \infty$ and $p \neq 2$, has the GL-l.u.st. iff $\mathcal{M}$ is isomorphic, as Banach space, to $L^\infty(\Omega,\mu)$ for some measure space (Ω,μ). Moreover, this happens iff $L^p(\mathcal{M})$ is isomorphic to a subspace of a commutative L^p-space.*

Note that $\mathcal{M}$ is isomorphic, as Banach space, to a commutative L^∞ iff $\mathcal{M}$ is the direct sum (ℓ_∞ sense) of finitely many algebras of the form $L^\infty(\mu; B(H))$ $(= L^\infty(\mu) \otimes B(H))$ with $\dim(H) < \infty$.

Another striking divergence from the classical case is provided by the uniform approximation property (UAP in short): by an extremely complicated construction, Szankowski proved that $B(\ell_2)$ fails the approximation property (AP in short), and moreover ([189])

that S^p (or $S^{p'}$) fails the UAP for $p > 80$. It remains a challenging open problem to prove this for any $p \neq 2$.

We will describe another striking difference in Section 7, that is, a non-commutative L^p-space $L^p(\mathcal{M})$, $0 < p \leqslant 1$, is never an analytic UMD space except when $\mathcal{M}$ is isomorphic, as Banach space, to a commutative L^∞-space.

Surprisingly, by [122], "stability" provides us with one more sharp contrast. Recall that a Banach space X is stable (in Krivine–Maurey's sense) if for any bounded sequences $\{x_m\}_{m\geqslant 1}$, $\{y_n\}_{n\geqslant 1}$ in X and any ultrafilters U, V on $\mathbb{N}$

$$\lim_{m\in U}\lim_{n\in V}\|x_m + y_n\| = \lim_{n\in V}\lim_{m\in U}\|x_m + y_n\|.$$

It is well known that any commutative L^p-space ($1 \leqslant p < \infty$) is stable (cf. [109]). This is no longer true in the non-commutative setting. In fact, we have the following characterization of stable non-commutative L^p-spaces.

THEOREM 4.3. *Let $1 \leqslant p < \infty$, $p \neq 2$. Then $L^p(\mathcal{M})$ is stable iff $\mathcal{M}$ is of type* I.

The "if" part of Theorem 4.3 was independently proved by Arazy [8] and Raynaud [166]. The "only if" part is due to Marcolino [122]. Marcolino's proof is divided into two steps. The first one (the proof of which is relatively easy) is that $L^p(R)$, $p \neq 2$, is *not* stable (recalling that R is the hyperfinite II_1 factor). The second step is the above Theorem 3.5.

5. Uniform convexity (real and complex) and uniform smoothness

The fact that $L^p(\mathcal{M})$, $1 < p < \infty$, is uniformly convex and smooth immediately follows from the following Clarkson type inequalities.

THEOREM 5.1. *Let $1 < p, p' < \infty$ with $1/p + 1/p' = 1$. Then*
 (i) *if $1 \leqslant p \leqslant 2$,*

$$\Big[\frac{1}{2}\big(\|x+y\|_p^{p'} + \|x-y\|_p^{p'}\big)\Big]^{1/p'} \leqslant \big(\|x\|_p^p + \|y\|_p^p\big)^{1/p}, \quad x, y \in L^p(\mathcal{M}); \tag{5.1}$$

 (ii) *if $2 \leqslant p \leqslant \infty$,*

$$\Big[\frac{1}{2}\big(\|x+y\|_p^{p} + \|x-y\|_p^{p}\big)\Big]^{1/p} \leqslant \big(\|x\|_p^{p'} + \|y\|_p^{p'}\big)^{1/p'}, \quad x, y \in L^p(\mathcal{M}). \tag{5.2}$$

Inequalities (5.1) and (5.2), of course, have their origin in the classical Clarkson inequalities for commutative L^p-spaces. In the non-commutative setting, some partial or particular cases of (5.1) and (5.2) were obtained in [47,128] (see also [49] for additional references). (5.1) and (5.2), as stated above were proved in [78] and [106] (see also [191] and [64]).

PROOF OF THEOREM 5.1. The proof is almost obvious via the complex interpolation. Indeed, (5.1) (resp. (5.2)) is trivially true for $p=1,2$ (resp. $p=2,\infty$). Then the complex interpolation yields (5.1) and (5.2). We also note that (5.1) and (5.2) are dual to each other. □

Let δ_X (resp. ρ_X) denote the modulus of convexity (resp. smoothness) of a Banach space X. Theorem 5.1 implies the following

COROLLARY 5.2. *Let $1<p<\infty$. Then $L^p(\mathcal{M})$ is uniformly convex and smooth; more precisely, we have*

(i) *for $1<p\leqslant 2$*

$$\delta_{L^p(\mathcal{M})}(\varepsilon) \geqslant \frac{1}{p'2^{p'}}\varepsilon^{p'}, \quad 0<\varepsilon<2, \quad \textit{and} \quad \rho_{L^p(\mathcal{M})}(t) \leqslant \frac{1}{p}t^p, \quad t>0;$$

(ii) *for $2<p<\infty$*

$$\delta_{L^p(\mathcal{M})}(\varepsilon) \geqslant \frac{1}{p2^p}\varepsilon^{p}, \quad 0<\varepsilon<2, \quad \textit{and} \quad \rho_{L^p(\mathcal{M})}(t) \leqslant \frac{1}{p'}t^{p'}, \quad t>0.$$

The reader can find some applications of the uniform convexity of $L^p(\mathcal{M})$, e.g., in [107,108].

Let us comment on the estimate for the modulus of convexity given by Corollary 5.2 (the same comment, of course, applies to the modulus of smoothness as well). This estimate is best possible only in the case of $2<p<\infty$. We should also point out that in this case the relevant constant $1/(p2^p)$ is optimal (for it is already so in the commutative case; see [115], p. 63). Keeping in mind the well-known result on the modulus of convexity of commutative L^p-spaces, one would expect that the order of $\delta_{L^p(\mathcal{M})}(\varepsilon)$ for $1<p<2$ be $O(\varepsilon^2)$. This is indeed the case (cf. [193]). In fact, we have a more precise result as follows.

THEOREM 5.3. *Let $1<p<\infty$. Then*

(i) *for $1<p\leqslant 2$*

$$\begin{aligned}&\left(\|x\|_p^2+(p-1)\|y\|_p^2\right)^{1/2}\\ &\quad\leqslant\left[\frac{1}{2}\left(\|x+y\|_p^p+\|x-y\|_p^p\right)\right]^{1/p}, \quad \forall x,y\in L^p(\mathcal{M});\end{aligned} \tag{5.3}$$

(ii) *for $2<p<\infty$,*

$$\begin{aligned}&\left[\frac{1}{2}\left(\|x+y\|_p^p+\|x-y\|_p^p\right)\right]^{1/p}\\ &\quad\leqslant\left(\|x\|_p^2+(p-1)\|y\|_p^2\right)^{1/2}, \quad \forall x,y\in L^p(\mathcal{M}).\end{aligned} \tag{5.4}$$

Moreover, the constant $p-1$ is optimal in both (5.3) *and* (5.4).

This theorem was proved in [14] for Schatten classes. As pointed out by the authors, the arguments there work for semifinite von Neumann algebras as well. Then the general case follows by Theorem 3.6. We should emphasize that the optimality of the constant $p-1$ in (5.3) and (5.4) has important applications to hypercontractivity. We will later illustrate this by discussing the Fermionic hypercontractivity.

Note that if one does not care about the best constants, one can deduce Theorem 5.3 from the optimal order of $\delta_{L^p(\mathcal{M})}(\varepsilon)$ and $\rho_{L^p(\mathcal{M})}(t)$ obtained in [193] (at least, for Schatten classes). Note also that (5.3) and (5.4) are equivalent by duality. We will include a very simple proof of (5.4) for $p=2^n$ ($n \in \mathbb{N}$), and so by interpolation for all $2<p<\infty$ with some constant C_p instead of $p-1$.

Theorem 5.3 gives the optimal estimates for $\delta_{L^p(\mathcal{M})}(\varepsilon)$ $(1<p<2)$ and $\rho_{L^p(\mathcal{M})}(t)$ $(2<p<\infty)$.

COROLLARY 5.4. *We have, for any* $0<\varepsilon<2$ *and* $t>0$

$$\delta_{L^p(\mathcal{M})}(\varepsilon) \geqslant \frac{p-1}{8}\varepsilon^2, \quad 1<p\leqslant 2, \quad \text{and}$$

$$\rho_{L^p(\mathcal{M})}(t) \leqslant \frac{p-1}{2}t^2, \quad 2\leqslant p<\infty.$$

REMARK. The constants $(p-1)/8$ and $(p-1)/2$ in the above estimates are optimal (see [115], p. 63 for the commutative case).

Corollaries 5.2 and 5.4 yield the type and cotype of $L^p(\mathcal{M})$ for $1<p<\infty$.

COROLLARY 5.5. *Let* $1<p<\infty$. *Then* $L^p(\mathcal{M})$ *is of type* $\min(2,p)$ *and cotype* $\max(2,p)$.

The type and cotype of $L^p(\mathcal{M})$ were determined in [193] for Schatten classes, and in [63] for the general case. We will see later that $L^p(\mathcal{M})$ is of cotype 2 for $0<p\leqslant 1$.

Now we turn to the application of the optimality of the constant $p-1$ in (5.3) and (5.4) to the Fermionic hypercontractivity. Before starting our discussion, we should point out, however, that in the scalar case (i.e., in the case where $\mathcal{M}=\mathbb{C}$) Theorem 5.3 is exactly Nelson's celebrated hypercontractivity inequality for the two point space (cf. [20] and [130]). This two point hypercontractivity inequality easily yields the optimal hypercontractivity for the classical Ornstein–Uhlenbeck semigroup. Carlen and Lieb used Theorem 5.3 (in the case of Schatten classes) to obtain the optimal Fermionic hypercontractivity, thus solving a problem left open since Gross' pioneer works in the domain (cf. [75]).

Let R be the hyperfinite II_1 factor. We recall that R is generated by a sequence $(\varepsilon_n)_{n\geqslant 1}$ of self-adjoint unitaries satisfying (CAR) (see Section 1). We also recall that $\{w_A\colon A\subset\mathbb{N},\ A \text{ finite}\}$ is an orthonormal basis of $L^2(R)$. We define the number operator N by $Nw_A=|A|w_A$ ($|A|$ denoting the cardinality of A). N is an unbounded positive self-adjoint operator on $L^2(R)$. It generates the Fermionic Ornstein–Uhlenbeck semigroup P_t: $P_t=\mathrm{e}^{-tN}$, $t\geqslant 0$. One can show that P_t is a contraction on $L^p(R)$ for all $1\leqslant p\leqslant\infty$. The optimal Fermionic hypercontractivity is contained in the following

THEOREM 5.6. *Let $1 < p < q < \infty$. Then P_t is a contraction from $L^p(R)$ to $L^q(R)$ iff $e^{-2t} \leqslant (p-1)/(q-1)$.*

Let us briefly comment on the proof of Theorem 5.6. First, since linear combinations of the w_A's are dense in $L^p(R)$, it suffices to prove Theorem 5.6 in the finite-dimensional case, that is, when P_t is restricted to the L^p-spaces based on the von Neumann algebra generated by $\{\varepsilon_1, \dots, \varepsilon_n\}$ $(n \in \mathbb{N})$. Second, by standard arguments as for the classical Ornstein–Uhlenbeck semigroup, one can reduce Theorem 5.6 to the special case where $2 = p < q$. Assuming these reductions, one can use (5.4) to prove Theorem 5.6 by induction on n (noting that the case $n = 1$ corresponds to Nelson's two point hypercontractivity). We refer to [35] for the details.

REMARKS. (i) Theorem 5.6 implies, and in fact, is equivalent to the optimal Fermionic logarithmic Sobolev inequality, see [76] and [35].

(ii) Biane [18] obtained the analogue of Theorem 5.6 for the free Ornstein–Uhlenbeck semigroup. Note that this latter semigroup is also ultracontractive (cf. [23,24]).

We end the discussion on the uniform convexity and smoothness by providing a simple proof for Theorem 5.3 (except for the best constant). We need only to consider (5.4). We are going to show that for $2 \leqslant p < \infty$ there is a constant C_p, depending only on p, such that

$$\begin{aligned}&\left[\frac{1}{2}\left(\|x+y\|_p^p + \|x-y\|_p^p\right)\right]^{1/p} \\ &\quad \leqslant \left(\|x\|_p^2 + C_p\|y\|_p^2\right)^{1/2}, \quad \forall x, y \in L^p(\mathcal{M}).\end{aligned} \tag{5.4_p}$$

To that end, by Theorem 3.6, we can assume that $\mathcal{M}$ is semifinite and equipped with a faithful normal semifinite trace τ. The key step in the proof of (5.4_p) is the implication "$(5.4_p) \Rightarrow (5.4_{2p})$". Let us show this. Assume (5.4_p). Let $x, y \in L^{2p}(\mathcal{M})$, and set $a = x^*x + y^*y$, $b = x^*y + y^*x$. Then $a, b \in L^p(\mathcal{M})$ and

$$\begin{aligned}\frac{1}{2}\left(\|x+y\|_{2p}^{2p} + \|x-y\|_{2p}^{2p}\right) &= \frac{1}{2}\left(\|a+b\|_p^p + \|a-b\|_p^p\right) \\ &\leqslant \left(\|a\|_p^2 + C_p\|b\|_p^2\right)^{p/2} \quad (\text{by } (5.4_p)) \\ &\leqslant \left[\left(\|x\|_{2p}^2 + \|y\|_{2p}^2\right)^2 + 4C_p\|x\|_{2p}^2\|y\|_{2p}^2\right]^{p/2} \\ &\leqslant \left(\|x\|_{2p}^2 + (2C_p+1)\|y\|_{2p}^2\right)^p;\end{aligned}$$

whence (5.4_{2p}) with $C_{2p} \leqslant 2C_p + 1$. Therefore, starting with the trivial case $p = 2$ (noting that $C_2 = 1$), and by iteration, we get $C_{2^n} \leqslant 2^n - 1$ (in fact, $C_{2^n} = 2^n - 1$). Thus for these special values of p we obtain the best constant in (5.4). Then for any other value of p, say, $2^n < p < 2^{n+1}$, by complex interpolation, we deduce (5.4_p) with

$$C_p \leqslant \left(2^n - 1\right)^{1-\theta}\left(2^{n+1} - 1\right)^{\theta},$$

where $1/p = (1-\theta)/2^n + \theta/2^{n+1}$. $\square$

Now we pass on to the uniform PL-convexity and Hardy convexity of non-commutative L^p-spaces. This time, we admit quasi-normed spaces (so $p < 1$ is allowed).

Let X be a (complex) quasi-Banach space. Let $\mathbb{T}$ be the unit circle equipped with normalized Lebesgue measure. For $0 < p < \infty$ we denote by $L^p(\mathbb{T}, X)$ the usual L^p-space of Bochner measurable functions with values in X. Note that $L^p(\mathbb{T}, L^p(\mathcal{M}))$ is just the non-commutative L^p-space based on the von Neumann algebra tensor product $L^\infty(\mathbb{T}) \otimes \mathcal{M}$. Let $\mathcal{P}(X)$ denote the family of all complex polynomials with coefficients in X:

$$\mathcal{P}(X) = \left\{ \sum_{k=0}^{n} x_k z^k \colon x_k \in X,\ 0 \leqslant k \leqslant n,\ n \in \mathbb{N} \right\}.$$

DEFINITION. Let X be a quasi-Banach space. Let $0 < p < \infty$ and $\varepsilon > 0$. We define

$$H_X(\varepsilon) = \inf\bigl\{ \|x + zy\|_{L^1(\mathbb{T},X)} - 1 \colon \|x\| = 1,\ \|y\| \geqslant \varepsilon,\ x, y \in X \bigr\}$$

and

$$h_X^p(\varepsilon) = \inf\bigl\{ \|f\|_{L^p(\mathbb{T},X)} - 1 \colon \|f(0)\| = 1,\ \|f - f(0)\|_{L^p(\mathbb{T},X)} \geqslant \varepsilon,\ f \in \mathcal{P}(X) \bigr\}.$$

X is said to be uniformly PL-convex (resp. H^p-convex) if $H_X(\varepsilon) > 0$ (resp. $h_X^p(\varepsilon) > 0$) for all $\varepsilon > 0$. $H_X(\varepsilon)$ (resp. $h_X^p(\varepsilon)$) is called the modulus of PL-convexity (resp. H^p-convexity) of X.

The uniform PL-convexity was introduced and studied in [45]. It was shown there that in the definition of $H_X(\varepsilon)$ above, if the L^1-norm is replaced by an L^p-norm, then the resulting modulus is equivalent to $H_X(\varepsilon)$. The uniform H^p-convexity was explicitly introduced in [199]; however, it is already implicit in [80]. It was proved in [202] that if X is uniformly H^p-convex for one $p \in (0, \infty)$, then so is it for all $p \in (0, \infty)$. Thus we say that X is *uniformly H-convex* if it is uniformly H^p-convex for some p. The uniform PL-convexity (resp. H-convexity) is closely related to inequalities satisfied by analytic (resp. Hardy) martingales with values in X. The Enflo–Pisier renorming theorem about the uniform (real) convexity admits analogues for these uniform complex convexities. We refer to [45,199,201,202] and [149] for more information.

REMARKS. (i) For any given $0 < p < \infty$ there is a constant $\alpha_p > 0$ such that for all quasi-Banach spaces X

$$H_X(\varepsilon) \geqslant \alpha_p h_X^p(\alpha_p \varepsilon), \quad 0 < \varepsilon \leqslant 1.$$

Consequently, the uniform H-convexity implies the uniform PL-convexity.

(ii) If a Banach space X is uniformly convex, it is uniformly H-convex.

THEOREM 5.7. *Assume* $0 < p, q < \infty$. *Let* $\mathcal{M}$ *be a von Neumann algebra. Then*

$$h^p_{L^q(\mathcal{M})}(\varepsilon) \geqslant \alpha\varepsilon^r, \quad 0 < \varepsilon \leqslant 1,$$

where $r = \max(2, p, q)$ *and* $\alpha > 0$ *is a constant depending only on* p, q.

REMARKS. (i) In the case $q > 1$, Theorem 5.7 easily follows from Corollaries 5.2 and 5.4. Thus the non-trivial part of Theorem 5.7 lies in the case $q \leqslant 1$.

(ii) Theorem 5.7 implies, of course, that the same estimate holds for the modulus of PL-convexity.

(iii) In the case of $q = 1$, Theorem 5.7 is contained in [80]. In fact, it is this result which motivated the introduction of the uniform H-convexity.

Theorem 5.7, as stated above, was proved in [201]. The ingredient of the proof is the Riesz type factorization for Hardy spaces of analytic functions with values in non-commutative L^p-spaces. In Section 8 below we will discuss such a factorization in a more general context.

The following corollary completes Corollary 5.5. Thus the non-commutative L^p-spaces have the same type and cotype as the commutative L^p-spaces.

COROLLARY 5.8. $L^p(\mathcal{M})$ *is of cotype* 2 *for any* $0 < p \leqslant 1$ *and any von Neumann algebra* $\mathcal{M}$.

This corollary was proved in [193] for $p = 1$ and in [201] for $0 < p < 1$.

We recall that a quasi-Banach space X has the analytic Radon–Nikodym property (abbreviated as analytic RNP) if any bounded analytic function $F: D \to X$ has a.e. radial limits in X, where D denotes the unit disc (cf. [30,59], and also [32] for additional references). It is known that the uniform H-convexity implies the analytic RNP. Thus we get the

COROLLARY 5.9. $L^p(\mathcal{M})$ *has the analytic RNP for any* $0 < p \leqslant 1$ *and any von Neumann algebra* $\mathcal{M}$.

The results discussed in this section have all been extended to symmetric operator spaces. We refer to [70,195] for the cotype, uniform convexity, PL-convexity and smoothness in the unitary ideals, and in the general case, to [200] for the uniform convexity and smoothness, to [203,204] for the uniform H-convexity, RNP and analytic RNP (see also [129]). Finally, we mention that [39] contains related results, especially those on the local uniform convexity for symmetric operator spaces.

6. Non-commutative Khintchine inequalities

This section is devoted to the non-commutative Khintchine inequalities and the closely related Grothendieck-type factorization theorems. Although all results in this section hold for the general non-commutative L^p-spaces, we will restrict ourselves to the semifinite ones, i.e., those constructed from an *n.s.f.* trace. Letters A_p, $B_p, \ldots,$ will denote positive constants depending only on p.

Let $(\varepsilon_n)_{n\geqslant 1}$ be a Rademacher (or Bernoulli) sequence, i.e., a sequence of independent random variables on some probability space $(\Omega, \mathcal{F}, P)$ such that $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$ for all $n \geqslant 1$. We first recall the classical Khintchine inequalities. Let $0 < p < \infty$. Then for all finite sequences (a_n) of complex numbers

$$A_p^{-1}\Big\|\sum_{n\geqslant 1} a_n\varepsilon_n\Big\|_{L^p(\Omega,P)} \leqslant \Big\|\sum_{n\geqslant 1} a_n\varepsilon_n\Big\|_{L^2(\Omega,P)} \leqslant B_p\Big\|\sum_{n\geqslant 1} a_n\varepsilon_n\Big\|_{L^p(\Omega,P)}. \tag{6.1}$$

(Note that obviously $\|\sum_{n\geqslant 1} a_n\varepsilon_n\|_{L^2(\Omega,P)} = (\sum_{n\geqslant 1}|a_n|^2)^{1/2}$.) These inequalities remain valid (suitably modified) when the coefficients a_n's are vectors from a Banach space X. In that case they are due to Kahane, and are usually called "Khintchine–Kahane inequalities": for all finite sequences (a_n) in X

$$\begin{aligned} A_p^{-1}\Big\|\sum_{n\geqslant 1} a_n\varepsilon_n\Big\|_{L^p(\Omega,P;X)} &\leqslant \Big\|\sum_{n\geqslant 1} a_n\varepsilon_n\Big\|_{L^2(\Omega,P;X)} \\ &\leqslant B_p\Big\|\sum_{n\geqslant 1} a_n\varepsilon_n\Big\|_{L^p(\Omega,P;X)}. \end{aligned}\tag{6.2}$$

In particular, if X is a commutative L^p-space, say $X = L^p$ over $(0, 1)$, (6.2) implies that for all finite sequences (a_n) in $L^p(0, 1)$

$$\begin{aligned} A_p^{-1}\Big\|\Big(\sum_{n\geqslant 1}|a_n|^2\Big)^{1/2}\Big\|_{L^p} &\leqslant \Big\|\sum_{n\geqslant 1} a_n\varepsilon_n\Big\|_{L^2(\Omega,P;L^p)} \\ &\leqslant B_p\Big\|\Big(\sum_{n\geqslant 1}|a_n|^2\Big)^{1/2}\Big\|_{L^p}. \end{aligned}\tag{6.3}$$

It is (6.3) that we will extend to the non-commutative setting.

Now let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with an *n.s.f.* trace τ. Let $a = (a_n)$ be a finite sequence in $L^p(\mathcal{M})$ (recalling that by our convention, $L^p(\mathcal{M}) = L^p(\mathcal{M}, \tau)$). Define

$$\|a\|_{L^p(\mathcal{M};l^2_C)} = \Big\|\Big(\sum_{n\geqslant 0}|a_n|^2\Big)^{1/2}\Big\|_p, \qquad \|a\|_{L^p(\mathcal{M};l^2_R)} = \Big\|\Big(\sum_{n\geqslant 0}|a_n^*|^2\Big)^{1/2}\Big\|_p.$$

This gives two norms (or quasi-norms if $p < 1$) on the family of all finite sequences in $L^p(\mathcal{M})$. The corresponding completions (relative to the w^*-topology for $p = \infty$) are denoted by $L^p(\mathcal{M}; l^2_C)$ and $L^p(\mathcal{M}; l^2_R)$, respectively. The reader is referred to [156] for a discussion of these norms.

Now we can state the non-commutative Khintchine inequalities.

THEOREM 6.1. *Let $1 \leqslant p < \infty$, and let $\mathcal{M}$ be a semifinite von Neumann algebra. Let $a = (a_n)_{n\geqslant 0}$ be a finite sequence in $L^p(\mathcal{M})$.*

(i) *If $2 \leqslant p < \infty$, there is a constant B'_p (depending only on p) such that*

$$\|a\|_{L^p(\mathcal{M};l^2_C)\cap L^p(\mathcal{M};l^2_R)} \leqslant \Big\|\sum_{n\geqslant 0} \varepsilon_n a_n\Big\|_{L^p(\Omega,P;L^p(\mathcal{M}))} \leqslant B'_p \|a\|_{L^p(\mathcal{M};l^2_C)\cap L^p(\mathcal{M};l^2_R)}. \tag{6.4}$$

(ii) *If $1 \leqslant p < 2$, there is an absolute constant $A > 0$ (independent of p and a) such that*

$$A\|a\|_{L^p(\mathcal{M};l^2_C)+L^p(\mathcal{M};l^2_R)} \leqslant \Big\|\sum_{n\geqslant 0} \varepsilon_n a_n\Big\|_{L^p(\Omega,P;L^p(\mathcal{M}))} \leqslant \|a\|_{L^p(\mathcal{M};l^2_C)+L^p(\mathcal{M};l^2_R)}. \tag{6.5}$$

For the convenience of the reader we recall the norms in $L^p(\mathcal{M}; l^2_C) \cap L^p(\mathcal{M}; l^2_R)$ and $L^p(\mathcal{M}; l^2_C) + L^p(\mathcal{M}; l^2_R)$:

$$\|a\|_{L^p(\mathcal{M};l^2_C)\cap L^p(\mathcal{M};l^2_R)} = \max\big\{\|a\|_{L^p(\mathcal{M};l^2_C)}, \|a\|_{L^p(\mathcal{M};l^2_R)}\big\}$$

and

$$\|a\|_{L^p(\mathcal{M};l^2_C)+L^p(\mathcal{M};l^2_R)} = \inf\big\{\|b\|_{L^p(\mathcal{M};l^2_C)} + \|c\|_{L^p(\mathcal{M};l^2_R)}\big\},$$

where the infimum runs over all decompositions $a = b + c$ with $b \in L^p(\mathcal{M}; l^2_C)$ and $c \in L^p(\mathcal{M}; l^2_R)$.

This result was first proved in [117] for $1 < p < \infty$ in the case of the Schatten classes. The general statement as above (including $p = 1$) is contained in [121]. Modulo the classical fact that in all preceding inequalities the sequence (ε_n) can be replaced by a lacunary sequence, say, by $(z^{2^n})_{n\geqslant 1}$ on the unit circle $\mathbb{T}$, the main ingredient of the proof in [121] is a Riesz type factorization theorem (see Theorem 8.3 below).

REMARKS. (i) Like in the classical Khintchine inequalities (6.1), the constant B'_p in (6.4) is of order $\sqrt{p}$ (the best possible) as $p \to \infty$ (cf. [154, p. 106]).

(ii) We have already mentioned that in Theorem 6.1, the sequence (ε_n) can be replaced by a lacunary sequence. It is also classical that (ε_n) can be replaced by a sequence of independent standard Gaussian variables.

(iii) More generally, Theorem 6.1 holds when (ε_n) is replaced by certain sequences in a non-commutative L^p-space $L^p(\mathcal{N})$ and $\varepsilon_n a_n$ is replaced by $\varepsilon_n \otimes a_n$ in $L^p(\mathcal{N} \otimes \mathcal{M})$, for instance, this holds for the generators of a free group, for a free semi-circular system (in

Voiculescu's sense [196]) and for a sequence of CAR operators (as in Section 1). Note that in the free cases, (6.4) even holds for $p = \infty$! The reader is referred to [81,154] for more information, and also to [27,29] for some related results and for the best constants in these inequalities.

(iv) Theorem 6.1 also holds for non-commutative L^p-spaces associated with a general von Neumann algebra (cf. [100,101]). Note that [100,101] contains more inequalities related to (6.4) and (6.5).

COROLLARY 6.2. *Let (ε_{ij}) be an independent collection (indexed by $\mathbb{N} \times \mathbb{N}$) of mean zero ± 1-valued random variables on $(\Omega, \mathcal{F}, P)$. For any $2 \leqslant p < \infty$, there is a constant C_p such that for any finitely supported function $x : \mathbb{N}^2 \to \mathbb{C}$, we have*

$$|\!|\!| x |\!|\!|_p \leqslant \Big\| \sum_{i,j} \varepsilon_{ij} x(i,j) e_{ij} \Big\|_{L^p(\Omega, P; S^p)} \leqslant C_p |\!|\!| x |\!|\!|_p, \tag{6.6}$$

where

$$|\!|\!| x |\!|\!|_p = \max\left\{ \left(\sum_i \left(\sum_j |x(i,j)|^2 \right)^{p/2} \right)^{1/p}, \left(\sum_j \left(\sum_i |x(i,j)|^2 \right)^{p/2} \right)^{1/p} \right\}. \tag{6.7}$$

A fortiori this implies

$$\Big\| \sum \varepsilon_{ij} x(i,j) e_{ij} \Big\|_{L^p(\Omega, P; S^p)} \leqslant C_p \inf_{\varepsilon(i,j) = \pm 1} \Big\| \sum \varepsilon(i,j) x(i,j) e_{ij} \Big\|_{S^p}. \tag{6.8}$$

PROOF. Take $L^p(\mathcal{M}) = S^p$. Let $a_{ij} = x(i,j) e_{ij}$. Then $(\sum_{ij} a_{ij}^* a_{ij})^{1/2} = \sum_j \lambda_j e_{jj}$ and $(\sum a_{ij} a_{ij}^*)^{1/2} = \sum \mu_i e_{ii}$ where $\lambda_j = (\sum_i |x(i,j)|^2)^{1/2}$ and $\mu_i = (\sum_j |x(i,j)|^2)^{1/2}$. Thus (6.6) is a special case of (6.4). $\square$

REMARK 6.3. The preceding result remains valid with the same proof when $1 \leqslant p < 2$ provided one changes the definition of $|\!|\!| x |\!|\!|_p$ to the following one (dual to the other):

$$|\!|\!| x |\!|\!|_p = \inf\left\{ \|y\|_{\ell_p(\ell_2)} + \left\| {}^t z \right\|_{\ell_p(\ell_2)} \right\},$$

where the infimum runs over all possible decompositions of the form $x = y + z$.

REMARK. [101] contains more inequalities of type (6.6). Here we just mention one of them, which is an extension of (6.6). Let (f_{ij}) be an independent collection of mean zero random variables in $L^p(\Omega, \mathcal{F}, P)$ $(2 \leqslant p < \infty)$. Then

$$\Big\| \sum f_{ij} e_{ij} \Big\|_{L^p(\Omega, P; S^p)} \approx \max\left\{ \left(\sum_{i,j} \|f_{ij}\|_p^p \right)^{1/p}, \left(\sum_i \left(\sum_j \|f_{ij}\|_2^2 \right)^{p/2} \right)^{1/p}, \right.$$
$$\left. \left(\sum_j \left(\sum_i \|f_{ij}\|_2^2 \right)^{p/2} \right)^{1/p} \right\},$$

where the equivalence constants depend only on p.

In the case of $0 < p < 1$, it is easy to check that the second inequality of (6.5) still holds. However, this is not clear for the first one.

PROBLEM 6.4. Does the first inequality of (6.5) hold for $0 < p < 1$ (with some constant depending on p)? Does Remark 6.3 extend to $p < 1$?

Like in the commutative setting, the non-commutative Khintchine inequalities are closely related to non-commutative Grothendieck type factorization theorems. Indeed, it was shown in [121] that (6.5) in the case of $p = 1$ is equivalent to the non-commutative little Grothendieck theorem. To go further, we need one more definition.

DEFINITION. Let $1 \leqslant p \leqslant \infty$, $0 < q \leqslant r < \infty$. Let Y be a Banach space, and let $u : L^p(\mathcal{M}) \to Y$ be an operator ($\mathcal{M}$ being a semifinite von Neumann algebra). u is said to be (r, q)-concave if there is a constant C such that for all finite sequences (a_n) in $L^p(\mathcal{M})$

$$\Big(\sum \|ua_n\|^r\Big)^{1/r} \leqslant C\Big\|\Big(\sum |a_n|_s^q\Big)^{1/q}\Big\|_p,$$

where $|a|_s = ((a^*a + aa^*)/2)^{1/2}$ denotes the symmetric modulus of an operator a. If $q = r$, u is simply said to be q-concave.

In the case of $p = \infty$ (then $\mathcal{M}$ can be any C^*-algebra), the above notion reduces to that of (r, q)-C^*-summing operators introduced in [144] and [147]. The following is an easy consequence of the Hahn–Banach theorem (cf. [144] for a proof).

PROPOSITION 6.5. *Let $\mathcal{M}$ be a semifinite von Neumann algebra. Let $1 \leqslant q < p \leqslant \infty$ and $s = p/q$. Then for any operator $u : L^p(\mathcal{M}) \to Y$ the following assertions are equivalent*

(i) *u is q-concave;*

(ii) *there are a constant C and $f \in (L^s(\mathcal{M}))^*$, $f \geqslant 0$, such that*

$$\|ua\| \leqslant C\big(f\big(|a|_s^q\big)\big)^{1/q}, \quad \forall a \in L^p(\mathcal{M}).$$

The following Grothendieck-type factorization theorem (when Y is a Hilbert space) is equivalent to (6.5) with p' in place of p.

THEOREM 6.6. *Let $2 < p \leqslant \infty$, and let Y be a Banach space of cotype 2. Then any operator $u : L^p(\mathcal{M}) \to Y$ is 2-concave, equivalently (via Proposition 6.5), there are a constant C and $f \in (L^{p/2}(\mathcal{M}))^*$ with $f \geqslant 0$ such that*

$$\|ua\| \leqslant C\|u\|\big(f\big(|a|_s^2\big)\big)^{1/2}, \quad \forall a \in L^p(\mathcal{M}).$$

Moreover, C can be chosen to depend only on the cotype 2 constant of Y.

REMARK. The basic case $p = \infty$ (=non-commutative Grothendieck theorem), is proved in [147] (see also [144,146]). In this case, $\mathcal{M}$ can actually be any C^*-algebra. In the case of

$p < \infty$, Theorem 6.6 is essentially the main result in [118]. More generally, [118] proves this for operators $u : E(\mathcal{M}, \tau) \to H$, where H is a Hilbert space and E is a 2-convex *r.i.* space with an additional mild condition. This, together with [121], implies that Theorem 6.1 can be extended to some symmetric operator spaces. The main difficulty in [118] is to obtain Theorem 6.6 with a constant C independent of p, or equivalently which remains bounded when $p \to \infty$. If we ignore this important point, it is very easy to deduce Theorem 6.6 from (6.4), as follows.

PROOF OF THEOREM 6.6 FOR $p < \infty$ WITH $C = C_p$. Since $L^p(\mathcal{M})$ is of type 2, by Kwapień's theorem (cf. [146], Theorem 3.2), u factors through a Hilbert space. Thus we may assume Y itself is a Hilbert space. Let (a_n) be a finite sequence in $L^p(\mathcal{M})$. Then

$$\begin{aligned}\Big(\sum \|u(a_n)\|^2\Big)^{1/2} &= \Big\|\sum u(a_n)\varepsilon_n\Big\|_{L^2(\Omega, P; Y)} \\ &\leqslant \|u\| \Big\|\sum a_n \varepsilon_n\Big\|_{L^2(\Omega, P; L^p(\mathcal{M}))} \\ &\leqslant C_p \Big\|\Big(\sum |a_n|_s^2\Big)^{1/2}\Big\|_p \quad \text{(by (6.4)).}\end{aligned}$$

Therefore, u is 2-concave. □

Unfortunately, the preceding proof does not work for $p = \infty$. The main difficulty in this case is to show that an operator u from $\mathcal{M}$ into a space of cotype 2 factors through a Hilbert space. This was done in [147]. The proof given there relies on another result of independent interest, that we state as follows.

THEOREM 6.7. *Let $1 < q < \infty$. Let $u : A \to Y$ be an operator from a C^*-algebra A into a Banach space Y. Then the following assertions are equivalent*

(i) *u is $(q, 1)$-C^*-summing*;

(ii) *there are a constant C and a state f on A such that*

$$\|ua\| \leqslant C\|u\|\big(f(|a|_s)\big)^{1/q}\|a\|^{1-1/q}, \quad \forall a \in A;$$

(iii) *for any $1 \leqslant r < q$ there are a constant C and a state f on A such that*

$$\|ua\| \leqslant C\|u\|\big(f(|a|_s^r)\big)^{1/q}\|a\|^{1-r/q}, \quad \forall a \in A;$$

(iv) *u is (q, r)-C^*-summing for any $1 \leqslant r < q$.*

Thus Theorem 6.7 gives a characterization of (q, r)-C^*-summing operators defined on a C^*-algebra. (ii) and (iii) above can be reformulated as a Pietsch-type factorization of u through a non-commutative Lorentz space $L^{q,1}$, constructed from the state f via the real interpolation in the spirit of Kosaki's construction presented in Section 3. The resulting spaces, denoted by $L^{q,1}(f)$, possess properties similar to the usual Lorentz spaces. The reader is referred to [147] for more information.

REMARK. There does not seem to be a known characterization similar to that in Theorem 6.7 for (q, r)-concave operators defined on $L^p(\mathcal{M})$ $(p < \infty)$.

Let us close this section by an application of Theorem 6.7.

THEOREM 6.8. *Let $\mathcal{M}$ be any von Neumann algebra and $X \subset \mathcal{M}_*$ a reflexive subspace. Then there are a normal state f of $\mathcal{M}$ and $p > 1$ such that X embeds isomorphically into $L^{p,p}(f)$, where $L^{p,p}(f)$ is the non-commutative L^p-space referred to above.*

This theorem, proved in [147], is a non-commutative version of a classical theorem due to Rosenthal in the commutative setting. Its proof uses Theorem 6.7 and a previous result in [92] that any reflexive subspace of $\mathcal{M}_*$ is superreflexive. Note that the real interpolation space $L^{p,p}(f)$ can be replaced by the corresponding complex interpolation space.

REMARK. Let A be a C^*-algebra, and let $T: A \to \ell_2$ be absolutely summing (in the usual sense). If A is commutative, it is well known that T factors as $T = T_1 T_2$, where $T_2: A \to \ell_2$ is bounded and $T_1 \in S^2$. In [161] it is shown that for a general C^*-algebra A, one can get a factorization $T = T_1 T_2$, where $T_2: A \to \ell_2$ is bounded and $T_1: \ell_2 \to \ell_2$ belongs to the Schatten class S_4 (the exponent 4 is optimal). A fortiori T is compact. In particular, there is no embedding of ℓ_2 into a non-commutative L_1-space with absolutely summing adjoint. See [146, p. 68] for background on embeddings of this kind.

7. Non-commutative martingale inequalities

This section deals with non-commutative martingale inequalities. The reader is referred to [71] for the classical (= commutative) martingale inequalities. In what follows, $\mathcal{M}$ will be a von Neumann algebra equipped with a normal faithful finite normalized trace τ. We begin with some necessary definitions. Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra. The non-commutative L^p-space associated with $(\mathcal{N}, \tau|_{\mathcal{N}})$ is naturally identified with a subspace of $L^p(\mathcal{M})$. There is a unique normal faithful conditional expectation $\mathcal{E}: \mathcal{M} \to \mathcal{N}$ preserving the trace τ, i.e., $\tau(\mathcal{E}(x)) = \tau(x)$ for all $x \in \mathcal{M}$. For any $1 \leqslant p \leqslant \infty$, $\mathcal{E}$ is extended to a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{N})$, still denoted by $\mathcal{E}$.

Now let $(\mathcal{M}_n)_{n \geqslant 0}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of all the $\mathcal{M}_n$'s is w^*-dense in $\mathcal{M}$. Let $\mathcal{E}_n$ be the conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_n$. Then as usual, we define a non-commutative martingale (with respect to $(\mathcal{M}_n)_{n \geqslant 0}$) as a sequence $x = (x_n)_{n \geqslant 0}$ in $L^1(\mathcal{M})$ such that

$$\mathcal{E}_n(x_{n+1}) = x_n, \quad \forall n \geqslant 0.$$

If additionally all x_n's are in $L^p(\mathcal{M})$, x is called an L^p-martingale. Then we set

$$\|x\|_p = \sup_{n \geqslant 0} \|x_n\|_p.$$

If $\|x\|_p < \infty$, x is called a bounded L^p-martingale. The difference sequence of x is defined as $dx = (dx_n)_{n \geqslant 0}$ with $dx_0 = x_0$ and $dx_n = x_n - x_{n-1}$ for all $n \geqslant 1$.

REMARK. Let $x_\infty \in L^p(\mathcal{M})$. Set $x_n = \mathcal{E}_n(x_\infty)$ for all $n \geqslant 0$. Then $x = (x_n)$ is a bounded L^p-martingale and $\|x\|_p = \|x_\infty\|_p$; moreover, x_n converges to x_∞ in $L^p(\mathcal{M})$ (relative to the w^*-topology in the case $p = \infty$). Conversely, if $1 < p < \infty$, every bounded L^p-martingale converges in $L^p(\mathcal{M})$, and so is given by some $x_\infty \in L^p(\mathcal{M})$ as previously. Thus one can identify the space of all bounded L^p-martingales with $L^p(\mathcal{M})$ itself in the case $1 < p < \infty$.

The main result of [156] can be stated as follows. Recall that $A_p, B_p, \ldots$, denote constants depending only on p.

THEOREM 7.1. *Let $\mathcal{M}$ and $(\mathcal{M}_n)_{n\geqslant 0}$ be as above. Let $1 < p < \infty$, and let $x = (x_n)_{n\geqslant 0}$ be a finite L^p-martingale with respect to $(\mathcal{M}_n)_{n\geqslant 0}$. Then*

$$A_p^{-1} S_p(x) \leqslant \|x\|_p \leqslant B_p S_p(x), \tag{7.1}$$

where for $2 \leqslant p < \infty$,

$$S_p(x) = \|dx\|_{L^p(\mathcal{M};l^2_C)\cap L^p(\mathcal{M};l^2_R)},$$

and for $1 < p < 2$,

$$S_p(x) = \inf\bigl\{\|dy\|_{L^p(\mathcal{M};l^2_C)} + \|dz\|_{L^p(\mathcal{M};l^2_R)}\bigr\},$$

the infimum being taken over all decompositions $x = y + z$ with L^p-martingales y and z.

This is the non-commutative Burkholder–Gundy inequalities. Note that in the commutative case, $S_p(x)$ is the L^p-norm of the usual square function of x (so that the above difference between the cases $2 \leqslant p < \infty$ and $1 < p < 2$ disappears). The proof of Theorem 7.1 in [156] is rather tortuous, due to the fact that the usual techniques from classical martingale theory, such as maximal functions, stopping times, etc., are no longer available in the non-commutative setting. See [156] and [19] for applications to non-commutative stochastic integrals. For Clifford martingales, some particular cases of Theorem 7.1 also appear in [34].

REMARK 7.2. The second inequality in (7.1) holds for $p = 1$ too. This follows from the duality between $\mathcal{H}^1$ and $\mathcal{BMO}$, proved in [156].

Like in the commutative case, Theorem 7.1 implies the unconditionality of non-commutative martingale differences. Let us record this explicitly as follows.

COROLLARY 7.3. *With the same assumptions as in Theorem 7.1, we have*

$$\Bigl\|\sum_{n\geqslant 0} \varepsilon_n dx_n\Bigr\|_p \leqslant C_p \Bigl\|\sum_{n\geqslant 0} dx_n\Bigr\|_p, \quad \forall \varepsilon_n = \pm 1. \tag{7.2}$$

Some rather particular cases of (7.2) also appear in [65,66]. Note that in the case of $2 \leqslant p < \infty$, (7.2) is equivalent to (7.1), modulo the non-commutative Khintchine inequalities. However, in the case of $1 < p < 2$, to prove that (7.2) implies (7.1), one needs a non-commutative version of a classical inequality due to Stein. We refer to [156] for more details.

REMARK. If p is an even integer, the second inequality of (7.1) was extended in [155] to sequences more general than martingale difference sequences (the so-called p-orthogonal sequences); moreover, for these values of p, the method of [155] yields that the order of the constant B_p in (7.1) is $\mathrm{O}(p)$ (for even integers p), which is optimal as $p \to \infty$.

For the convenience of the reader, we recall the optimal order of the constants A_p and B_p in the commutative case (cf., e.g., [31]): B_p is bounded as $p \to 1$ and $\mathrm{O}(p)$ as $p \to \infty$; A_p is $\mathrm{O}((p-1)^{-1})$ as $p \to 1$ and $\mathrm{O}(p^{1/2})$ as $p \to \infty$. The constants A_p and B_p in (7.1) obtained in [156] are not satisfactory at all (they are of exponential type as $p \to \infty$). Thus finding the optimal order of A_p and B_p in Theorem 7.1 seemed a very interesting question. Very recently, major progress on this was achieved by Randrianantoanina [165], as follows.

THEOREM 7.4 ([165]). *There is a constant C such that for any finite non-commutative martingale x in $L^1(\mathcal{M})$ and any sequence (ε_n) of signs*

$$\Big\| \sum_{n\geqslant 0} \varepsilon_n dx_n \Big\|_{1,\infty} \leqslant C \Big\| \sum_{n\geqslant 0} dx_n \Big\|_1 .$$

By interpolation, this implies the optimal order of the constant C_p in (7.2), namely, $C_p = \mathrm{O}(p)$ as $p \to \infty$. This, in turn, combined with Theorem 6.1, yields better estimates for A_p, B_p in (7.1), namely A_p is $\mathrm{O}((p-1)^{-2})$ when $p \to 1$ and both A_p and B_p are $\mathrm{O}(p)$ when $p \to \infty$ (which for B_p is optimal). It was also shown in [102] that $\mathrm{O}(p)$ is the optimal order of A_p as $p \to \infty$. Note that this order is the square of what it is in the commutative case. On the other hand, it was proved in [100] that B_p remains bounded as $p \to 1$.

We will now discuss two other inequalities: the Burkholder and Doob inequalities.

THEOREM 7.5. *With the same assumptions as in Theorem 7.1, we have*

$$A_p^{-1} s_p(x) \leqslant \|x\|_p \leqslant B_p s_p(x), \tag{7.3}$$

where for $2 \leqslant p < \infty$,

$$s_p(x) = \max\Big\{ \Big(\sum_{n\geqslant 0} \|dx_n\|_p^p \Big)^{1/p}, \Big\| \Big(\sum_{n\geqslant 0} \mathcal{E}_{n-1}(|dx_n|^2) \Big)^{1/2} \Big\|_p, \Big\| \Big(\sum_{n\geqslant 0} \mathcal{E}_{n-1}(|dx_n^*|^2) \Big)^{1/2} \Big\|_p \Big\}$$

and for $1 < p < 2$,

$$s_p(x) = \inf\left\{\left(\sum_{n\geqslant 0} \|dw_n\|_p^p\right)^{1/p} + \left\|\left(\sum_{n\geqslant 0} \mathcal{E}_{n-1}\left(|du_n|^2\right)\right)^{1/2}\right\|_p + \left\|\left(\sum_{n\geqslant 0} \mathcal{E}_{n-1}\left(|dv_n^*|^2\right)\right)^{1/2}\right\|_p\right\},$$

where the infimum runs over all decompositions $x = w + u + v$ *with* L^p*-martingales* w, u *and* v.

This theorem comes from [100]. It is the non-commutative analogue of the classical Burkholder inequality. Note that in the commutative case $(\sum \mathcal{E}_{n-1}(|\mathrm{d}x_n|^2))^{1/2}$ is the conditioned square function of x. Like in the commutative case, Theorem 7.5 implies a non-commutative analogue of Rosenthal's inequality concerning independent mean zero random variables; see [100,101] for more details and some applications.

THEOREM 7.6 ([97]). *Let* $\mathcal{M}$ *and* $(\mathcal{M}_n)$ *be as in Theorem* 7.1. *Let* $1 \leqslant p < \infty$. *Let* (a_n) *be a finite sequence of positive elements in* $L^p(\mathcal{M})$. *Then*

$$\left\|\sum_{n\geqslant 0} \mathcal{E}_n(a_n)\right\|_p \leqslant C_p \left\|\sum_{n\geqslant 0} a_n\right\|_p. \tag{7.4}$$

Note that in the commutative case, (7.4) is the dual reformulation of Doob's classical maximal inequality. Although it is clearly impossible to define the maximal function of a non-commutative martingale as in the commutative setting, Junge found in [97] a substitute, consistent with [154], which enables him to formulate a non-commutative analogue of Doob's inequality itself, which is dual to (7.4). Note that the latter result immediately implies the almost everywhere convergence of bounded non-commutative martingales in $L^p(\mathcal{M})$ for all $p > 1$. Results of this kind on the almost everywhere convergence of non-commutative martingales go back to Cuculescu [43]. The reader is referred to [44] and [90,91] for more information.

REMARKS. (i) Like the constants in (7.1) the constants in (7.3) and (7.4) obtained in [100, 97] are not satisfactory at all. In fact, they depend on those in (7.1) since the proofs of (7.3) and (7.4) in [97] and [100] use (7.1). The more recent results of [165] imply better estimates for these constants.

(ii) It was proved in [102] that the optimal order of the constant C_p in (7.4) is $\mathrm{O}(p^2)$ as $p \to \infty$. This is in strong contrast with the commutative case for, in the commutative case, the optimal order of the corresponding constant is $\mathrm{O}(p)$ as $p \to \infty$. The same phenomenon occurs for the optimal order of the best constant in the non-commutative Stein inequality proved in [156], namely, this optimal order is $\mathrm{O}(p)$ as $p \to \infty$; again it is the square of what it is in the commutative case. We refer to [102] for more information.

(iii) All the preceding results hold in the non-tracial case as well (cf. [100,101,97]).

In the rest of this section, we briefly discuss the UMD property and the analytic UMD property of non-commutative L^p-spaces, a subject closely related to inequality (7.2). Applying Corollary 7.3 to *commutative martingales* with values in $L^p(\mathcal{M})$, $1 < p < \infty$, we get the unconditionality of *commutative martingale* differences with values in $L^p(\mathcal{M})$, that is, $L^p(\mathcal{M})$ is a UMD space in Burkholder's sense (cf. [32] for information on UMD spaces). This is a well-known fact, proved in [21] and [16]. In fact, these authors proved that the Hilbert transform extends to a bounded map on $L^p(\mathbb{T}; L^p(\mathcal{M}))$ for any $1 < p < \infty$; but this property (called "HT" in short) is equivalent to UMD. We also refer to the next section for discussions on Hilbert type transforms. Together with Theorem 3.6 we obtain the

COROLLARY 7.7. *$L^p(\mathcal{M})$ is a UMD space for any $1 < p < \infty$ and any von Neumann algebra $\mathcal{M}$.*

We mention an open problem circulated in the non-commutative world for almost two decades on the UMD property for symmetric operator spaces.

PROBLEM 7.8. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with an *n.s.f.* trace τ, and let E be a UMD *r.i.* space on $(0, \infty)$. Is $E(\mathcal{M}, \tau)$ a UMD space?

We now turn to the analytic UMD property. Let $\mathbb{T}^{\mathbb{N}}$ be the infinite torus equipped with the product measure, denoted by dm_∞. Let Ω_n be the σ-field generated by the coordinates $(z_0, \dots, z_n)$, $n \geqslant 0$. Let X be a quasi-Banach space. By a Hardy martingale in $L^p(\mathbb{T}^{\mathbb{N}}; X)$ $(0 < p \leqslant \infty)$, we mean any sequence $f = (f_n)$ satisfying the following: for any $n \geqslant 0$, $f_n \in L^p(\mathbb{T}^{\mathbb{N}}, \Omega_n; X)$ and f_n is analytic in the last variable z_n, i.e., f_n admits an expansion as follows

$$f_n(z_0, \dots, z_{n-1}, z_n) = \sum_{k\geqslant 1} \varphi_{n,k}(z_0, \dots, z_{n-1}) z_n^k,$$

where $\varphi_{n,k} \in L^p(\mathbb{T}^{\mathbb{N}}, \Omega_{n-1}; X)$ for $n \geqslant 0, k \geqslant 0$. If in addition, $\varphi_{n,k} = 0$ for all $k \geqslant 2$, f is called an analytic martingale. Note that if X is a Banach space and $1 \leqslant p \leqslant \infty$, any Hardy martingale in $L^p(\mathbb{T}^{\mathbb{N}}; X)$ is a martingale in the usual sense.

DEFINITION. X is called an analytic UMD space if for some $0 < p < \infty$ (or equivalently for all $0 < p < \infty$) there is a constant C such that all finite Hardy martingales f in $L^p(\mathbb{T}^{\mathbb{N}}; X)$ satisfy

$$\left\| \sum_{n\geqslant 0} \varepsilon_n df_n \right\|_p \leqslant C \left\| \sum_{n\geqslant 0} df_n \right\|_p, \quad \forall \varepsilon_n = \pm 1.$$

This notion was introduced in [69]. The apparent weakening obtained by requiring the above inequality be verified only for analytic martingales, is actually an equivalent definition of analytic UMD spaces (cf. [69]). Typical examples of Banach spaces which are analytic UMD but not UMD are commutative L^1-spaces. In fact, all commutative

L^p-spaces, $0 < p \leqslant 1$, are analytic UMD spaces. We refer to [69] for more information (see also [32]).

However, this no longer holds in the non-commutative setting:

PROPOSITION 7.9. *Let $\mathcal{M}$ be a von Neumann algebra and $0 < p \leqslant 1$. Then $L^p(\mathcal{M})$ is an analytic UMD space iff $\mathcal{M}$ is isomorphic, as Banach space, to a commutative L^∞-space.*

It was proved in [80] that the trace class S^1 is not an analytic UMD space. The ingredient of the proof there is the unboundedness of the triangular projection on S^1 (cf. [111]). (This projection is in fact a non-commutative Riesz projection in the context of the next section.) The same idea also shows that S^p is not an analytic UMD space for $0 < p < 1$. Noting that the analytic UMD property is "local", we then deduce the general case from Theorem 3.5.

8. Non-commutative Hardy spaces

A classical theorem of Szegö says that if w is a positive function on the unit circle $\mathbb{T}$ such that $\log w \in L^1(\mathbb{T})$, there is an outer function φ such that $|\varphi| = w$ a.e. on $\mathbb{T}$. A lot of effort has been made to extend this theorem to operator valued functions, not only for its intrinsic interest, but also because it is the gateway to many useful applications (cf., e.g., [85,86, 46,179,198]). This problem makes sense in the broader context of subdiagonal algebras, introduced by Arveson in the 60's in order to unify several frequently used non-selfadjoint algebras such as triangular matrices and bounded analytic operator valued functions. In this section we will present the extension to this general context of some classical results for analytic functions in the unit disc, including Szegö's theorem, boundedness of the Hilbert transform and the Riesz factorization theorem.

Throughout this section, unless explicitly indicated otherwise, $\mathcal{M}$ will denote a finite von Neumann algebra equipped with a normal faithful finite normalized trace τ. Let $\mathcal{D}$ be a von Neumann subalgebra of $\mathcal{M}$. Let $\mathcal{E}$ be the (unique) normal faithful conditional expectation of $\mathcal{M}$ with respect to $\mathcal{D}$ which leaves τ invariant.

DEFINITION. A w^*-closed subalgebra $H^\infty(\mathcal{M})$ of $\mathcal{M}$ is called a finite subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$ (or to $\mathcal{D}$) if

(i) $\{x + y^*\colon x, y \in H^\infty(\mathcal{M})\}$ is w^*-dense in $\mathcal{M}$;

(ii) $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y)$, $\forall x, y \in H^\infty(\mathcal{M})$;

(iii) $\{x\colon x, x^* \in H^\infty(\mathcal{M})\} = \mathcal{D}$.

$\mathcal{D}$ is then called the diagonal of $H^\infty(\mathcal{M})$.

This notion can be generalized further (see [12]). However, the theory we will give below is, on one hand, satisfactory only for finite subdiagonal algebras as above, and on the other, interesting enough to cover many important cases.

REMARKS. (i) If $H^\infty(\mathcal{M})$ is a finite subdiagonal algebra of $\mathcal{M}$, it is automatically maximal in the sense that it is contained in no proper subdiagonal algebra with respect to $\mathcal{E}$ other than itself (see [61]).

(ii) Consequently, $H^\infty(\mathcal{M})$ admits the following useful characterization (cf. [12])

$$H^\infty(\mathcal{M}) = \{x \in \mathcal{M}\colon\ \tau(xy) = 0,\ \forall y \in H_0^\infty(\mathcal{M})\},$$

where

$$H_0^\infty(\mathcal{M}) = \{x \in H^\infty(\mathcal{M})\colon\ \mathcal{E}(x) = 0\}.$$

Here are some examples (see [12] for more).

(i) *Triangular matrices.* Let M_n be the full algebra of all complex $n \times n$ matrices equipped with the normalized trace. Let $\mathcal{T}_n$ be the algebra of all upper triangular matrices in M_n. Then $\mathcal{T}_n$ is a finite subdiagonal algebra of M_n. In this case, the theory we will give below is partly contained in [73].

(ii) *Nest algebras.* Let $\mathcal{P}$ be a totally ordered family of projections in $\mathcal{M}$ containing 0 and 1. Let

$$\mathcal{N}(\mathcal{P}) = \{x \in \mathcal{M}\colon\ xe = exe,\ \forall e \in \mathcal{P}\}.$$

Then $\mathcal{N}(\mathcal{P})$ is a finite subdiagonal algebra of $\mathcal{M}$. The above example on triangular matrices is a special case of nest algebras.

(iii) *Analytic operator valued functions.* Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. Let $(L^\infty(\mathbb{T}), dm) \otimes (\mathcal{M}, \tau)$ be the von Neumann algebra tensor product (recalling that $\mathbb{T}$ is the unit circle equipped with normalized Lebesgue measure dm). Let $H^\infty(\mathbb{T}, \mathcal{M})$ be the subalgebra of $(L^\infty(\mathbb{T}), dm) \otimes (\mathcal{M}, \tau)$ consisting of all functions f such that

$$\int \tau\big(xf(z)\big)\bar{z}^n\, dm(z) = 0, \quad \forall x \in L^1(\mathcal{M}),\ \forall n \in \mathbb{Z},\ n < 0.$$

Then $H^\infty(\mathbb{T}, \mathcal{M})$ is a finite subdiagonal algebra of $(L^\infty(\mathbb{T}), dm) \otimes (\mathcal{M}, \tau)$. This is the algebra of "analytic" functions with values in $\mathcal{M}$. More precisely, each element f in $H^\infty(\mathbb{T}, \mathcal{M})$ can be extended, using Poisson integrals, to an $\mathcal{M}$-valued function, analytic and bounded in the unit disc admitting f as its (radial or non-tangential) weak-$*$ boundary values. The particularly interesting case $H^\infty(\mathbb{T}, M_n)$ or $H^\infty(\mathbb{T}, B(l_2))$ was extensively studied (cf., e.g., [85,46]). Note that $B(l_2)$ does not fit into our setting; however, for almost all problems we are concerned with, it can be recovered from M_n by approximation.

In the remainder of this section, unless specified otherwise, $H^\infty(\mathcal{M})$ will denote a finite subdiagonal algebra of $\mathcal{M}$ with diagonal $\mathcal{D}$. For $0 < p < \infty$ the corresponding Hardy space $H^p(\mathcal{M})$ is defined as the closure of $H^\infty(\mathcal{M})$ in $L^p(\mathcal{M})$. Many results on the classical Hardy spaces in the unit disc have been extended to the present setting. We refer, for instance, to [13,93,127,123,124,126,159,173,175] and [177] for more information and references. We now give some of these extensions. The first one is the Szegö type theorem.

THEOREM 8.1. *Suppose $w \in \mathcal{M}$ and $w^{-1} \in L^2(\mathcal{M})$. Then there are a unitary $u \in \mathcal{M}$ and $\varphi \in H^\infty(\mathcal{M})$ with $\varphi^{-1} \in H^2(\mathcal{M})$ such that $w = u\varphi$.*

This theorem, proved by Saito [175], improves a previous factorization theorem due to Arveson [12], in which both w and w^{-1} are supposed to belong to $\mathcal{M}$. Saito's proof essentially follows the same fashion set out by Arveson, although some extra technical difficulties appear.

REMARKS. (i) The above theorem can be still improved as follows: let $0 < p, q \leqslant \infty$ and $w \in L^p(\mathcal{M})$ with $w^{-1} \in L^q(\mathcal{M})$. Then there are a unitary $u \in \mathcal{M}$ and $\varphi \in H^p(\mathcal{M})$ with $\varphi^{-1} \in H^q(\mathcal{M})$ such that $w = u\varphi$.

(ii) In the classical case of analytic functions in the unit disc, for a positive function w on $\mathbb{T}$, the condition $\log w \in L^1(\mathbb{T})$ is necessary and sufficient for the existence of a factorization $w = u\varphi$, with $u \in L^\infty(\mathbb{T})$ unimodular and φ an outer function. It is an open problem to extend this to the non-commutative setting. Some partial results can be found in [46,86] and [198].

The following is an immediate consequence of Theorem 8.1 (and also of the remark (i) above).

COROLLARY 8.2. *Let $w \in L^1(\mathcal{M})$ such that $w \geqslant 0$ and $w^{-1} \in L^p(\mathcal{M})$ for some $0 < p \leqslant \infty$. Then there is $\varphi \in H^2(\mathcal{M})$ such that $w = \varphi^*\varphi$.*

By a rather standard argument, one can deduce from Theorem 8.1 the following Riesz factorization theorem, which was proved in [124] (see also [177] for the case where $p = q = 2$).

THEOREM 8.3. *Let $1 \leqslant p, q, r \leqslant \infty$ with $1/r = 1/p + 1/q$. Then any $x \in H^r(\mathcal{M})$ can be factored as $x = yz$ with $y \in H^p(\mathcal{M})$ and $z \in H^q(\mathcal{M})$; moreover,*

$$\|x\|_r = \inf\bigl\{\|y\|_p\|z\|_q \colon x = yz,\ y \in H^p(\mathcal{M}),\ z \in H^q(\mathcal{M})\bigr\}.$$

REMARKS. (i) It seems unclear whether the infimum above is attained.

(ii) With the notations in Theorem 8.3, one has the following more precise statement: for any $\varepsilon > 0$ there are $y \in H^p(\mathcal{M})$ and $z \in H^q(\mathcal{M})$ such that $x = yz$ and

$$\mu_t(y) \leqslant \bigl(\mu_t(x) + \varepsilon\bigr)^{r/p}, \qquad \mu_t(z) \leqslant \bigl(\mu_t(x) + \varepsilon\bigr)^{r/q}, \quad \forall t > 0.$$

In particular, if $x \in H^\infty(\mathcal{M})$, then $y, z \in H^\infty(\mathcal{M})$ and

$$\|y\|_p\|z\|_q = \|x\|_r + \mathrm{o}(1) \quad \text{as } \varepsilon \to 0.$$

This allows to partially extend Theorem 8.3 to the case of indices less than 1 (at least, for elements $x \in H^\infty(\mathcal{M}) \subset H^r(\mathcal{M})$). However, it is unknown whether Theorem 8.3, in its full generality, still holds for indices less than 1.

The reader can find applications of Theorem 8.3 to Hankel operators in [179,123], to invariant subspaces of $L^p(\mathcal{M})$ in [175], and to the uniform H-convexity in [201] and [203].

We will now describe the Hilbert transform and Riesz projection. Let $x \in \{a+b^*\colon a, b \in H^\infty(\mathcal{M})\}$. It is easy to see that x admits a unique decomposition

$$x = a + d + b^*, \quad \text{with } a, b \in H_0^\infty(\mathcal{M}),\ d \in \mathcal{D}.$$

Then we define the Hilbert transform H by

$$Hx = -\mathrm{i}\big(a - b^*\big).$$

Clearly, $x + \mathrm{i}Hx \in H^\infty(\mathcal{M})$; moreover if x is self-adjoint, Hx is the unique self-adjoint element in $\{a + b^*\colon a, b \in H^\infty(\mathcal{M})\}$ such that $x + \mathrm{i}Hx \in H^\infty(\mathcal{M})$ and $\mathcal{E}(Hx) = 0$. Note that

$$L^2(\mathcal{M}) = H_0^2(\mathcal{M}) \oplus L^2(\mathcal{D}) \oplus \big(H^2(\mathcal{M})\big)^\perp,$$

where $H_0^2(\mathcal{M}) = \{x \in H^2(\mathcal{M})\colon \mathcal{E}(x) = 0\}$. One easily checks that $H_0^2(\mathcal{M})$ (resp. $(H^2(\mathcal{M}))^\perp$) is the closure of $H_0^\infty(\mathcal{M})$ (resp. $\{x^*\colon x \in H_0^\infty(\mathcal{M})\}$) in $L^2(\mathcal{M})$. This decomposition of $L^2(\mathcal{M})$ shows that H extends to a contraction on $L^2(\mathcal{M})$, still denoted by H.

Now let P be the orthogonal projection of $L^2(\mathcal{M})$ onto $H^2(\mathcal{M})$ (i.e., P is the "Riesz projection"). Like in the classical case, H and P are linked together as follows

$$P = \frac{1}{2}(\mathrm{id}_{L^2(\mathcal{M})} + H) + \frac{1}{2}\mathcal{E}.$$

Thus, as far as boundedness is concerned, it suffices to consider one of them.

THEOREM 8.4. (i) *H extends to a bounded map on $L^p(\mathcal{M})$ for any $1 < p < \infty$; more precisely, one has*

$$\|Hx\|_p \leqslant C_p\|x\|_p, \quad \forall x = a + b^*,\ a, b \in H^\infty(\mathcal{M}),$$

where $C_p \leqslant Cp^2/(p-1)$ with C a universal constant.

(ii) *H also extends to a bounded map from $L^1(\mathcal{M})$ into $L^{1,\infty}(\mathcal{M})$ (the non-commutative weak L^1-space).*

This result was proved in [160]. Of course, (i) above (for the case $1 < p < 2$) follows by interpolation from (ii) and the L^2-boundedness of H (and by duality for the case $2 < p < \infty$). However, (i) admits a much simpler separate proof (see the discussion below).

In the case of triangular matrices, (i) above is often referred to as Matsaev's theorem (cf. [73]). In this case, the corresponding Riesz projection is the usual triangular projection (see [111] for more results on this projection; see also [208] for related results). Let us discuss another particularly interesting case, that of analytic operator valued functions. Then Theorem 8.4(i) is equivalent to the UMD property of $L^p(\mathcal{M})$ that we already saw in the last section. Indeed, considering the finite subdiagonal algebra $H^\infty(\mathbb{T}, \mathcal{M})$ of $L^\infty(\mathbb{T}) \otimes \mathcal{M}$, one sees that $H = \mathcal{H} \otimes \mathrm{id}_{L^2(\mathcal{M})}$, where $\mathcal{H}$ is the usual Hilbert transform on $\mathbb{T}$. Thus the boundedness of H on $L^p(L^\infty(\mathbb{T}) \otimes \mathcal{M})$ is equivalent to the fact that $\mathcal{H} \otimes \mathrm{id}_{L^2(\mathcal{M})}$ extends to a bounded map on $L^p(\mathbb{T}; L^p(\mathcal{M}))$ (noting that $L^p(L^\infty(\mathbb{T}) \otimes \mathcal{M}) = L^p(\mathbb{T}; L^p(\mathcal{M}))$). In other words, $L^p(\mathcal{M})$ has the "HT property", which is equivalent to the UMD property, as already mentioned in the last section. The main idea of the proof is an old trick due to Cotlar, which still works in the general setting as in Theorem 8.4. The ingredient is the following formula, whose proof is straightforward.

LEMMA 8.5. *For any $x = a + b^*$ with $a, b \in H_0^\infty(\mathcal{M})$*

$$(Hx)^* Hx = x^* x + H\big(x^* Hx + (Hx)^* x\big).$$

Using Lemma 8.5, we easily check that the boundedness of H on $L^p(\mathcal{M})$ implies that on $L^{2p}(\mathcal{M})$ (see also the proof of Theorem 5.3 above). Then starting from $p = 2$ and iterating, we deduce that H is bounded on $L^{2^n}(\mathcal{M})$ for all integers $n \geqslant 1$. Finally, interpolation and duality yield Theorem 8.4(i). We also point out that this argument gives the optimal order of the constant C_p as stated in Theorem 8.4.

REMARKS. (i) It was shown in [150] that in the case of triangular matrices or analytic operator valued functions, the non-commutative Hardy spaces form an interpolation scale with respect to the real and complex methods. The same arguments work in the general case as well. Thus for any $0 < p_0, p_1 \leqslant \infty$ and $0 < \theta < 1$

$$\big(H^{p_0}(\mathcal{M}), H^{p_1}(\mathcal{M})\big)_{\theta,p} = \big(H^{p_0}(\mathcal{M}), H^{p_1}(\mathcal{M})\big)_\theta = H^p(\mathcal{M}),$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$.

(ii) The Hilbert transform H enables us to identify the dual of $H^1(\mathcal{M})$ with the non-commutative analog of the space BMO (for bounded mean oscillation) as in Fefferman's classical result, namely the space $BMO(\mathcal{M})$ defined as follows:

$$BMO(\mathcal{M}) = \big\{x + Hy\colon x, y \in L^\infty(\mathcal{M})\big\}$$

equipped with the norm

$$\|z\| = \inf\big\{\|x\|_\infty + \|y\|_\infty\colon z = x + Hy, x, y \in L^\infty(\mathcal{M})\big\}.$$

We refer to [124,125] for more information.

We end this section by an open problem. A famous theorem due to Bourgain states that the quotient space $L^1(\mathbb{T})/H^1(\mathbb{T})$ is a GT space of cotype 2 (cf., e.g., [146]). It is not clear at all how to extend this theorem to the non-commutative case.

PROBLEM 8.6. Let $H^\infty(\mathcal{M})$ be a finite subdiagonal algebra in $\mathcal{M}$. Is $L^1(\mathcal{M})/H^1(\mathcal{M})$ of cotype 2? or merely of finite cotype?

In the case of triangular matrices or vector-valued analytic functions, this problem has been circulated in Banach space theory almost since Bourgain's theorem. By the way, note that any quotient of $L^1(\mathcal{M})$ by a reflexive subspace is of cotype 2. This is the non-commutative version of a theorem due to Kisliakov and Pisier (cf. [146]). It follows from [92] and [145].

9. Hankel operators and Schur multipliers

In general it is not so easy to compute (up to equivalence) the S^p-norm of an operator x in S^p, except when $x=(x_{ij})$ is a column (or row) matrix and when it is a diagonal one, as follows:

$$\Big\|\sum x_i e_{i1}\Big\|_{S^p} = \Big\|\sum x_i e_{1i}\Big\|_{S^p} = \Big(\sum |x_i|^2\Big)^{1/2}$$

and

$$\Big\|\sum x_i e_{ii}\Big\|_{S^p} = \Big(\sum |x_i|^p\Big)^{1/p}.$$

In view of their importance and ubiquity in Analysis, it was natural to wonder about the case when $x=(x_{ij})$ is a Hankel matrix, i.e., there is a (complex) sequence γ in ℓ_2 such that

$$x_{ij} = \gamma_{i+j}, \quad \forall i, j \geqslant 0. \tag{9.1}$$

This case was solved in Peller's remarkable paper [136] as follows.

THEOREM 9.1. *Let $x=(x_{ij})$ be given by (9.1) and let $1<p<\infty$. Let $\varphi(z)=\sum_{j\geqslant 0} z^j\gamma_j$ be "its symbol" and let*

$$\Delta_0\varphi(z) = x_{00}, \qquad \Delta_n\varphi(z) = \sum_{2^{n-1}\leqslant j<2^n} z^j\gamma_j, \quad \forall n \geqslant 1\ (z\in\mathbb{T}).$$

Then $x\in S^p$ iff $\sum_n 2^n\|\Delta_n\varphi\|_p^p<\infty$ (here $\|\cdot\|_p$ denotes the L^p-norm on the unit circle with normalized Lebesgue measure). Moreover, $\|x\|_{S^p}$ is equivalent to

$$\Big(\sum_{n\geqslant 0} 2^n\|\Delta_n\varphi\|_p^p\Big)^{1/p}. \tag{9.2}$$

Actually, Peller also solved the cases $p=1$ and $0<p<1$ but the solution is then a bit more complicated to state (cf. [136] and [138]). The case $0<p<1$ was obtained independently by Semmes (see [139] for this and for additional references). More generally, Peller [137] proved an extension of this result for Hankel matrices $x = (x_{ij})$ with entries $x_{ij} = \gamma_{i+j}$ in $S^p(H)$. In that case, Peller proved that $\|x\|_{S^p(\ell_2(H))}$ is equivalent to

$$\left(\sum_{n\geqslant 0} 2^n \|\Delta_n \varphi\|^p_{L^p(S^p(H))}\right)^{1/p}. \tag{9.3}$$

Unfortunately, while (9.2) is very easy to use, in general the norm of $\Delta_n\varphi$ in $L^p(S^p)$ seems as untractable as that of x in $S^p(\ell_2(H))$. However, when the spectrum of the symbol $\varphi = \sum z^j \gamma_j$ is restricted to be in a suitably "thin" set of integers Λ (meaning that the Fourier coefficients $\gamma_j \in S^p$ are zero when $j \notin \Lambda$), then (9.3) can effectively be used, as shown in the recent paper [83] (see (9.10) below).

To explain this, we will work in a (possibly non-commutative) discrete group Γ but the case of $\Gamma = \mathbb{Z}$ is the most interesting one.

DEFINITION 9.2. Let $p \geqslant 2$ be an integer and let $\Lambda \subset \Gamma$ be a subset. Let $\varepsilon = +1$ if p is even and $\varepsilon = -1$ otherwise.

(i) Λ is called a $B(p)$-set if whenever two p-tuples (s_i) and (t_i) in Λ satisfy $s_1 t_1^{-1} s_2 t_2^{-1} \cdots s_p t_p^{-1} = e$ we have necessarily $\{s_1, \dots, s_p\} = \{t_1, \dots, t_p\}$ with multiplicity (meaning that if an element is repeated, it appears the same number of times in both sets).

(ii) For any t in Γ, we denote by $R_p(t, \Lambda)$ the number of p-tuples $t_1, t_2, \dots, t_p$ in Γ with $t_i \neq t_j$ for all $i \neq j$ such that $t_1^{-1} t_2 t_3^{-1} \cdots t_p^{\varepsilon} = t$; moreover we let $Z(p, \Lambda) = \sup\{R_p(t, \Lambda) \mid t \neq e\}$. We say that Λ is a $Z(p)$-set if $Z(p, \Lambda) < \infty$.

The above is inspired by Zygmund's study of the sets (called here $Z(2)$-sets) $\Lambda \subset \mathbb{Z}^d$ such that $Z(2, \Lambda) = \sup_{t\neq 0} \operatorname{card}\{(n, m) \in \Lambda^2 \mid n - m = t\} < \infty$. As observed by Zygmund, the finite subsets $\Lambda_N = \{(n, m) \in \mathbb{Z}^2 \mid n^2 + m^2 = N\}$ are uniformly $Z(2)$-sets, more precisely we have

$$\sup_{N\geqslant 1} Z(2, \Lambda_N) \leqslant 2. \tag{9.4}$$

Actually, the same is true if, replacing $\mathbb{Z}^2$ by $\mathbb{R}^2$, we consider the circle of radius $\sqrt{N}$ instead of Λ_N. A mere look at the picture of such a circle then establishes (9.4).

The paper [83] also shows that (generic) random subsets of $[1, \dots, N]$ with cardinality $N^{1/2}$ are $Z(2)$-sets with constants uniformly bounded over N. On the other hand, as pointed out in [83], $B(p)$-sets are a fortiori $Z(p)$-sets and this provides examples of a different kind: for instance free sets as well as any subset $\Lambda \subset \Gamma$ which does not satisfy any non-trivial relation of length $\leqslant 2p$. More generally, the generators of the free Abelian groups such as $\mathbb{Z}^d$ or $\mathbb{Z}^{\mathbb{N}}$ are $B(p)$-sets. On the other hand, because of torsion, the Rademacher functions (=coordinates on $\{-1, 1\}^{\mathbb{N}}$), identified to a subset $R \subset \{-1, 1\}^{(\mathbb{N})}$, do *not* form a $B(p)$-set, but it is easy to see that they form a $Z(p)$-set for any $p \geqslant 2$ (with $Z(p, R) = p!$).

Thus, the following result in [83] can be viewed as an extension of Theorem 6.1 (=the non-commutative Khintchine inequalities). Recall that $vN(\Gamma)$ is equipped with its normalized trace τ_Γ (see Section 1).

THEOREM 9.3. *Let Γ be any discrete group. Let $p \geqslant 4$ be an even integer and let $\Lambda = \{t_n \mid n \geqslant 0\} \subset \Gamma$ be a $Z(p/2)$-subset. Then there is a constant C such that for any semifinite $\mathcal{M}$ and for any finite sequence $a = (a_n)_{n\geqslant 0}$ in $L^p(\mathcal{M})$ we have*

$$\|a\|_{L^p(\mathcal{M};l^2_C)\cap L^p(\mathcal{M};l^2_R)} \leqslant \Big\| \sum_{n\geqslant 0} \lambda(t_n)\otimes a_n \Big\|_{L^p(vN(\Gamma)\otimes\mathcal{M})} \leqslant C\|a\|_{L^p(\mathcal{M};l^2_C)\cap L^p(\mathcal{M};l^2_R)}. \tag{9.5}$$

Moreover, the left side of (9.5) *is actually valid for $\Lambda = \Gamma$.*

REMARK 9.4. When Γ is commutative and $\dim(\mathcal{M}) = 1$, the sets satisfying (9.5) are exactly the $\Lambda(p)$-sets in Rudin's sense (see [171] and [22]). Because of this, the sets satisfying (9.5) when $L^p(\mathcal{M}) = S^p$ are called $\Lambda(p)_{cb}$-sets and are studied in [83].

DEFINITION 9.5. Consider two (commutative) L^p-spaces $L^p(\mu), L^p(\nu)$ $(1 \leqslant p < \infty)$ and (closed) subspaces $E \subset L^p(\mu)$, $F \subset L^p(\nu)$. A linear mapping $u : E \to F$ is called completely bounded (in short c.b.) if there is a constant C such that

$$\Big\| \sum u(x_i)(\cdot)\, y_i \Big\|_{L^p(\nu;S^p)} \leqslant C \Big\| \sum x_i(\cdot)\, y_i \Big\|_{L^p(\mu;S^p)} \quad \forall x_i \in E,\ \forall y_i \in S^p.$$

We denote by $\|u\|_{cb}$ the smallest C for which this holds.

This definition is coherent with the one used in the theory of operator spaces (cf. [154]).

REMARK 9.6. More generally if $L^p(\mu), L^p(\nu)$ are non-commutative L^p-spaces associated to semifinite traces μ, ν the preceding definition still makes sense using $L_p(\mu\otimes \mathrm{Tr})$ and $L^p(\nu\otimes\mathrm{Tr})$ instead of $L^p(\mu; S^p)$ and $L^p(\nu; S^p)$.

COROLLARY 9.7. *Let Λ and p be as in Theorem* 9.3 (*more generally, what follows is valid for any $p > 2$ if Λ is assumed $\Lambda(p)_{cb}$*). *Let (ε_n) denote the Rademacher functions on $(\Omega, \mathcal{F}, P)$, as in Section* 6. *Let E_R (resp. E_Λ) be the closed subspace of $L^p(\Omega,\mathcal{F},P)$ (resp. $L^p(vN(\Gamma))$) generated by $\{\varepsilon_n \mid n \geqslant 0\}$ (resp. $\{\lambda(t_n) \mid n\geqslant 0\}$). Then the linear mappings u and u^{-1} defined on the linear spans by $u(\varepsilon_n) = \lambda(t_n)$ and $u^{-1}(\lambda(t_n)) = \varepsilon_n$ extend to c.b. maps $u : E_R \to E_\Lambda$ and $u^{-1}\colon E_\Lambda \to E_R$ with $\|u\|_{cb} \leqslant C$ and $\|u^{-1}\|_{cb} \leqslant B'_p$. Moreover, the* (*orthogonal*) *projection $P : L^p(vN(\Gamma)) \to E_\Lambda$, defined by $P(\lambda(t)) = \lambda(t)$ if $t \in \Lambda$, and $=0$ if $t \notin \Lambda$, is c.b. on $L^p(vN(\Gamma))$.*

The preceding results provide non-trivial new examples of c.b. Fourier multipliers on $L^p(\mathbb{T})$. We now turn to Schur multipliers. A linear map $T: S^p \to S^p$ (resp. $T: B(\ell_2) \to B(\ell_2)$) is called a Schur multiplier if it is of the form

$$T(x) = \left[\varphi(i, j) x_{ij}\right]$$

for some function $\varphi: \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. In this case, we write $T = M_\varphi$. The case $p = 2$ is of course trivial: we have then $\|M_\varphi\| = \sup_{i,j} |\varphi(i, j)|$. In the case $p = \infty$, it is well known (due essentially to Grothendieck) that bounded Schur multipliers $T = M_\varphi: B(\ell_2) \to B(\ell_2)$ are all of the following special form: there are bounded sequences (x_i) and (y_j) in ℓ_2 such that

$$\varphi(i, j) = \langle x_i, y_j \rangle. \tag{9.6}$$

Moreover, we have

$$\|M_\varphi\| = \inf\left\{\sup_i \|x_i\| \sup_j \|y_j\|\right\},$$

where the infimum runs over all possible (x_i) and (y_j) satisfying (9.6). This implies in particular (due to Haagerup) that bounded Schur multipliers on $B(\ell_2)$ (or on S^1) are "automatically" c.b. (see [152]). However, the following remains open (we conjecture that the answer is negative):

PROBLEM 9.8. Is every bounded Schur multiplier on S^p $(1 < p \neq 2 < \infty)$ c.b.?

Note that it is rather easy to give examples of bounded Fourier multipliers on $L^p(G)$ which are not c.b. when G is any compact infinite commutative group and $1 < p \neq 2 < \infty$ (see [83] or [152], p. 91).

PROBLEM 9.9. Is there a description of c.b. Schur multipliers on S^p extending (9.6) to $1 < p \neq 2 < \infty$?

It is known ([83], see also [206]) that the space of bounded (or c.b.) Schur multipliers of S^p $(2 < p < \infty)$ does *not* coincide with any interpolation space between the cases $p = 2$ and $p = \infty$.

DEFINITION 9.10. A subset $A \subset \mathbb{N} \times \mathbb{N}$ is called a $\sigma(p)$-set $(p \geqslant 2)$ if $\{e_{ij} \mid (i, j) \in A\}$ is an unconditional basic sequence in S^p.

A simple application of Corollary 6.2 shows that this holds iff there is a constant C such that for any finitely supported function $x: A \to \mathbb{C}$ we have

$$|||x|||_p \leqslant \left\| \sum_{(i,j)\in A} x(i, j) e_{ij} \right\|_{S^p} \leqslant C |||x|||_p, \tag{9.7}$$

where $|||x|||_p$ is as defined in (6.7) above. It is easy to see by interpolation that, if $2 \leqslant p \leqslant \infty$, we have $|||x|||_p \leqslant \|x\|_{S^p}$ for all x in S^p. Hence (9.7) implies that the idempotent Schur multiplier corresponding to the indicator function of A is bounded on S^p with norm $\leqslant C$.

For example, any set A for which either one of the two coordinate projections is one to one when restricted to A, is obviously a $\sigma(p)$-set. The following result provides much less trivial examples.

PROPOSITION 9.11 ([83]). *Let $p \geqslant 2$. Let $\Lambda \subset \mathbb{Z}$ be a $Z(p/2)$-set, or more generally a $\Lambda(p)_{cb}$-set. Then the set $A_\Lambda = \{(i,j) \in \mathbb{N}^2 \mid i+j \in \Lambda\}$ is a $\sigma(p)$-set.*

PROOF. This follows from (6.4), applied to the series $\varphi(z) = \sum_{n\in\Lambda} z^n (\sum_{i+j=n} x(i,j)e_{ij})$. Indeed for any z we have

$$\|\varphi(z)\|_{S^p} = \Big\| \sum_{(i,j)\in A_\Lambda} x(i,j)e_{ij} \Big\|_{S^p} = \Big\| \sum_{n\in\Lambda} a_n \Big\|_{S^p},$$

where $a_n = \sum_{i+j=n} x(i,j)e_{ij}$. By the $\Lambda(p)_{cb}$-property of Λ (see Remark 9.4) there is a constant C such that:

$$\begin{aligned}\frac{1}{C}\Big\|\sum z^n a_n\Big\|_{L^p(S^p)} &\leqslant \max\Big\{\Big\|\Big(\sum a_n^* a_n\Big)^{1/2}\Big\|_{S^p}, \Big\|\Big(\sum a_n a_n^*\Big)^{1/2}\Big\|_{S^p}\Big\} \\ &\leqslant \Big\|\sum z^n a_n\Big\|_{L^p(S^p)}. \end{aligned} \tag{9.8}$$

But as we just observed we have $\|\sum z^n a_n\|_{L^p(S^p)} = \|\sum a_n\|_{S^p}$ and

$$\sum a_n^* a_n = \sum_j \sum_i |x(i,j)|^2 e_{jj}, \qquad \sum a_n a_n^* = \sum_i \sum_j |x(i,j)|^2 e_{ii}.$$

Hence (9.8) implies (9.7) with $A = A_\Lambda$. □

REMARK. In the situation of Proposition 9.11, the same argument shows that if $A = A_\Lambda$ then for any finitely supported function $x : A \to S^p$ we have

$$Q(x) \leqslant \Big\| \sum_{(i,j)\in A} e_{ij} \otimes x(i,j) \Big\|_{S^p(\ell_2 \otimes \ell_2)} \leqslant CQ(x), \tag{9.9}$$

where

$$\begin{aligned} Q(x) = \max\Big\{&\Big(\sum_j \Big\|\Big(\sum_i x(i,j)^* x(i,j)\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}, \\ &\Big(\sum_i \Big\|\Big(\sum_j x(i,j) x(i,j)^*\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}\Big\}. \end{aligned}$$

A set $A \subset \mathbb{N}^2$ satisfying (9.9) for some constant C is called a $\sigma(p)_{cb}$-set. Equivalently, this means, by (6.4), that $\{e_{ij} \mid (i,j) \in A\}$ is "completely unconditional" (see Remark 9.6), i.e., for any choice of signs $\varepsilon_{ij} = \pm 1$, the transformations

$$\sum_{(i,j)\in A} x(i,j)e_{ij} \mapsto \sum_{(i,j)\in A} \varepsilon_{ij} x(i,j)e_{ij}$$

are c.b. on the closure in S^p of $\{e_{ij} \mid (i,j) \in A\}$. Since the left side of (9.9) remains valid for $A = \mathbb{N}^2$, (9.9) implies that the indicator function of A is a c.b. Schur multiplier on S^p.

In particular, if $x = (x(i,j))$ is Hankelian, i.e., $x(i,j) = \gamma(i+j)$ for some finitely supported function $\gamma : \Lambda \to S^p$, then (9.9) implies

$$q(\gamma) \leqslant \left\| \sum_{n\in\Lambda} \gamma(n) \otimes \sum_{i+j=n} e_{ij} \right\|_{S^p(l_2 \otimes l_2)} \leqslant Cq(\gamma), \tag{9.10}$$

where

$$q(\gamma) = \max\left\{ \left(\sum_j \left\| \left(\sum_{n\geqslant j} \gamma(n)^*\gamma(n) \right)^{1/2} \right\|_{S^p}^p \right)^{1/p}, \right.$$
$$\left. \left(\sum_j \left\| \left(\sum_{n\geqslant j} \gamma(n)\gamma(n)^* \right)^{1/2} \right\|_{S^p}^p \right)^{1/p} \right\}.$$

Thus we can "compute" (up to C) the norm of a Hankel operator with "spectrum" in Λ.

COROLLARY 9.12 ([83]). *Let $p > 2$ be an even integer.*

(i) *There are $\delta > 0$ and C such that, for any n, there is a subset $A_n \subset [1,\ldots,n]^2$ with $|A_n| \geqslant \delta n^{1+2/p}$ such that $\{e_{ij} \mid (i,j) \in A_n\}$ is C-unconditional in S^p, i.e., a $\sigma(p)$-set.*

(ii) *There is an idempotent Schur multiplier T (idempotent means here $T^2 = T$) which is bounded on S^p but unbounded on S^q for any $q > p$.*

PROOF. The proof combines Theorem 9.3 with Rudin's (combinatorial and number theoretic) construction of a $B(p/2)$ set $\Lambda \subset \mathbb{Z}$ such that

$$\limsup_{N\to\infty} \sup_{a,b\in\mathbb{N}} N^{-2/p} \left| \Lambda \cap [a, a+bN] \right| > 0. \qquad \square$$

REMARKS. (i) It is shown in [84] that, for any $p > 2$, $n^{1+2/p}$ is the maximal possible order of growth in the first part of Corollary 9.12.

(ii) The preceding corollary almost surely remains valid when $p > 2$ is *not* an even integer, but no proof is known at the time of this writing.

(iii) It is proved in [136] (see also [105] for related estimates on the case $p = \infty$) that the orthogonal projection from S^2 onto the subspace of all Hankel matrices (i.e., the averaging projection) is bounded on S^p iff $1 < p < \infty$, and for $p = 1$, it is bounded from S^1 to $S^{1,2}$ (a fortiori it is of "weak" type $(1, 1)$). See [2] for more recent results on (Hankel and Toeplitz) Schur multipliers, in particular for the case S^p with $p < 1$.

10. Isomorphism and embedding

In this section we discuss isomorphism and embedding of non-commutative L^p-spaces. Unless explicitly stated otherwise, we will assume all L^p-spaces considered in this section are separable and infinite-dimensional, or equivalently, the underlying von Neumann algebras are infinite-dimensional and act on separable Hilbert spaces.

Throughout this section, L^p denotes the classical commutative L^p-space on $[0, 1]$. The isomorphic classification of *commutative* L^p-spaces is extremely simple, for there are only two non-isomorphic commutative L^p-spaces: l^p and L^p. However, in the non-commutative setting, the situation is far from simple. In fact, it is impossible to list all non-commutative L^p-spaces up to isomorphism. It even seems very hard to classify them according to the different types of the underlying von Neumann algebras. Despite these difficulties, considerable progress has been achieved in the last few years.

Let K^p denote the direct sum in the l^p-sense of the S_n^p's, i.e.,

$$K^p = \Big(\bigoplus_{n\geqslant 1} S_n^p\Big)_p .$$

Note that K^p is the non-commutative L^p-space associated with the von Neumann algebra $\mathcal{M} = \bigoplus_{n\geqslant 1} M_n$, the direct sum of the matrix algebras M_n, $n \geqslant 1$. We also recall that if X is a Banach space, $L^p(X)$ stands for the usual L^p-space of Bochner measurable p-integrable functions on $[0, 1]$ with values in X. If $X = L^p(\mathcal{M})$, $L^p(X)$ is just the non-commutative L^p-space associated with $L^\infty(0, 1) \otimes \mathcal{M}$. We should call the reader's attention to the two different notations for the Schatten classes, equally often used in the literature: S^p in our notation is sometimes denoted by C^p, and K^p by S^p! Recall that R denotes the hyperfinite II_1 factor.

THEOREM 10.1. *Let $\mathcal{M}$ be a hyperfinite semifinite von Neumann algebra. Let $1 \leqslant p < \infty$, $p \neq 2$. Then $L^p(\mathcal{M})$ is isomorphic to precisely one of the following thirteen spaces:*

$$l^p,\ L^p,\ K^p,\ S^p,\ L^p \oplus K^p,\ L^p \oplus S^p,\ L^p(K^p),\ S^p \oplus L^p(K^p),$$
$$L^p(S^p),\ L^p(R),\ S^p \oplus L^p(R),\ L^p(S^p) \oplus L^p(R),\ L^p(R \otimes B(l^2)).$$

REMARKS. (i) The first nine spaces in the above list give precisely all non-commutative L^p-spaces, up to isomorphism, associated with von Neumann algebras of type I.

(ii) Theorem 10.1 is proved in [82]. Prior to that, the case of type I was studied in [187].

[82] also contains results on non-commutative L^p-spaces associated with hyperfinite factors of type III and free group von Neumann algebras. More precisely, it is shown there

that the non-commutative L^p-spaces associated with hyperfinite factors of type III_λ for all $\lambda \in (0, 1]$ are isomorphic, and the non-commutative L^p-space associated with a free group von Neumann algebra is independent, up to isomorphism, of the number of generators as soon as this number is not less than 2. We refer the interested reader to [82] for more information.

The proof of Theorem 10.1 can be reduced to the non-embedding of one non-commutative L^p-space into another. The main general result on this is the following

THEOREM 10.2. *Let $0 < p < \infty$, $p \neq 2$. Let $\mathcal{M}$ be a finite von Neumann algebra. Then S^p does not embed (isomorphically) into $L^p(\mathcal{M})$.*

We get immediately the following corollary.

COROLLARY 10.3. *Let $0 < p < \infty$, $p \neq 2$. Let $\mathcal{M}$ be a finite von Neumann algebra and $\mathcal{N}$ an infinite von Neumann algebra. Then $L^p(\mathcal{N})$ does not embed into $L^p(\mathcal{M})$.*

Theorem 10.2 was proved in [186] for $p > 2$, in [82] for $1 \leqslant p < 2$, and in [188] for $p < 1$. Note that in the special case where $\mathcal{M} = L^\infty(0, 1)$, Theorem 10.2 was established by McCarthy in the pioneering paper [128]. His result was considerably improved in [74]. In particular, Theorem 4.1 implies that K^p does not embed into L^p. In the converse direction, it was proved in [11] that L^p does not embed into S^p.

In the case of $0 < p < 2$, we have the following result, much stronger than Theorem 10.2.

THEOREM 10.4. *Let $0 < p < 2$. Let $\mathcal{M}$ be a finite von Neumann algebra. Let $(u_{i,j})_{i,j\geqslant 1}$ be an infinite matrix of elements in $L^p(\mathcal{M})$ such that $\sup_{i,j} \|u_{i,j}\|_p < \infty$. Suppose that all rows, columns and generalized diagonals of $(u_{i,j})_{i,j\geqslant 1}$ are unconditional. Then one of the following three alternatives holds*

(i) *Some row or column has a subsequence equivalent to the canonical basis of l^p;*
(ii) *There is a constant $\lambda > 0$ such that for every integer n some row and some column contain n elements λ-equivalent to the canonical basis of l^p_n;*
(iii) *There is a generalized diagonal $(u_{i_k,j_k})_{k\geqslant 1}$ such that*

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \left\| \sum_{k=1}^{n} u_{i_k,j_k} \right\|_p = 0.$$

Here by a generalized diagonal of $(u_{i,j})_{i,j\geqslant 1}$ we mean a sequence $(u_{i_k,j_k})_{k\geqslant 1}$ with $i_1 < i_2 < \cdots$ and $j_1 < j_2 < \cdots$. Theorem 10.4 was proved in [82] for $1 \leqslant p < 2$ and in [188] for $p < 1$. Using Theorem 10.4, we can deduce the following refinement of Theorem 10.1, which comes from [82] for $1 \leqslant p < 2$, and from [188] for $p < 1$.

THEOREM 10.5. *Let $\mathcal{M}$ be as in Theorem 10.1, and let $0 < p < 2$. If $X \neq Y$ are listed in the tree in the following figure, then X embeds into Y iff X can be joined to Y through a descending branch*

$$
\begin{tikzcd}[column sep=small]
 & & \ell^p \arrow[dl] \arrow[dr] & & \\
 & K^p \arrow[dl] \arrow[dr] & & L^p \arrow[dl] & \\
S^p \arrow[dr] & & K^p \oplus_p L^p \arrow[dl] \arrow[dr] & & \\
 & S^p \oplus_p L^p \arrow[dr] & & L^p(K^p) \arrow[dl] \arrow[dr] & \\
 & & S^p \oplus_p L^p(K^p) \arrow[dl] \arrow[dr] & & L^p(R) \arrow[dl] \\
 & L^p(S^p) \arrow[dr] & & S^p \oplus_p L^p(R) \arrow[dl] & \\
 & & L^p(S^p) \oplus_p L^p(R) \arrow[d] & & \\
 & & L^p(R \overline{\otimes} B(\ell^2)) & &
\end{tikzcd}
$$

Several non-embeddings in Theorem 10.5 are already contained in Corollary 10.3 and the discussion just after it. On the other hand, the non-embedding of $L^p(K^p)$ into $L^p \oplus S^p$ was established in [187], and that of $L^p(R)$ into $L^p(S^p)$ in [157]. The proof for the first non-embedding in [187] uses the classical result that L^p contains a subspace isomorphic to l^q for all $0 < p < q < 2$. This classical result admits a non-commutative version, which is a remarkable result recently obtained by Junge (see Corollary 10.12 below), and which is the main ingredient for the non-embedding of $L^p(R)$ into $L^p(S^p)$. The remaining non-embeddings in Theorem 10.5 can be reduced to the following

THEOREM 10.6. *Let $0 < p < 2$, and let $\mathcal{M}$ and $\mathcal{N}$ be finite von Neumann algebras. Let $X \subset L^p(\mathcal{M})$ be a closed subspace which contains no subspace isomorphic to l^p, and let Y be a quasi-Banach space which contains no subspace isomorphic to X. Then $X \otimes_p S^p$ does not embed into $Y \oplus L^p(\mathcal{N})$, where $X \otimes_p S^p$ denotes the closure of the algebraic tensor product $X \otimes S^p$ in $L^p(\mathcal{M} \otimes B(l^2))$.*

Theorem 10.6 was proved in [188]. It extends some results in [82]. Like in [82], its proof heavily relies upon Theorem 10.4. Using this theorem and Corollary 10.12 below (and its commutative counterpart, cited above), we deduce that $L^p(S^p)$ (resp. $L^p(R \otimes B(l^2))$) does not embed in $S^p \oplus L^p(R)$ (resp. $L^p(S^p) \oplus L^p(R)$).

Subspaces of $L^p(\mathcal{M})$, which have no copy isomorphic to l^p, can be characterized as follows.

THEOREM 10.7. *Let $0 < p < \infty$, $p \neq 2$. Let $\mathcal{M}$ be a finite von Neumann algebra and $X \subset L^p(\mathcal{M})$ a closed subspace. Then the following assertions are equivalent:*

(i) *X contains a subspace isomorphic to l^p.*
(ii) *For any $\lambda > 1$, X contains a subspace λ-isomorphic to l^p.*
(iii) *X contains l_n^p's uniformly.*
(iv) *For any q such that $0 < q < p$ the norms $\|\cdot\|_q$ and $\|\cdot\|_p$ are not equivalent on X.*

REMARK. If one of the preceding assertions holds, then X contains a perturbation of a normalized sequence formed of operators with disjoint support; consequently, if $p \geqslant 1$, X contains, for any $\lambda > 1$, a subspace λ-isomorphic to l^p and λ-complemented in $L^p(\mathcal{M})$.

The above theorem is the extension to the non-commutative setting of the classical Kadets–Pełczyński results for commutative L^p-spaces (cf. [103,170]). It was proved in [186] for $p > 2$, in [82] for $1 \leqslant p < 2$, and in [163] and [188] for $0 < p < 1$. In the case $p > 2$, Theorem 10.7 yields the non-commutative analogue of the following striking dichotomy:

COROLLARY 10.8. *Let $\mathcal{M}$ and X be as in Theorem 10.7 with $2 < p < \infty$. Then either X is isomorphic to a Hilbert space or X contains a subspace isomorphic to l^p.*

REMARKS. (i) The above corollary is easier for subspaces of S^p, and there it holds for all $0 < p < \infty$ (cf. [68]).

(ii) More generally, Theorem 10.7 was extended in [169] to non-commutative L^p-spaces associated with any von Neumann algebra.

(iii) [162,164] and [169] contain more results closely related to Theorem 10.7 and Corollary 10.8.

There are many open problems on the subject discussed above. Below we give two of them. Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras of type λ and μ, respectively, where $\lambda, \mu \in \{\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$. Combining Corollary 10.3, Theorem 10.5 and Theorem 3.5, we see that if $\lambda < \mu$ and $(\lambda, \mu) \neq (\mathrm{II}_\infty, \mathrm{III})$, then $L^p(\mathcal{M})$ and $L^p(\mathcal{N})$ are not isomorphic for all $0 < p < \infty$, $p \neq 2$. It is unknown whether this is still valid for $(\lambda, \mu) = (\mathrm{II}_\infty, \mathrm{III})$.

PROBLEM 10.9. Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras of type II_∞ and III, respectively. Are $L^p(\mathcal{M})$ and $L^p(\mathcal{N})$ isomorphic for $p \neq 2$?

Theorem 10.5 solves the embedding problem for all spaces listed there in the case of $p < 2$. On the other hand, Corollary 10.3 provides some partial solutions in the case of $p > 2$. However, we do not know whether Theorem 10.5 holds in full generality for $p > 2$. Below we state three of the most important cases left unsolved in Theorem 10.5.

PROBLEM 10.10. Let $p > 2$, and let (X, Y) be one of the three couples $(L^p(K^p), S^p \oplus L^p)$, $(L^p(S^p), S^p \oplus L^p(K^p))$ and $(L^p(R), L^p(S^p))$. Does X embed into Y?

All previous non-embedding results deal with a couple of non-commutative L^p-spaces with the same index p. However, Junge's theorem already mentioned above says that S^q does embed into $L^p(R)$ for $p < q < 2$. In fact, Junge [96] proved the following striking result, much stronger than the embedding of S^q into $L^p(R)$.

THEOREM 10.11. *Let $0 < p < q < 2$. Then $L^q(R \otimes B(l^2))$ embeds isometrically into $L^p(R)$.*

As an immediate consequence, we get the

COROLLARY 10.12. *Let $0 < p < q < 2$. Then both S^q and $L^q(R)$ embed isometrically into $L^p(R)$.*

In the commutative case, it is well-known that any l_n^q embeds (uniformly over n) into some l_N^p. Junge also obtained the non-commutative version of this in [95].

THEOREM 10.13. *Let $0 < p < q < 2, \varepsilon > 0, n \in \mathbb{N}$. Then there is $N = N(p,q,\varepsilon,n)$ such that S_N^p contains a subspace $(1+\varepsilon)$-isomorphic to S_n^q.*

Like in the commutative case, Junge's arguments for the preceding results are probabilistic. They use non-commutative analogues of p-stable or Poisson processes. The reader is referred to [95,96] for more details and more embedding results.

We conclude this section by a few words about the local theory of the non-commutative L^p-spaces, very recently developed in [99], in analogy with the classical $\mathcal{L}_p$-space theory. Actually, it is better (and more convenient in some sense) to develop this theory in the operator space framework. Then the corresponding $\mathcal{L}_p$-spaces are called $\mathcal{O}L_p$-spaces in [60]. Many classical results concerning $\mathcal{L}_p$-spaces have been transferred to this non-commutative setting. In particular, any separable $\mathcal{O}L_p$-space (with an additional assumption) has a basis. It was also proved that $L^p(\mathcal{M})$ $(1 < p < \infty)$ is an $\mathcal{O}L_p$-space when $\mathcal{M}$ is injective or the von Neumann algebra of a free group (in the former case, p can be equal to 1). Consequently, these non-commutative L^p-spaces have bases. In the case of $p = \infty$, it was shown that any separable nuclear C^*-algebra has a basis. The interested reader is referred to [99] for more information.

References

[1] C.A. Akemann, J. Anderson and G.K. Pedersen, *Triangle inequalities in operator algebras*, Linear and Multilinear Algebra **11** (1982), 167–178.

[2] A.B. Alexandrov and V.V. Peller, *Hankel and Toeplitz–Schur multipliers*, Math. Ann. **324** (2002), 277–327.

[3] H. Araki and T. Masuda, *Positive cones and L_p-spaces for von Neumann algebras*, Publ. Res. Inst. Math. Sci. **18** (1982), 759–831 (339–411).

[4] J. Arazy, *Some remarks on interpolation theorems and the boundedness of the triangular projections in unitary matrix spaces*, Integral Equations Operator Theory **1** (1978), 453–495.

[5] J. Arazy, *On the geometry of the unit ball of unitary matrix spaces*, Integral Equations Operator Theory **4** (1978), 151–171.

[6] J. Arazy, *More on convergence in unitary matrix spaces*, Proc. Amer. Math. Soc. **83** (1981), 44–48.

[7] J. Arazy, *Basic sequences, embeddings, and the uniqueness of the symmetric structure in unitary matrix spaces*, J. Funct. Anal. **40** (1981), 302–340.

[8] J. Arazy, *On stability of unitary matrix spaces*, Proc. Amer. Math. Soc. **87** (1983), 317–321.

[9] J. Arazy and Y. Friedman, *Contractive Projections in C_p*, Mem. Amer. Math. Soc. 95 (459) (1992).

[10] J. Arazy and P.-K. Lin, *On p-convexity and q-concavity of unitary matrix spaces*, Integral Equations Operator Theory **8** (1985), 295–313.

[11] J. Arazy and J. Lindenstrauss, *Some linear topological properties of the spaces C_p of operators on Hilbert spaces*, Compositio Math. **30** (1975), 81–111.
[12] W.B. Arveson, *Analyticity in operator algebras*, Amer. J. Math. **89** (1967), 578–642.
[13] W.B. Arveson, *Interpolation problems in nest algebras*, J. Funct. Anal. **20** (1975), 208–233.
[14] K. Ball, E. Carlen and E. Lieb, *Sharp uniform convexity and smoothness inequalities for trace norms*, Invent. Math. **115** (1994), 463–482.
[15] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, Berlin (1976).
[16] E. Berkson, T.A. Gillespie and P.S. Muhly, *Abstract spectral decompositions guaranteed by the Hilbert transform*, Proc. London Math. Soc. **53** (1986), 489–517.
[17] R. Bhatia, *Matrix Analysis*, Graduate Texts in Math. 169, Springer, New York (1997).
[18] Ph. Biane, *Free hypercontractivity*, Comm. Math. Phys. **184** (1997), 457–474.
[19] Ph. Biane and R. Speicher, *Stochastic calculus with respect to free Brownian motion and analysis on Wigner space*, Probab. Theory Related Fields **112** (1998), 373–409.
[20] A. Bonami, *Etude des coefficients de Fourier des fonctions de $L^p(G)$*, Ann. Inst. Fourier **20** (1970), 335–402.
[21] J. Bourgain, *Vector valued singular integrals and the H^1-BMO duality*, Probability Theory and Harmonic Analysis, Chao and Woyczynski, eds, Dekker, New York (1986), 1–19.
[22] J. Bourgain, *Bounded orthogonal systems and the $\Lambda(p)$-set problem*, Acta Math. **162** (1989), 227–245.
[23] M. Bożejko, *A q-deformed probability, Nelson's inequality and central limit theorems*, Nonlinear Fields, Classical, Random, Semiclassical, P. Garbecaki and Z. Popowci, eds, World Scientific, Singapore (1991), 312–335.
[24] M. Bożejko, *Untracontractivity and strong Sobolev inequality for q-Ornstein–Uhlenbeck semigroup* $(-1 < q < 1)$, Infinite Dimensional Analysis, Quantum Probability and Related Topics **2** (1999), 203–220.
[25] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics II*, Springer (1981).
[26] L.G. Brown, *Lidskii's theorem in the type* II *case*, Geometric Methods in Operator Algebras (Kyoto, 1983), Pitman Res. Notes in Math. Ser. 123, Harlow (1986), 1–35.
[27] A. Buchholz, *Norm of convolution by operator-valued functions on free groups*, Proc. Amer. Math. Soc. **127** (1999), 1671–1682.
[28] A. Buchholz, *Operator Khintchine inequality in non-commutative probability*, Math. Ann. **319** (2001), 1–16.
[29] A. Buchholz, *Operator Khintchine inequality for Fermions and Rademachers*, in preparation.
[30] A.V. Bukhvalov and A.A. Danilevich, *Boundary properties of analytic and harmonic functions with values in Banach spaces*, Math. Notes **31** (1982), 104–110.
[31] D. Burkholder, *Sharp inequalities for martingales and stochastic integrals*, Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987), Astérisque 157–158 (1988), 75–94.
[32] D. Burkholder, *Martingales and singular integrals in Banach spaces*, Handbook of the Geometry of Banach Spaces, Vol. 1, W.B. Johnson and J. Lindenstrauss, eds, Elsevier, Amsterdam (2001), 233–269.
[33] A.P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. **26** (1966), 273–299.
[34] E.A. Carlen and P. Krée, *On martingale inequalities in non-commutative stochastic analysis*, J. Funct. Anal. **158** (1998), 475–508.
[35] E. Carlen and E. Lieb, *Optimal hypercontractivity for Fermi fields and related non-commutative integration inequalities*, Comm. Math. Phys. **155** (1993), 27–46.
[36] E. Carlen and E. Lieb, *A Minkowski type trace inequality and strong subadditivity of quantum entropy*, Differential Operators and Spectral Theory, Amer. Math. Soc. Transl. Ser. 2 189, Amer. Math. Soc., Providence, RI (1999), 59–68.
[37] V.I. Chilin, P.G. Dodds, A.A. Sedaev and F.A. Sukochev, *A characterization of Kadets–Klee properties in symmetric spaces of measurable functions*, Trans. Amer. Math. Soc. **348** (1996), 4895–4918.
[38] V.I. Chilin, P.G. Dodds and F.A. Sukochev, *The Kadets–Klee property in symmetric spaces of measurable operators*, Israel J. Math. **97** (1997), 203–219.
[39] V.I. Chilin, A.W. Krygin and F.A. Sukochev, *Uniform and local uniform convexity of symmetric spaces of measurable operators*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 355–368.
[40] V.I. Chilin, A.W. Krygin and F.A. Sukochev, *Extreme points of convex fully symmetric sets of measurable operators*, Integral Equations Operator Theory **15** (1992), 186–226.

[41] V.I. Chilin and F.A. Sukochev, *Weak convergence in non-commutative symmetric spaces*, J. Operator Theory **31** (1994), 35–65.

[42] F. Cobos and T. Schonbek, *On a theorem by Lions and Peetre about interpolation between a Banach space and its dual*, Houston J. Math. **24** (1998), 325–344.

[43] I. Cuculescu, *Martingales on von Neumann algebras*, J. Multivariate Anal. **1** (1971), 17–27.

[44] I. Cuculescu and A.G. Oprea, *Noncommutative Probability*, Kluwer Academic Publishers (1994).

[45] W.J. Davis, D.J.H. Garling and N. Tomczak-Jaegermann, *The complex convexity of quasi-normed linear spaces*, J. Funct. Anal. **55** (1984), 110–150.

[46] A. Devinatz, *The factorization of operator valued functions*, Ann. Math. **73** (1961), 458–495.

[47] J. Dixmier, *Formes linéaires sur un anneau d'opérateurs*, Bull. Soc. Math. France **81** (1953), 9–39.

[48] J. Dixmier, *Les Algèbres d'Opérateurs dans l'Espace Hilbertien (Algèbres de von Neumann)*, Deuxième édition, revue et augmentée. Cahiers Scientifiques, Fasc. XXV, Gauthier-Villars, Paris (1969).

[49] P.G. Dodds and T.K. Dodds, *Some aspects of the theory of symmetric operator spaces*, Quaestiones Math. **18** (1995), 47–89.

[50] P.G. Dodds, T.K. Dodds, P.M. Dowling, C.J. Lennard and F.A. Sukochev, *A uniform Kadets–Klee property for symmetric operator spaces*, Math. Proc. Cambridge Philos. Soc. **118** (1995), 487–502.

[51] P.G. Dodds, T.K. Dodds and B. de Pagter, *Non-commutative Banach function spaces*, Math. Z. **201** (1989), 583–597.

[52] P.G. Dodds, T.K. Dodds and B. de Pagter, *Weakly compact subsets of symmetric operator spaces*, Math. Proc. Cambridge Philos. Soc. **110** (1991), 169–182.

[53] P.G. Dodds, T.K. Dodds and B. de Pagter, *Fully symmetric operator spaces*, Integral Equations Operator Theory **15** (1992), 941–972.

[54] P.G. Dodds, T.K. Dodds and B. de Pagter, *Non-commutative Köthe duality*, Trans. Amer. Math. Soc. **339** (1993), 717–750.

[55] P.G. Dodds, T.K. Dodds, B. de Pagter and F.A. Sukochev, *Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces*, J. Funct. Anal. **148** (1997), 28–69.

[56] P.G. Dodds, T.K. Dodds, B. de Pagter and F.A. Sukochev, *Lipschitz continuity of the absolute value in preduals of semifinite factors*, Integral Equations Operator Theory **34** (1999), 28–44.

[57] P.G. Dodds, T.K. Dodds and F.A. Sukochev, *Lifting of Kadets–Klee properties to symmetric spaces of measurable operators*, Proc. Amer. Math. Soc. **125** (1997), 1457–1467.

[58] P.G. Dodds, G. Schlüchtermann and F.A. Sukochev, *Weak compactness criteria in symmetric spaces of measurable operators*, Proc. Cambridge Philos. Soc. **131** (2001), 363–384.

[59] E.A. Edgar, *Analytic martingale convergence*, J. Funct. Anal. **69** (1986), 268–280.

[60] E. Effros and Z.J. Ruan, *$\mathcal{OL}_p$-spaces*, Contemp. Math. **228** (1998), 51–77.

[61] R. Exel, *Maximal subdiagonal algebras*, Amer. J. Math. **110** (1988), 775–782.

[62] T. Fack, *Sur la notion de valeur caractéristique*, J. Operator Theory **7** (1982), 307–333.

[63] T. Fack, *Type and cotype inequalities for non-commutative L^p-spaces*, J. Operator Theory **17** (1987), 255–279.

[64] T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pacific J. Math. **123** (1986), 269–300.

[65] S.V. Ferleger and A.F. Sukochev, *Harmonic analysis in UMD-spaces: applications to basis theory*, Mat. Zametki **58** (1995), 890–905.

[66] S.V. Ferleger and A.F. Sukochev, *On the contractibility to a point of the linear groups of reflexive non-commutative L^p-spaces*, Math. Proc. Cambridge Philos. Soc. **119** (1996), 545–560.

[67] A. Figá-Talamanca and M. Picardello, *Harmonic Analysis on Free Groups*, Marcel Dekker, New York (1983).

[68] Y. Friedman, *Subspaces of $LC(H)$ and C_p*, Proc. Amer. Math. Soc. **53** (1975), 117–122.

[69] D.J. Garling, *On martingales with values in a complex Banach space*, Math. Proc. Cambridge Philos. Soc. **104** (1988), 399–406.

[70] D.J. Garling and N. Tomczak-Jaegermann, *The cotype and uniform convexity of unitary ideals*, Israel J. Math. **45** (1983), 175–197.

[71] A.M. Garsia, *Martingale inequalities*, Seminar Notes on Recent Progress, Benjamin Inc., New York (1973).

[72] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Non-selfadjoint Operators*, Transl. Math. Monographs 18, Amer. Math. Soc., Providence, RI (1969).

[73] I.C. Gohberg and M.G. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, Transl. Math. Monographs 24, Amer. Math. Soc., Providence, RI (1970).

[74] Y. Gordon and D.R. Lewis, *Absolutely summing operators and local unconditional structures*, Acta Math. **133** (1974), 27–48.

[75] L. Gross, *Existence and uniqueness of physical ground states*, J. Funct. Anal. **10** (1972), 52–109.

[76] L. Gross, *Hypercontractivity and logarithmic Sobolev inequalities for the Clifford–Dirichlet form*, Duke Math. J. **42** (1975), 383–396.

[77] A. Grothendieck, *Réarrangements de fonctions et inégalités de convexité dans les algèbres de von Neumann munies d'une trace*, Sém. Bourbaki **113** (1955), 1–13.

[78] U. Haagerup, *L^p-spaces associated with an arbitrary von Neumann algebra*, Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977), Colloq. Internat. CNRS 274, CNRS, Paris (1979), 175–184.

[79] U. Haagerup, *Non-commutative integration theory*, Lecture given at the Symposium in Pure Mathematics of the Amer. Math. Soc., Queens University, Kingston, Ontario (1980).

[80] U. Haagerup and G. Pisier, *Factorization of analytic functions with values in non-commutative L_1-spaces and applications*, Canad. Math. J. **41** (1989), 882–906.

[81] U. Haagerup and G. Pisier, *Bounded linear operators between C^*-algebras*, Duke Math. J. **71** (1993), 889–925.

[82] U. Haagerup, H. Rosenthal and F.A. Sukochev, *Banach embedding properties of non-commutative L^p-spaces*, Mem. Amer. Math. Soc., to appear.

[83] A. Harcharras, *Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets*, Studia Math. **137** (1999), 203–260.

[84] A. Harcharras, S. Neuwirth and K. Oleszkiewicz, *Lacunary matrices*, Indiana Univ. Math. J. **50** (2001), 1675–1689.

[85] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York (1964).

[86] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. **106** (1961), 175–213.

[87] M. Hilsum, *Les espaces L^p d'une algèbre de von Neumann (théorie spatiale)*, J. Funct. Anal. **40** (1980), 151–169.

[88] H. Izumi, *Constructions of non-commutative L^p-spaces with a complex parameter arising from modular actions*, Internat. J. Math. **8** (1997), 1029–1066.

[89] H. Izumi, *Natural bilinear forms, natural sesquilinear forms and the associated duality on non-commutative L^p-spaces*, Internat. J. Math. **9** (1998), 975–1039.

[90] R. Jajte, *Strong Limit Theorems in Noncommutative Probability*, Lecture Notes in Math. 1110, Springer, Berlin (1985).

[91] R. Jajte, *Strong Limit Theorems in Noncommutative L_2-Spaces*, Lecture Notes in Math. 1477, Springer, Berlin (1991).

[92] H. Jarchow, *Weakly compact operators on C^*-algebras*, Math. Ann. **273** (1986), 341–343.

[93] G. Ji and K.-S. Saito, *Factorization in subdiagonal algebras*, J. Funct. Anal. **159** (1998), 191–202.

[94] W.B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, Handbook of the Geometry of Banach Spaces, Vol. 1, W.B. Johnson and J. Lindenstrauss, eds, Elsevier, Amsterdam (2001), 1–84.

[95] M. Junge, *Embeddings of non-commutative L_p-spaces into non-commutative L_1-spaces, $1 < p < 2$*, Geom. Funct. Anal. **10** (2000), 389–406.

[96] M. Junge, *The non-commutative Poisson process*, in preparation.

[97] M. Junge, *Doob's inequality for non-commutative martingales*, J. Reine Angew. Math. **549** (2002), 149–190.

[98] M. Junge and A. Defant, *Almost everywhere convergence of series in non-commutative L_q-spaces*, Preprint (2000).

[99] M. Junge, N.J. Nielsen, Z.-J. Ruan and Q. Xu, *Local theory of non-commutative L_p-spaces*, Preprint (2000).

[100] M. Junge and Q. Xu, *Non-commutative Burkholder/Rosenthal inequalities*, Ann. Probab., to appear.

[101] M. Junge and Q. Xu, *Non-commutative Burkholder/Rosenthal inequalities: applications*, to appear.

[102] M. Junge and Q. Xu, *The optimal order of the best constants in certain non-commutative martingale inequalities*, in preparation.

[103] M.I. Kadets and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces* L_p, Studia Math. **21** (1962), 161–176.
[104] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras I and II*, Academic Press (1983) and (1986).
[105] S.V. Kislyakov, *Projectors onto the set of Hankel matrices*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **126** (1983), 109–116 (in Russian).
[106] H. Kosaki, *Applications of the complex interpolation method to a von Neumann algebra*, J. Funct. Anal. **56** (1984), 29–78.
[107] H. Kosaki, *Applications of uniform convexity of noncommutative* L^p*-spaces*, Trans. Amer. Math. Soc. **283** (1984), 265–282.
[108] H. Kosaki, *On the continuity of the map* $\varphi \to |\varphi|$ *from the predual of a* W^**-algebra*, J. Funct. Anal. **59** (1984), 123–131.
[109] J. Krivine and B. Maurey, *Espaces de Banach stables*, Israel J. Math. **39** (1981), 273–295.
[110] R. Kunze, L^p*-Fourier transforms in locally compact unimodular groups*, Trans. Amer. Math. Soc. **89** (1958), 519–540.
[111] S. Kwapień and A. Pełczyński, *The main triangle projection in matrix spaces and its applications*, Studia Math. **34** (1970), 43–67.
[112] E. Lieb, *Convex trace functions and the Wigner–Yanase–Dyson conjecture*, Adv. Math. **11** (1973), 267–288.
[113] M. Leinert, *Integration with respect to a weight*, Internat. J. Math. **2** (1991), 177–182.
[114] M. Leinert, *Integration with respect to a trace*, Arch. Math. **59** (1992), 475–486.
[115] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer (1979).
[116] J.L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Etudes Sci. Publ. Math. **19** (1964), 5–68.
[117] F. Lust-Piquard, *Inégalités de Khintchine dans* C_p $(1 < p < \infty)$, C.R. Acad. Sci. Paris **303** (1986), 289–292.
[118] F. Lust-Piquard, *A Grothendieck factorization theorem on 2-convex Schatten spaces*, Israel J. Math. **79** (1992), 331–365.
[119] F. Lust-Piquard, *Riesz transforms associated with the number operator on the Walsh system and the fermions*, J. Funct. Anal. **155** (1998), 263–285.
[120] F. Lust-Piquard, *Riesz transforms on deformed Fock spaces*, Comm. Math. Phys. **205** (1999), 519–549.
[121] F. Lust-Piquard and G. Pisier, *Non-commutative Khintchine and Paley inequalities*, Ark. Mat. **29** (1991), 241–260.
[122] J. Marcolino, *La stabilité des espaces* L^p *non-commutatifs*, Math. Scand. **81** (1997), 212–218.
[123] M. Marsalli, *Noncommutative* H^2 *spaces*, Proc. Amer. Math. Soc. **125** (1997), 779–784.
[124] M. Marsalli and G. West, *Noncommutative* H^p *spaces*, J. Operator Theory **40** (1998), 339–355.
[125] M. Marsalli and G. West, *The dual of noncommutative* H^1, Indiana Univ. Math. J. **47** (1998), 489–500.
[126] M. Marsalli and G. West, *Toeplitz operators with noncommuting symbols*, Integral Equations Operator Theory **32** (1998), 65–74.
[127] M. McAsey, P.S. Muhly and K.-S. Saito, *Nonselfadjoint crossed products, I*, Trans. Amer. Math. Soc. **248** (1979), 381–409; II, J. Math. Soc. Japan **33** (1981), 485–495; III, J. Operator Theory **12** (1984), 3–22.
[128] C.A. McCarthy, c_p, Israel J. Math. **5** (1967), 249–271.
[129] C. Muscalu, *Limit of lacunary Fourier series with coefficients in non-commutative symmetric spaces*, Proc. Amer. Math. Soc. **125** (1997), 541–546.
[130] E. Nelson, *The free Markoff field*, J. Funct. Anal. **12** (1973), 211–227.
[131] E. Nelson, *Notes on non-commutative integration*, J. Funct. Anal. **15** (1974), 103–116.
[132] V.I. Ovchinnikov, *s-numbers of measurable operators*, Functional Anal. Appl. **4** (1970), 236–242.
[133] V.I. Ovchinnikov, *Symmetric spaces of measurable operators*, Soviet Math. Dokl. **11** (1970), 448–451.
[134] V.I. Ovchinnikov, *Symmetric spaces of measurable operators*, Trudy Inst. Matem. VGU **3** (1971), 88–107.
[135] J. Peetre and G. Sparr, *Interpolation and non-commutative integration*, Ann. Mat. Pura Appl. **104** (1976), 187–207.
[136] V. Peller, *Hankel operators of class* C_p *and their applications*, Math. USSR-Sb. **41** (1982), 443–479.
[137] V. Peller, *Vectorial Hankel operators, commutators and related operators of the Schatten–von Neumann class* γ_p, Integral Equations Operator Theory **5** (1982), 244–272.

[138] V. Peller, *Description of Hankel operators of the class C_p for $p > 0$, investigation of the rate of rational approximation and other applications*, Math. USSR-Sb. **50** (1985), 465–494.

[139] V. Peller, *An excursion into the theory of Hankel operators*, Holomorphic Spaces (Berkeley, CA, 1995), Math. Sci. Res. Inst. Publ. 33, Cambridge Univ. Press, Cambridge (1998), 65–120.

[140] H. Pfitzner, *Weak compactness in the dual of a C^*-algebra is determined commutatively*, Math. Ann. **298** (1994), 349–371.

[141] A. Pietsch, *Interpolationsfunktoren, Folgenideale und Operatorenideale*, Czechoslovak Math. J. **21** (1971), 644–652.

[142] A. Pietsch and H. Triebel, *Interpolationstheorie für Banachideale von beschränkten linearen Operatoren*, Studia Math. **31** (1968), 95–109.

[143] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Compositio Math. **37** (1978), 3–19.

[144] G. Pisier, *Grothendieck's theorem for noncommutative C^*-algebras, with an appendix on Grothendieck's constants*, J. Funct. Anal. **29** (1978), 397–415.

[145] G. Pisier, *Quotients of Banach spaces of cotype q*, Proc. Amer. Math. Soc. **85** (1982), 32–36.

[146] G. Pisier, *Factorization of Linear Operators and Geometry of Banach Spaces*, CBMS Regional Conf. Series in Math. 60, Amer. Math. Soc., Providence, RI (1986).

[147] G. Pisier, *Factorization of operators through $L_{p\infty}$ or L_{p1} and noncommutative generalizations*, Math. Ann. **276** (1986), 105–136.

[148] G. Pisier, *Riesz transforms: a simpler analytic proof of P.-A. Meyer's inequality*, Séminaire de Probabilités, XXII, Lecture Notes in Math. 1321, Springer (1988), 485–501.

[149] G. Pisier, *Factorization of operator valued analytic functions*, Adv. Math. **93** (1992), 61–125.

[150] G. Pisier, *Interpolation between H^p spaces and noncommutative generalizations I*, Pacific J. Math. **155** (1992), 341–368; II, Rev. Mat. Iberoamericana **9** (1993), 281–291.

[151] G. Pisier, *Multipliers and lacunary sets in non-amenable groups*, Amer. J. Math. **117** (1995), 337–376.

[152] G. Pisier, *Similarity Problems and Completely Bounded Maps*, Lecture Notes in Math. 1618, Springer (1996).

[153] G. Pisier, *The Operator Hilbert Space OH, Complex Interpolation and Tensor Norms*, Mem. Amer. Math. Soc. 122 (585) (1996).

[154] G. Pisier, *Non-commutative vector valued L_p-spaces and completely p-summing maps*, Astérisque **247** (1998).

[155] G. Pisier, *An inequality for p-orthogonal sums in non-commutative L_p*, Illinois J. Math. **44** (2000), 901–923.

[156] G. Pisier and Q. Xu, *Non-commutative martingale inequalities*, Comm. Math. Phys. **189** (3) (1997), 667–698; see also the note C.R. Acad. Sci. Paris **323** (1996), 817–822.

[157] G. Pisier and Q. Xu, Unpublished manuscript (1999).

[158] R.J. Plymen and P.L. Robinson, *Spinors in Hilbert Space*, Cambridge Univ. Press (1994).

[159] S. Power, *Factorization in analytic operator algebras*, J. Funct. Anal. **67** (1986), 414–432.

[160] N. Randrianantoanina, *Hilbert transform associated with finite maximal subdiagonal algebras*, J. Austral. Math. Soc. Ser. A **65** (1998), 388–404.

[161] N. Randrianantoanina, *Factorization of operators on C^*-algebras*, Studia Math. **128** (1998), 273–285.

[162] N. Randrianantoanina, *Sequences in non-commutative L^p-spaces*, J. Operator Theory **48** (2002), 255–272.

[163] N. Randrianantoanina, *Embedding of l_p in non-commutative spaces*, J. Austral. Math. Soc., to appear.

[164] N. Randrianantoanina, *Kadets–Pełczyński decomposition for Haagerup L^p-spaces*, Math. Proc. Cambridge Philos. Soc. **132** (2002), 137–154.

[165] N. Randrianantoanina, *Non-commutative martingale transforms*, J. Funct. Anal. **194** (2002), 181–212.

[166] Y. Raynaud, *Stabilité des espaces d'opérateurs C_E*, Sém. Géom. Espacess de Banach, Paris VII (1983), 1–12.

[167] Y. Raynaud, *On ultrapowers of non-commutative L_1 spaces*, Colloquim del Departamento de Analisis Matematico (1997–98), Univ. Complutense Madrid.

[168] Y. Raynaud, *On ultrapowers of non-commutative L_p spaces*, J. Operator Theory **48** (2002), 41–68.

[169] Y. Raynaud and Q. Xu, *On the structure of subspaces of non-commutative L_p-spaces*, C.R. Acad. Sci. Paris **333** (2001), 213–218.

[170] H.P. Rosenthal, *On subspaces of L_p*, Ann. Math. **97** (1973), 225–344.

[171] W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–228.

[172] G.I. Russu, *Intermediate symmetrically normed ideals*, Functional Anal. Appl. **3** (1969), 94–95.

[173] K.S. Saito, *The Hardy spaces associated with a periodic flow on a von Neumann algebra*, Tohoku Math. J. **29** (1977), 69–75.

[174] K.S. Saito, *On non-commutative Hardy spaces associated with flows on finite von Neumann algebras*, Tohoku Math. J. **29** (1977), 585–595.

[175] K.S. Saito, *A note on invariant subspaces for finite maximal subdiagonal algebras*, Proc. Amer. Math. Soc. **77** (1979), 348–352.

[176] K.S. Saito, *Noncommutative L^p-spaces with $0 < p < 1$*, Math. Proc. Cambridge Philos. Soc. **89** (1981), 405–411.

[177] K.S. Saito, *Toeplitz operators associated with analytic crossed products*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 539–549.

[178] S. Sakai, *C^*-Algebras and W^*-Algebras*, Springer, Heidelberg (1971).

[179] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.

[180] C. Schütt, *Unconditionality in tensor products*, Israel J. Math. **31** (1978), 209–216.

[181] I. Segal, *A non-commutative extension of abstract integration*, Ann. Math. **57** (1953), 401–457.

[182] B. Simon, *Trace Ideals and Their Applications*, Cambridge Univ. Press (1979).

[183] W. Stinespring, *Integration theorems for gauges and duality for unimodular groups*, Trans. Amer. Math. Soc. **90** (1959), 15–56.

[184] S. Stratila, *Modular Theory in Operator Algebras*, Abacus Press (1981).

[185] S. Stratila and L. Zsidó, *Lectures on von Neumann Algebras*, Abacus Press (1979).

[186] F. Sukochev, *Non-isomorphism of L_p-spaces associated with finite and infinite von Neumann algebras*, Proc. Amer. Math. Soc. **124** (1996), 1517–1527.

[187] F. Sukochev, *Linear topological classification of separable L_p-spaces associated with von Neumann algebras of type I*, Israel J. Math. **115** (2000), 137–156.

[188] F. Sukochev and Q. Xu, *Embedding of non-commutative L_p-spaces, $p < 1$*, Arch. Math., to appear.

[189] A. Szankowski, *On the uniform approximation property in Banach spaces*, Israel J. Math. **49** (1984), 343–359.

[190] M. Takesaki, *Theory of Operator Algebras I*, Springer, New York (1979).

[191] M. Terp, *L^p spaces associated with von Neumann algebras*, Notes, Math. Institute, Copenhagen Univ. (1981).

[192] M. Terp, *Interpolation spaces between a von Neumann algebra and its predual*, J. Operator Theory **8** (1982), 327–360.

[193] N. Tomczak-Jaegermann, *The moduli of smoothness and convexity and the Rademacher averages of trace classes S_p $(1 \leqslant p < \infty)$*, Studia Math. **50** (1974), 163–182.

[194] N. Tomczak-Jaegermann, *On the differentiability of the norm in trace classes S_p*, Sém. Maurey–Schwartz 1974–1975: Espaces L_p, applications radonifiantes et géométrie des espaces de Banach, Exp. No. XXII, Centre Math. Ecole Polytech., Paris (1975).

[195] N. Tomczak-Jaegermann, *Uniform convexity of unitary ideals*, Israel J. Math. **48** (1984), 249–254.

[196] D. Voiculescu, K. Dykema and A. Nica, *Free Random Variables*, CRM Monogr. Ser. 1, Amer. Math. Soc., Providence, RI.

[197] F. Wattbled, *Interpolation complexe d'un espace de Banach et son antidual*, C.R. Acad. Sci. Paris **321** (1995), 1437–1440.

[198] N. Wiener and P. Masani, *The prediction theory of multivariate stochastic processes I*, Acta Math. **98** (1957), 111–150; II, **99** (1958), 93–137.

[199] Q. Xu, *Inégalités pour les martingales de Hardy et renormage des espaces quasi-normés*, C.R. Acad. Sci. Paris **306** (1988), 601–604.

[200] Q. Xu, *Convexité uniforme des espaces symétriques d'opérateurs mesurables*, C.R. Acad. Sci. Paris **309** (1989), 251–254.

[201] Q. Xu, *Applications du théorème de factorisation pour des fonctions à valeurs opérateurs*, Studia Math. **95** (1990), 273–292.

[202] Q. Xu, *Convexités uniformes et inégalités de martingales*, Math. Ann. **287** (1990), 193–211.

[203] Q. Xu, *Analytic functions with values in lattices and symmetric spaces of measurable operators*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 541–563.

[204] Q. Xu, *Radon–Nikodým property in symmetric spaces of measurable operators*, Proc. Amer. Math. Soc. **115** (1992), 329–335.
[205] Q. Xu, *Interpolation of operator spaces*, J. Funct. Anal. **139** (1996), 500–539.
[206] Q. Xu, *Interpolation of Schur multiplier spaces*, Math. Z. **235** (2000), 707–715.
[207] F.J. Yeadon, *Non-commutative L^p-spaces*, Math. Proc. Cambridge Philos. Soc. **88** (1980), 91–102.
[208] L. Zsidó, *On spectral subspaces associated to locally compact Abelian groups of operators*, Adv. Math. **36** (1980), 213–276.