# Mathematics 5324 (Topology)

Midterm 1

If you use anything proved in the notes, make sure to write down a complete and precise statement of what you are using, and indicate explicitly that it is proved in the notes.

1. Prove or disprove: any map of simplices $f: \mathbf{m} \to \mathbf{n}$ can be presented as a composition of an injective map $g: \mathbf{m} \to \mathbf{p}$ followed by a surjective map $h: \mathbf{p} \to \mathbf{n}$, i.e., $f = h \circ g$.

2. Fix a simplex $\mathbf{p}$. Consider the simplicial set X such that $X_{\mathbf{m}}$ is the set of all maps of simplices $\mathbf{m} \to \mathbf{p}$ that are not surjective and the simplicial structure map $X_f: X_{\mathbf{n}} \to X_{\mathbf{m}}$ sends $\alpha \in X_{\mathbf{n}}$ to $\alpha \circ f$, where $f: \mathbf{m} \to \mathbf{n}$ is a map of simplices.
  - Prove that this data indeed defines a simplicial set, i.e., the maps X_f indeed land in $X_{\mathbf{m}}$ and satisfy the functoriality properties.
  - What are the nondegenerate simplices of X? Give an explicit description and compute their number in terms of $\dim \mathbf{p}$.
  - Draw a picture of X for all $\mathbf{p}$ such that $\dim \mathbf{p} \in [0, 3]$.

3. Prove or disprove: if two simplicial maps $f, g: X \to Y$ coincide on all nondegenerate simplices of X, then $f = g$.

4. Prove or disprove: if X is a simplicial set, $s: \Delta^{\mathbf{n}} \to X$ is an arbitrary $\mathbf{n}$-simplex of X, and $f, g: \mathbf{m} \to \mathbf{n}$ are surjective maps of simplices such that $s \circ \Delta^f = s \circ \Delta^g$, then $f = g$.

5. A nondegenerate simplex $s: \Delta^{\mathbf{m}} \to X$ of a simplicial set X is *proper* if it is not a face of any other nondegenerate simplex, i.e., $s \neq \mathrm{d}_i(t)$ for any nondegenerate simplex t. For instance, the simplicial set $\Delta^{\mathbf{m}}$ has a single proper nondegenerate simplex, namely, the nondegenerate $\mathbf{m}$-simplex. Prove or disprove: there is a nonempty simplicial set X that does not have any proper nondegenerate simplices.

6. Consider a disk with two holes:
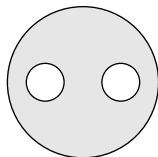
Draw a simplicial set that models it and write down a presentation in terms of generators and relations.