Tsit-Yuen Lam. Representation theory.

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Introduction

All representations are finite-dimensional.

Most groups will be finite.

We assume that k is a commutative ring (later: field).

Classic Definition. An n-dimensional representation of a group G is a group homomorphism $D: G \to \mathrm{GL}_n(k)$. (D comes from Darstellung.) Two representations D and D' are said to be equivalent if there is a $T \in \mathrm{GL}_n(k)$ such that for all $g \in G$ we have $T^{-1}D(g)T = D'(g)$. The character of D is a function $\chi_D: G \to k$ such that $\chi_D(g) = \mathrm{tr}(D(g))$ for all $g \in G$.

Note. If two representations are equivalent, they have the same characters. If two elements of the group are conjugate ($g \sim g'$), then $\chi_D(g) = \chi_D(g')$.

We call χ_D a "class function" because it is constant on each conjugacy class.

Definition. A representation D is faithful if the corresponding homomorphism is injective.

Modern Definition. (80 years old.) A G-module (over k) is a k-module (usually finitely generated) with a k-linear G-action. (Note: g must act as a k-automorphism because g^{-1} exists.)

Isomorphisms of G-modules make obvious sense. We say that G acts faithfully if the only element that acts as identity is the unit.

To link these definitions we do the following. Suppose that we have a homomorphism $D: G \to \mathrm{GL}_n(k)$. We set $g \cdot v = D(g)(v)$. Conversely, if we have a G-module, then we define $D(g)(v) = g \cdot v$.

Propositions. Two representations D and D' are equivalent iff two associate G-modules are equivalent.

Ring theoretic perspective

Form a group ring $kG = k[G]$, which is a k-algebra.

Observation. G-modules over k are the same as kG-modules.

Definition. Representation afforded by a G-module V over k is irreducible iff $V \neq 0$ and V has no non-trivial kG-submodules. Representation afforded by a G-module V over k is indecomposable iff $V \neq 0$ and $V \neq V_1 \oplus V_2$ for non-trivial kG-modules V_1 and V_2.

An irreducible representation is indecomposable but not vice versa. One dimensional representations are always irreducible and indecomposable.

Example. Let $k = \mathbb{F}_2$, let G be a cyclic group of order 2 generated by element σ, and let $V = kG$ with a left action. This module is not irreducible because $\{0, \sigma\}$ is a non-trivial kG-submodule. This module is indecomposable because $\{0, \sigma\}$ is the only non-trivial kG-submodule.

Matricial perspective

Suppose that D is a reducible representation with $V_0 \subset V$ being a non-trivial kG-invariant submodule. Choose a basis of V_0 and supplement it to a basis of V. We have two representations: $g \to D_1(g)$ afforded by V_0 and $g \to D_2(g)$ afforded by V/V_0. Then the matrix corresponding to $D(g)$ has the form $\begin{pmatrix} D_1(g) & E(g) \\ 0 & D_2(g) \end{pmatrix}$. If D is decomposable, then $E(g) = 0$.

Note. One-dimensional representation is a homomorphism $G \to k^* = \mathrm{GL}_1(k)$. Two such representations are equivalent iff they are equal.

Composition factors

If V is a kG-module then there exists a composition series $0 = V_0 \subset V_1 \subset \cdots \subset V_m = V$, with all V_k different. Here V_{k+1}/V_k are simple kG-modules. The sequence of these composition factors is unique up to a permutation.

In our earlier example the composition factors are trivial one-dimensional representations.

Direct sums

We have $\chi_{V\oplus V'} = \chi_V + \chi_{V'}$.

Scalar extensions. If we have an extension of fields K/k and a representation V over k, then we have a representation over K, which satisfies the equation $V^K = K \otimes_k V$.

The character of a representation is the sum of the characters of its composition factors.

If we recall our example, we can easily see that the equivalence class of kG-module is not determined by its character, because the character of the module in the example is 0 and the character of the direct sum of two copies of k is also 0.

Example. If G is a finite group, then we have a left regular representation. Its character χ satisfies the following equations: $\chi(1) = |G|$ and $\chi(g) = 0$ for $g \neq 1$.

Review of simple modules. Let R be any ring. An R-module V is simple iff $V \neq 0$ and every element of V generates V. An R-module V is simple iff it is isomorphic to R/m for some maximum left ideal m of R. Here m is uniquely determined if R is commutative. If R is a finite-dimensional k-algebra (k is a field), then there are only finitely many simple R-modules up to isomorphism. Proof: Look at the left regular module of R. By Jordan-Hölder theorem we have finitely many composition factors $V_1, \ldots, V_n$. If V is simple, we have $V = R/m$. We can complete the two-element series m and R to a Jordan-Hölder series.

Master Theorem. (To be proved.) If G is a finite group with r conjugacy classes and k is the field of complex numbers, then the number of irreducible representations is equal to r. If n_i is the dimension of ith irreducible representation of G, then it divides $|G|$. Also $\sum_k n_k^2 = |G|$ (magic equation). Every finite-dimensional kG-module is uniquely a direct sum of irreducible representations. (The direct sum itself is not unique.)

Example. Let $G = \langle \sigma, \phi \mid \sigma^7 = \phi^3 = 1, \phi^{-1}\sigma\phi = \sigma^2 \rangle$. We have $|G| = 21$. We have 3 one-dimensional representations of G. Representatives of conjugacy classes: 1, σ, σ^3, ϕ, ϕ^2. By magic equation we discover that there are only two more irreducible representations, which are 3-dimensional.

For finite abelian group we get a character table in the following way: let $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ be the character group of G. The character table consists of these characters line by line. We have non-canonical isomorphism between G and G^* and canonical isomorphism between G and G^{**}. All irreducible representations have dimension 1.

Another perspective (without Master Theorem).

Theorem. Let G be a finite abelian group, k be algebraically closed field. Then any simple kG-module V is 1-dimensional.

Proof. It suffices to show that for any $g \in G$ the operator $D(g)$ is a scalar multiplication. Let $\lambda \in k$ be an eigenvalue of $D(g)$ and $E(\lambda)$ be the λ-eigenspace of $D(g)$. Obviously, it is invariant under G-action. Therefore, $D(g) = \lambda I_n$.

Remark. We need only assume that the polynomial $x^e - 1$ splits over k, where e is the exponent of G. But if k is arbitrary, the theorem does not work. The cyclic group of order 3 acts irreducible on $\mathbb{Q}^2$ by rotations by $2\pi/3$.

Example. Character table of quaternion group (of order 8). We use a complex matrix model of $\mathbb{H}$. It is easy to see that $D(1)$, $D(i)$, $D(j)$, $D(k)$ are linearly independent. Therefore, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ is isomorphic to $\mathbb{M}_2(\mathbb{C})$. Restriction to G gives us a 2-dimensional complex representation. It is irreducible, since $D(G)$ $\mathbb{C}$-spans $\mathbb{M}_2(\mathbb{C})$. We also have four obvious 1-dimensional representations.

A construction idea. Denote by k be a field and by D a division algebra over k. Denote by G a subgroup of D^* such that G spans D over k. Then D is a simple kG-module.

Proof. Trivial.

Example. Denote by G the cyclic group of order n. Construct all simple $\mathbb{Q}G$-modules.

Solution. Denote by d a divisor of n. Take V_d. By previous theorem, adjoining a primitive dth root of 1 to $\mathbb{Q}$ we obtain a simple $\mathbb{Q}G$-module. We have $\dim_{\mathbb{Q}} V_d = \varphi(d)$. Composition factors of $\mathbb{Q}G$ regarded as a left module include all V_d. Since the sum of their dimensions is equal to n, we have listed all simple modules over $\mathbb{Q}G$.

Example. Now we want to find all simple $\mathbb{R}G$-modules, where G is the quaternion group.

Solution. At dimension 1 we have 4 $\mathbb{R}G$-representations. Now denote by $\mathbb{H}$ the real quaternions. They form a simple $\mathbb{R}G$-module. If we tensor multiply this by $\mathbb{C}$, it splits into two irreducible modules.

The multiplicative group of quaternions has other finite subgroups apart from the classical quaternion group. For example, it contains generalized quaternion group of order $4m$. Take inside $\mathbb{C}^*$ a cyclic subgroup of order $2m$ and adjoin j to it. We also have other subgroups: binary tetrahedral group (order 24), binary octahedron group (order 48), binary icosahedral group (order 120). Hurwitz defined the ring of integral quaternions inside rational quaternions. It consists of all quaternions with integer coefficients and with simultaneously semi-integer coefficients.

We want to obtain all irreducible complex representations of tetrahedron group A_4 and binary tetrahedron group BT. The conjugacy classes of A_4 are 1 (1), τ (3), σ (4), σ^2 (4), where $\tau = (12)(34)$ and $\sigma = (123)$. The character table is

	1	τ	σ	σ^2
	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

The first three characters are 1-dimensional representations. The remaining representation must have dimension 3. It comes from representation of A_4 as group of symmetries of tetrahedron.

Theorem. Binary tetrahedron group is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$.

Proof. We sketch two different approaches. First, easy counting shows that $\mathrm{SL}_2(\mathbb{F}_3)$ has 24 elements. Recall that BT_{24} is a semidirect product of Q_8 and C_3, we just find the same structure inside $\mathrm{SL}_2(\mathbb{F}_3)$. We write down a unique 2-Sylow subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$. Then we write down a matrix σ_0 of order 3 and compute the conjugation action of it on 2-Sylow subgroup. Another way to prove this fact is as follows: Recall 2-dimensional irreducible complex representation: $D(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $D(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The rest is left to reader as an exercise.

Dihedral group. $D_n = \langle r, s \mid r^n = s^2 = 1 \text{ and } srs^{-1} = r^{-1} \rangle$. If n is odd, then $[G,G] = \langle r \rangle$ and $G/[G,G] = \langle s \rangle$. We have two 1-dimensional representations. If n is even, then $[G,G] = \langle r^2 \rangle$ and $G/[G,G] = \langle r \rangle \times \langle s \rangle$. We have four 1-dimensional representations. For $n = 2$ we have one 2-dimensional representation coming from symmetries of n-gon on the plane. We now see that Q_8 and D_4 have the same character tables but are not isomorphic. For higher n we get the following results:

Future theorem. If $\operatorname{char} k$ does not divide $n!$, then the reduced representation is irreducible over k.

Suppose E is a finite G-set. We can easily construct a G-module with E as a basis. This module is reducible, since elements with equal coefficients form a nontrivial submodule. Denote by $\bar{V}$ the corresponding factor module. We have $\chi_{\bar{V}}(g) = |\mathrm{Fix}_E(g)| - 1$. We will show that if $\operatorname{char} k$ does not divide $n!$ and G is a symmetric group, then the reduced module is simple. We can use this fact to understand representation of S_3 and S_4.

We work out all irreducible complex representations of S_4. Two of them are trivial one-dimensional representations. Inside S_4 we have normal Klein 4-group. Its factor group is S_3, therefore, one two-dimensional representation is obtained via pullback from S_3 representation. Two 3-dimensional representations come from tetrahedron and octahedron.

Chapter 1

Notational Convention: Write homs of modules on the opposite side of scalars.

Homo-law: $(rm)f = r(mf)$. If we let $E = \mathrm{End}({}_RM)$, then $M = {}_RM_E$ is (R,E)-bimodule. We have $\mathrm{End}({}_RR) = R$: $r \to [x \to xr]$. Similarly, $\mathrm{End}(R_R) = R$: $r \to [x \to rx]$.

Let $E = \mathrm{End}({}_RM)$. Then $M^n = nM = \oplus_{n \text{ copies}} M$. We have $\mathrm{End}(M^n) = \mathbb{M}_n(E)$.

Schur's lemma: endomorphism ring of a simple module is a division ring.

A module M is called semisimple iff every submodule splits iff it is a (direct) sum of simple submodules iff it is a (direct) sum of all of its simple submodules.

A semisimple module M is indecomposable iff it is simple. Semisimple modules are closed under arbitrary direct sums, submodules and quotient modules. If M is semisimple, then M is finitely generated iff it has finite length iff it is a simple sum of simple modules. In this case, M is a direct sum of its composition factors.

A ring R is called left-semisimple iff ${}_RR$ is semisimple. A ring is semisimple iff all R-modules are semisimple.

A ring R is called simple iff $R \neq 0$ and the only (two-sided) ideals are 0 and R. (Warning: Semisimple rings are not simple in general.)

Ideals of a ring R are in bijective correspondence with ideals of the ring $\mathbb{M}_n(R)$. Therefore, if R is simple, then so is $\mathbb{M}_n(R)$. In particular, matrices over division ring form simple ring.

Moreover, matrices over division ring form semisimple ring, more precisely: $\mathbb{M}_n(D) = nD^n$ as left modules. Therefore, D^n is the only simple $\mathbb{M}_n(D)$-module.

Now note that $\mathrm{End}({}_{\mathbb{M}_n(D)}D^n) = D$.

Artin-Wedderburn theorem. A ring R is left semisimple iff it is isomorphic to finite direct product of matrix rings over division ring.

Proof. Write ${}_RR$ as a direct sum of simple modules $\oplus_k n_k M_k$. Now we see that $\mathrm{End}(\oplus_k n_k M_k) = \prod_k \mathbb{M}_{n_k}(\mathrm{End}(M_k))$.

Definition. A bimodule ${}_SV_T$ is faithfully balanced if the ring homomorphisms $S \to \mathrm{End}(V_T)$ and $T \to \mathrm{End}({}_SV)$ are both isomorphisms.

Omnipresence: Start with any module ${}_SV$. (We may replace S by its image in $\mathrm{End}\, V$. Now S acts faithfully.) Now let $T = \mathrm{End}({}_SV)$, so that $V = {}_SV_T$. Then replace S by $\mathrm{End}(V_T)$. Now this bimodule is faithfully balanced.

Important example. For any ring R the ring ${}_{\mathbb{M}_n(R)}R^n_R$ is faithfully balanced bimodule. This includes for $n = 1$ the case ${}_RR_R$. Nice observation: ${}_{\mathbb{M}_n(R)}R^n$ is simple iff $n \neq 0$ and R is a division ring. If R is division ring, the statement is trivial. To prove the other implication we use Schur's lemma.

Artin-Wedderburn theorem. A ring R is a left semisimple ring iff R is a finite direct product of matrix rings over division rings: $\mathbb{M}_{n_i}(D_i)$.

Proof of uniqueness. It suffices to describe n_i and D_i in terms of the ring R. Note that $D_i^{n_i}$ is a simple R-module if we let other components of R act trivially. Therefore, the number of rings in the decomposition is equal to the number of simple left R-modules. D_i is the endomorphism ring of ith simple module. Now n_i is the dimension of this module over D_i. Also n_i is the multiplicity of V_i as composition factor. Moreover, n_i is the matrix size of the ith component in the Artin-Wedderburn decomposition.

Corollary. A left semisimple ring is also right semisimple ring and vice versa. Also a left semisimple ring is Artinian, and, therefore, Noetherian.

Corollary. A semisimple ring is a direct sum of its indecomposable ideals, which are uniquely determined up to a permutation.

Corollary. A commutative ring is semisimple iff it is isomorphic to a finite direct product of fields.

Now we want to relate simple rings to semisimple.

Theorem. If R is a simple ring, then R is semisimple iff $R = \mathbb{M}_n(D)$ where D is a division ring iff R is left artinian.

Proof. We only need to prove that an artinian simple ring is semisimple. Take a left minimal ideal I of R. Now take let B be the sum of all left ideals that are isomorphic to this one. We easily see that this sum is also a right ideal, therefore it coincides with the whole ring. By artinity we obtain the desired result.

Note that there is no simple classification of left Noetherian rings.

Maschke's theorem (1899). Suppose k is a field and G is a group. Then kG is semisimple iff the characteristic of k does not divide $|G|$ and G is finite. If kG is semisimple, we call this case ordinary. Otherwise we call it modular.

Modern proof. In the ordinary case we need to prove that every exact sequence of kG-modules splits. Fix a k-homomorphism $\lambda\colon V \to W$ such that λ is identity on W. Now average λ over all elements of G, obtaining a kG-homomorphism.

Assume that kG is semisimple. G may be infinite. Consider the augmentation map $\epsilon\colon kG \to k$, $\epsilon(\sum a_g g) = \sum_{g\in G} a_g$. Now take the kernel of ϵ. This kernel splits. Denote by J its complement, which is a left ideal. For any $\alpha \in J$ such that $\alpha \neq 0$ we have $(g-1)\alpha \in (\ker\epsilon) \cap J = 0$. Therefore, $\alpha = g\alpha$ for all g, hence $\alpha = a\sum_{h\in G} h$. Moreover, G is finite. Now note that $\epsilon(\alpha) = a|G| \neq 0$, therefore $\operatorname{char} k$ does not divide $|G|$.

Maschke's original approach involved hermitean forms and orthogonal complements. In real case replace the hermitean product by ordinary inner product.

Proposition. Every complex representation is equivalent to a unitary representation. Every real representation is equivalent to a orthogonal representation.

Proof. Average an arbitrary hermitean form over all elements of group.

Corollary. Every real or complex G-submodule splits.

Proof. Take an orthogonal complement.

From now on we assume that $\operatorname{char} k$ does not divide $|G|$. Denote by $M_i = D_i^{n_i}$ the simple G-modules. We have $D_i = \operatorname{End}_R(M_i)$. Let $m_i = \dim_k M_i$ and $d_i = \dim_k D_i$. We have $m_i = n_i d_i$, the Wedderburn components of kG are $\operatorname{End}(M_i)$. We have ${}_RR = \oplus_k n_k M_k$. Now we obtain the general magic equation $|G| = \sum_i n_i m_i = \sum_i n_i^2 d_i \leq \sum_i m_i^2$.

If k is algebraically closed, then $d_i = 1$ and we obtain the usual magic equation.

The number r is equal to the number of conjugacy classes. To prove this note that the number of conjugacy classes is equal to the dimension of the center of kG.

Example. Suppose G is an abelian group. We have $n_i = 1$ and every D_i is a field, therefore kG is a direct product of fields. Every D_i supports a simple kG-module. If k has all necessary roots of unity, then $D_i = k$. Therefore, if G and H are abelian and $|G| = |H|$, then $\mathbb{C}G$ is isomorphic to $\mathbb{C}H$.

Example. Construct the Wedderburn decomposition of $\mathbb{Q}G$, where G is a cyclic group of order n. We have $\mathbb{Q}G = \mathbb{Q}[t]/(t^n - 1)$. By Chinese remainder theorem we have $\mathbb{Q}G = \prod_{d\backslash n} Q[t]/(\Phi_d(t)) = \prod_{d\backslash n} \mathbb{Q}(\zeta_d)$. The last product is the Wedderburn decomposition.

Now we replace $\mathbb{Q}$ by $\mathbb{R}$. We have two cases. In the first case the order of group is even. In this case it is easy to see that the Wedderburn decomposition is $\mathbb{R}\times\mathbb{R}\times\mathbb{C}^{n/2-1}$. In the remaining case the Wedderburn decomposition is $\mathbb{R}\times\mathbb{C}^{(n-1)/2}$.

G	$\mathbb{Q}G$	$\mathbb{R}G$	$\mathbb{C}G$
C_{12}	$\mathbb{Q}^2 \times \mathbb{Q}(\omega)^2 \times \mathbb{Q}(i) \times \mathbb{Q}(\zeta_{12})$	$\mathbb{R}^2 \times \mathbb{C}^5$	$\mathbb{C}^{12}$
S_3	$\mathbb{Q}^2 \times \mathbb{M}_2(\mathbb{Q})$	$\mathbb{R}^2 \times \mathbb{M}_2(\mathbb{R})$	$\mathbb{C}\times\mathbb{C}\times\mathbb{M}_2(\mathbb{C})$
Q_8	$\mathbb{Q}^4 \times \mathbb{H}_{\mathbb{Q}}$	$\mathbb{R}^4 \times \mathbb{H}$	$\mathbb{C}^4 \times \mathbb{M}_2(\mathbb{C})$
D_4	$\mathbb{Q}^4 \times \mathbb{M}_2(\mathbb{Q})$	$\mathbb{R}^4 \times \mathbb{M}_2(\mathbb{R})$	$\mathbb{C}^4 \times \mathbb{M}_2(\mathbb{C})$
A_4	$\mathbb{Q}\times\mathbb{Q}(\omega)\times\mathbb{M}_3(\mathbb{Q}(\sqrt{-7}))$	$\mathbb{R}\times\mathbb{C}\times\mathbb{M}_3(\mathbb{R})$	$\mathbb{C}^3 \times \mathbb{M}_3(\mathbb{C})$
$\langle\phi,\sigma\rangle$	$\mathbb{Q}\times\mathbb{Q}(\omega)\times$		$\mathbb{C}^3 \times \mathbb{M}_3(\mathbb{C})^2$
S_4			$\mathbb{C}^2 \times \mathbb{M}_2(\mathbb{C}) \times \mathbb{M}_3(\mathbb{C})^2$

Theorem of splitting fields

How simple modules behave under extension of scalars?

Theorem. A left artinian ring R is simple iff there exists a faithful simple left R-module M.

Proof. (Jacobson radical argument.) Assume that there is a faithful simple left module M over left artinian ring R. First we show that R is semisimple. Check that ${}_RR$ is a semisimple module. It suffices to show that any minimal left ideal $I \subset {}_RR$ splits. Fix $0 \neq a \in I$. Then $am \neq 0$ for some $m \in M$. Hence $Ram = M$. In particular, $m = ram$ and $(1-ra)m = 0$, therefore $R(1-ra)$ is a proper left ideal of R. Denote by J a maximal left ideal that contains $R(1-ra)$. Note that $I \cap J = 0$, therefore ${}_RR = I \oplus J$.

Burnside's theorem. Denote by D a skew field. If we take a ring R we have $D \subset R \subset \mathbb{M}_n(D)$. Let $M = D^n$. Then $R = \mathbb{M}_n(D)$ iff ${}_RM$ is simple and $\operatorname{End}({}_RM)) = D$.

Proof. Certainly, R is a left artinian ring. Moreover, M is simple and faithful. By the theorem above, R is simple. The Wedderburn theory applies to R, hence $R = \mathbb{M}_n(D)$.

Homomorphism theorem. Let R be a k-algebra (not necessarily finite-dimensional) and $K \supset k$ be a field extension. Let M and N be left R-modules. Then the natural map $\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{R^K}(M^K, N^K)$ is a K-vector space isomorphism. Here $R^K = K \otimes_k R$ and $M^K = K \otimes_k M$.

Proof. First Course, page 104.

Theorem. Let R be a k-algebra (not necessarily finite-dimensional). Let ${}_RM$ be a simple R-module. We have $\operatorname{End}({}_RM) = k$ iff $R \to \operatorname{End}(M_k)$ is surjective iff for any field extension $K \supset k$ the module M^K is a simple R^K-module iff there is an algebraically closed field extension $E \supset k$ such that M^E is a simple R^E module.

Proof. (2) follows from (1) applied to image of $R \to \operatorname{End}(M_k)$. (3) follows from (2) because we have surjective morphism $R^K \to (\operatorname{End}(M_k))^K = \operatorname{End}_K(M^K)$, hence M^K is R^K-simple. (1) follows from (4): $\operatorname{End}_{R^E}(M^E) = E$ is isomorphic to $(\operatorname{End}_R(M))^E$, therefore $\operatorname{End}({}_RM) = k$.

Definition. Let R be a finite-dimensional k-algebra. Field extension $K \supset k$ is a splitting field for R if every simple R^K-module is absolutely simple.

Absolutely simple means that its isomorphism ring is the ring of scalars. Alternatively, it stays simple under any extension.

Algebraic closure is an example of splitting field.

A simple module over semisimple ring is absolutely simple if its Wedderburn component looks like $\mathbb{M}_n(k)$.

Proposition 1. If $k \subset K$ is a field extension, R is a finite-dimensional k-algebra, then any simple R^K-module U is a composition factor of V^K for some simple R-module V.

Proof. Insert U into composition series of ${}_RR^K$.

Proposition 2. Splitting field represents "stable state". If $k \subset K \subset L$ are field extensions, K is a splitting field for k-algebra R and U_i is a complete set of R^K simple modules, then U_i^L is a complete set of R^L simple modules. And L is a splitting field for R. Moreover, assume that L is a splitting field. Then K is a splitting field iff every simple R^L-module is defined over K (is isomorphic to tensor product of L and some R^K-simple module.

Theorem. For any finite-dimensional k-algebra there is a finite field extension K/k that splits R.

Proof. Take $L = \bar{k}$ and construct K. Take maximal left ideals A_i in R^L. Then $M_i = R^L/A_i$ gives a complete set of R^L-simples. Take a big finite extension K within L such that $A_i \cap R^K$ contains an L-basis of A_i.

Back to groups. If $\operatorname{char} k$ does not divide $|G|$ and M_i is a complete set of kG-simple modules. Then $|G| \leq \sum_i (\dim_k M_i)^2$ with equality iff k is a splitting field.

Definition. A set $I \subset R$ is called nil if every element of I is nilpotent.

Theorem. Any nil left ideal $J \subset R$ is nilpotent. The sum of two nil ideals is also nil.

Proof. Trivial. The first part uses descending chain condition on left ideals and Nakayama lemma.

Proposition. Let $\operatorname{rad}(R)$ (Wedderburn Radical) be the sum of all nil ideals of R. Then this is a nilpotent ideal that contains all nil left (or right) ideals J.

Corollary. The ring $R/\operatorname{rad}(R)$ is semisimple. The Wedderburn radical is the set of all elements that kill all simple modules.

Proof. After replacing R by $R/\operatorname{rad}(R)$ we may assume $\operatorname{rad}(R)=0$. We want ${}_RR$ to be semisimple. Since it is artinian, it is sufficient to check that every minimal left ideal E splits off. Note that $E^2 \neq 0$, since otherwise it would be nil. Hence there is an $a \in E$ such that $Ea \neq 0$. We must have $Ea = E$. Write $a = ea$ where $e \in E$. Consider left ideal $X = \{x \in E \mid xa = 0\}$. We have $X \subset E$ and $X \neq E$, hence $X = 0$. We have $a = ea = e^2 a$, therefore $e = e^2$. Then $E = Re$ splits.

Let $I = \operatorname{rad}(R)$. Want $IM = 0$ for all simple M. Otherwise we have $IM = M$ and $M = I^N M = 0$. Contradiction.

Let $\bar{R} = R/\operatorname{rad}(R)$. $\bar{R}$ is semisimple. So simple $\bar{R}$-modules are certainly simple R-modules. The direct sum of a complete set of simple modules over $\bar{R}$ is faithful over $\bar{R}$. If r kills all R-simples, then it kills all $\bar{R}$ simples, hence $\bar{r} = 0$, therefore $r \in \operatorname{rad}(R)$.

We can define general $\operatorname{rad}(R)$ as the set of all elements that kill all simple modules. This is the Jacobson radical.

Nil radicals present an obstruction to semisimplicity. Factorization by largest nil ideal yields a semisimple ring. A ring and its factor ring by Wedderburn radical have the same simple modules.

Recall that the (Wedderburn) radical of a finite-dimensional algebra is the set of all elements that kill all simple modules. If we apply this definition for arbitrary ring, we obtain Jacobson radical.

Theorem. Suppose that k has characteristic p and G is a finite p-group. Let M be a simple kG-module. Then G acts trivially on M. In particular, $\dim_k M = 1$.

Proof. Suppose that $|G| = p^n$. We use induction on n. Fix a central element of order p. Obviously, $D(c) - I$ is nilpotent. Hence c acts trivially on its kernel. Now view M as simple $k[G/\langle c\rangle]$-module and induct.

Corollary. Under same hypothesis, $\operatorname{rad}(kG) = I$, where I is the augmentation ideal. And $I^{|G|} = 0$. Every proper left (right) ideal of kG is contained in I. So kG is a noncommutative local ring.

Proof. For every $g \in G$ the element $g - 1$ acts trivially on every single kG-module. Then $g - 1 \in \operatorname{rad}(kG)$. Therefore, $I \subset \operatorname{rad}(kG)$ and $I = \operatorname{rad}(kG)$. We know that this ideal is nilpotent. The composition series of ${}_{kG}kG$ has exactly $|G|$ factors. We know that I kills each composition factor. Hence, $I^{|G|} = 0$.

Refinement. Suppose that k has characteristic p and G is a finite group with a normal p-subgroup H. Then H acts trivially on any simple kG-modules.

Proof. Let M_0 be the submodule consisting of all elements that are invariant under action of center of G. It is a kG-submodule, hence it coincides with M.

Corollary. If k has characteristic p and G has a normal p-Sylow subgroup H. Then simple kG-modules are the same as simple $k(G/H)$-modules. We are back to non-modular representations. Moreover, $\operatorname{rad}(kG) = \sum_{h \in H} kG(h-1)$.

Proof. First we verify that RHS is an ideal. Then we verify that RHS is contained in $\operatorname{rad}(kG)$. At last we observe that if we mod out RHS we obtain a semisimple ring $k[G/H]$. Hence, $\operatorname{rad}(kG)$ is contained in RHS.

Chapter 2

Theory of Characters.

If M is a left module over finite dimensional k-algebra, such that M is finite-dimensional over k, then its character is a function $\chi = (r \to \operatorname{tr}(m \to rm))$. If $0 \to M' \to M \to M'' \to 0$ is exact, then $\chi_M = \chi_{M'} + \chi_{M''}$.

Theorem. Suppose that k has characteristic zero and R is finite-dimensional k-algebra. Then isomorphism classes of semisimple R-modules are determined by their characters.

Proof. Let $R/\operatorname{rad}(R) = \prod_i W_i$ be the Wedderburn decomposition. Let M_i be the corresponding R-simples. Let $M = \oplus_i l_i M_i$. We need to compute l_i in terms of χ_M. Fix $s \in R$ such that $s_j = [i = j]$. We have $\chi_M(s) = l_i \chi_{M_i}(s) = l_i \dim_k M_i$. Since the characteristic is zero, l_i is uniquely determined.

Back to groups. If k has characterstic zero, two representation of G are equivalent iff their characters coincide.

Proposition. In any characteristic if the order of g is equal to m, then $\chi_D(g)$ is equal to the sum of mth roots of unity in algebraic closure of k. In particular, if $k = \mathbb{C}$, then $\chi_D(g)$ is an algebraic integer in $\mathbb{Q}(\zeta_m)$.

Definition. The kernel of character is the set of all group elements g such that $\chi(g) = \chi(1) \in k$.

Theorem. If k has characteristic zero, then $\ker \chi = \ker(D)$. If χ_i are all the characters of kG-simples, then the intersection of their kernels is the trivial group.

Proof. We can assume that k is algebraically closed. If $g \in \ker \chi$, then $n = \chi(1) = \sum_i \lambda_i$, where λ_i are the eignevalues of $D(g)$. Clearly, λ_i. Moreover, $D(g)^{|G|} = 1$, hence $D(g)$ is diagonalizable. Hence $D(g) = 1$.

Definition. Let $\mathrm{Irr}(G)$ be the set of all characters of irredecuble representations. For a character χ define its center as the set of all elements g such that $|\chi(g)| = \chi(1)$.

Theorem. An element g belongs to the center of a character iff $D(g) = \lambda I$ iff $g \in Z(G/\ker \chi)$.

Proof. As before, conclude that $D(g)$ is diagonalizable and all of its eigenvalues are equal to each other. Conversely, if $g \in Z(G/\ker \chi)$ we conclude that g belongs to the center.

Notation. We always assume that $\mathrm{char}\, k$ does not divide $|G|$. We have $kG = \prod \mathbb{M}_{n_i}(D_i)$, $m_i = n_i d_i$ etc. Let χ_i be the ith irreducible character.

Centrally primitive idempotents theorem. Let e_i be the identities of the Wedderburn components. These are centrally primitive idempotents in kG. A central idempotent is called primitive if it is nonzero and we cannot represent it as a sum of two nonzero central idempotents which have zero product (are orthogonal). We have $e_i = n_i/|G| \sum_g \chi_i(g^{-1})g$. In particular, $\mathrm{char}\, k$ does not divide n_i. If, in addition, k is a splitting field, then $C_g = |g| \sum_i \chi_i(g)e_i/n_i$.

Proof. Suppose that $e_i = \sum_h a_{i,h}h$. If χ_r is the regular character, then we have $\chi_r(e_i g^{-1}) = a_{i,g}|G|$. We have $a_{i,g} = |G|^{-1}\chi_r(e_i g^{-1}) = |G|^{-1} n_i \chi_i(g^{-1})$. To prove the second relation, note that $C_g = \sum b_{g,i} e_i$. Applying χ_j on both sides, we have $|g|\chi_j(g) = b_{g,j} n_j$. (Under splitting assumption we have $d_i = 1$.) Hence, $b_{g,j} = |g|\chi_j(g)/n_j$.

Theorem. Frobenius integrality theorem. Suppose k has characteristic zero and is a splitting field. Then $|g|\chi_i(g)/n_i$ is an algebraic integer. Moreover, $\dim M_i = n_i$ divides $|G|$.

Proof. The center of kG is $\prod_i ke_i$. Now $C_g \in \mathbb{Z}G$ is a ring that is finitely generated as an abelian group. Projecting upon the ke_i we get images that are algebraic integers. Finally, $e_i = n_i |G|^{-1} \sum_g \chi_i(g^{-1})g$. We have $|G|/n_i \in \sum_g AC_g \subset \sum_g A\left(\sum_j Ae_j\right)\sum_j Ae_j$. Hence $|G|/n_i$ is a rational integer.

First orthogonality relation. (No splitting field assumption.) For all i and j we have

$$|G|^{-1} \sum_{g \in G} \chi_i(g^{-1})\chi_j(hg) = [i = j]\chi_i(h)/n_i.$$

Proof. Use the fact that $e_i e_j = [i = j]e_i$. Recall that $e_i = n_i|G|^{-1} \sum_g \chi_i(g^{-1})g$. Comparing coefficients of h^{-1} we find out that $n_i|G|^{-1} n_j |G|^{-1} \sum_g \chi_i(g^{-1})\chi_j(hg) = [i = j] n_i |G|^{-1} \chi_i(h)$.

Corollary. If k has characteristic zero (no splitting field assumption), then a represetation is absolutely irreducible iff $\sum_g \chi(g^{-1})\chi(g) = |G|$.

Proof. Say $\chi_D = \chi_i$ with $d_i = 1$. Apply FOR with $i = j$. To prove the converse, write $\chi = \sum_i p_i \chi_i$. Then $|G| = \sum_g (\sum_i p_i \chi_i(g^{-1}))(\sum_j p_j \chi_j(g)) = \sum_{i,j} p_i^2 d_i |G|$. Since $\mathrm{char}\, k = 0$, we have $\sum_i p_i^2 d_i = 1$. Hence $p_i = d_i = 1$ for some i and $p_j = 0$ for all other j.

Now consider the set of all functions $\mu\colon G \to k$ that are constant on conjugacy classes. Define a k-bilinear form $[\mu, \nu] = |G|^{-1} \sum_g \mu(g^{-1})\nu(g) \in k$.

Corollary. Assume that k is a splitting field. Then χ_i form an orthonormal k-basis. Moreover, for any $\mu \in F_k(G)$ we have $\mu = \sum_i [\mu, \chi_i]\chi_i$ (Fourier expansion). We also have Plancherel formula: for all μ and ν in $F_k(G)$ we have $[\mu, \nu] = \sum_i [\mu, \chi_i][\nu, \chi_i]$. Assuming characteristic zero we see that $\mu \in F_k(G)$ is of the form χ_M for some kG-module M iff $[\mu, \chi_i]$ are all nonnegative integers. Moreover, M is irrdeucible iff $[\mu, \mu] = 1$.

Second orthogonality relation. Suppose that k is a splitting field and g and h are two elements of G. Then $\sum_i \chi_i(g)\chi_i(h^{-1}) = [g \sim h]|C_G(g)|$.

Proof. Use CPI and CPI for splitting field case.

Applications to permutation characters.

Burnside's lemma. The number of orbits of an action of a finite group G on a set E is equal to $[1, \pi] = |G|^{-1} \sum_g \pi(g)$, where $\pi(g)$ is the number of elements that are fixed by the element g.

Proof. It is sufficient to prove the lemma for transitive case. We have $|G| = n|G_i|$. Now $\sum_g \pi(g) = n|G_i|$.

Theorem. Suppose G acts transitively on E and let t be the number of orbits of G_1. Then $t = [\pi, \pi]$.

Proof. Expand $|G| \cdot [\pi, \pi]$.

Lemma. Suppose G is transitive on E and $n \geq 2$. Then G is doubly transitive iff $t = 2$ where t is the number of G_1-orbits on E. Here G_1 is the stabilizer of an element.

Theorem. Suppose G is transitive on E. Then G is doubly transitive iff $\bar{V}$ is an absolutely irreducible kG-module. Here $\bar{V}$ is reduced kG-module corresponding to the factor of free k-module on E by all elements with zero sum.

Proof. Let $\chi = \chi_{\bar{V}} = \pi - 1$. Then $[\chi, \chi] = [\pi - 1, \pi - 1] = [\pi, \pi] - 2[\pi, 1] + [1, 1] = t - 2m + 1 = t - 1 \in k$. We know that $\bar{V}$ is absolutely irreducible iff $[\chi, \chi] = 1$ iff $t = 2$ iff G is doubly transistive.

Now assume that k is a splitting field. Let g_i be a set of representatives of conjugacy classes. The the character table $C_{i,j} = \chi_i(g_j)$. Let $B_{i,j} = |g_i| \cdot |G|^{-1}\chi_j(g_i^{-1})$.

Theorem. FOR holds iff $CB = 1$. SOR holds iff $BC = 1$. Hence, all the statements are equivalent to each other.

Proof. Trivial substitution.

If $k = \mathbb{C}$ and $\mu = \sum_i a_i\chi_i$ for real a_i, then $\overline{\mu(g)} = \mu(g^{-1})$. On $\mathbb{C}$ one usually uses a different pairing: $\langle \mu, \nu \rangle = [\mu, \bar{\nu}]$. This is a positive definite hermitean form. Irreducible characters form an orthonormal basis as before. We also have Fourier expansion and Plancherel formula.

Corollary. For any class function $\chi \in F(G)$ define $\mathbb{Q}(\chi) = \sum_g \mathbb{Q}\chi(g) \subset \mathbb{C}$. If χ is an irreducible character, then $\mathbb{Q}(\chi)$ is an algebraic number field.

This follows from char.pdf: for any irreducible character χ we have $\chi(g)\chi(h) = \chi(1)|G|^{-1} \sum_z \chi(gh^z)$.

Theorem. Every $\chi \in \mathrm{Irr}(G)$ satisfies char.pdf: $\chi(g)\chi(h) = \chi(1)|G|^{-1} \sum_z \chi(gh^z)$, where $h^z = z^{-1}hz$.

Corollary. If $\chi \in \mathrm{Irr}(G)$, then $\mathbb{Q}(\chi)$ is an abeliean field extension of $\mathbb{Q}$.

Proof of Corollary. $\mathbb{Q}(\chi)$ is a finite-dimensional $\mathbb{Q}$-domain, hence a field extension of $\mathbb{Q}$. Observe that $\chi(g) \in \mathbb{Q}(\zeta)$ where $\zeta = \exp(2\pi i|G|^{-1})$. We see that $\mathbb{Q}(\chi)$ is Galois over $\mathbb{Q}$ with Galois group $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) = G/H$ which is an abelian group.

Proof of Theorem. For $g \in G$ define $\alpha_{i,j,g} = \#\{(g', g'') \in G \times G \mid g = g'g'' \wedge g' \sim g_i \wedge g'' \sim g_j\}$. Note that $g \mapsto \alpha_{i,j,g}$ is a class-function. Let's write $\alpha_{i,j,p} = \alpha_{i,j,g}$ where g belongs to pth conjugacy class. Then $C_iC_j = \sum_p \alpha_{i,j,p}C_p$. Apply π_l to C_iC_j. We get $|g_i| \cdot |g_j|\chi_l(1)^{-1}\chi(g_i)\chi(g_j) = \sum_p \alpha_{i,j,p}|g_p|\chi(g_p)$.

Converse Theorem. An aribtrary function $\mu\colon G \to \mathbb{C}$ is a scalar multiple of irreducible character iff μ satisfies char.pdf.

Sketch of Proof. Last part: write $\mu = z\chi$ where $\chi \in \mathrm{Irr}(G)$. We have $\mu(1) = z\chi(1) \in \mathbb{R}^+$. Hence $z \in \mathbb{R}^+$. Also $1 = \langle \mu, \mu \rangle = z^2\langle \chi, \chi \rangle = z^2$. Hence $z = 1$ and $\mu = \chi$.

First part: suppose that char.pdf holds. Suppose $\mu(1) = 0$. Then $\mu = 0 \cdot 1_G$. Assume $\mu(1) \neq 0$. In char.pdf set $g = 1$. We have $\mu(h) = |G|^{-1} \sum_z \chi(h^z)$. Hence μ is a class function. Define $\pi\colon Z(\mathbb{C}G) \to \mathbb{C}$ by $\pi(C_i) = |g_i|\mu(g_i)\mu(1)^{-1} \in \mathbb{C}$. Check that π is a $\mathbb{C}$-algebra homomorphism. It is enough to check that $\pi(C_i)\pi(C_j) = \sum \alpha_{i,j,p}\pi(C_p)$ via char.pdf. After this, $\pi = \pi_l$ for some μ. Then you check $\mu = z\chi_l$.

New Representations from Old

(1) Twist a group representation by group automorphism. Only outer automorphisms yield nontrivial twistings. (2) Twist a group representation by field automorphism. (3) Twist a group character by field automorphism of character field.

In the last case we obtain a group character because char.pdf stays true under field automorphism. To connect (2) and (3) note that we can extend (non-uniquely) a field automorphism of a character field and obtain the twisted representation.

(4) If V is a kG-module, then V^* is also a kG-module: $(g\lambda)(v) = \lambda(g^{-1}v)$. Obviously, $\chi_{D'}(g) = \chi_D(g^{-1})$ for all g.

Note that the group inverse sends conjugacy classes to conjugacy classes. We say that a conjugacy class is real if it is invariant under group inverse. A character over complex numbers is real-valued if all of its values are real.

Burnside Theorem 1. The number of real conjugacy classes is equal to the number of real-valued characters.

Proof (Brauer). By permuting pairs of complex-conjugated characters we get a matrix PC. Similarly, by permuting pairs of conjugacy classes which are mutually inverse we get a matrix CQ. Now $C^{-1}PC = Q$ and $\text{tr}(P) = \text{tr}(Q)$.

Corollary, $|G|$ is even iff the is a real-valued irreducible character $\chi \neq 1$.

Proof. From Burnside's theorem we obtain that the existence of χ is equivalent to existence of self-inverse conjugacy class.

Proposition. Suppose that the columns of character table are permuted arbitrarily. Then we can compute which column corresponds to the identity class. Moreover we can determine the sizes of conjugacy classes.

Proof. Suppose that $\chi \in \text{Irr}(G)$. Note that $\ker\chi = \{g \in G \mid \chi(g) = \chi(1)\}$. We will also prove that $\ker\chi = \{g \in G \mid 0 < \chi(g) \geq |\chi(h)|\}$. One of the inclusions is clear. The other inclusion is also trivial. Now recall that the intersection of the kernels of all characters consists of trivial element. Hence, the first column is uniquely determined by the character table. Now we can find $|g_j|$ by second orthogonality relation.

Theorem. The character table determines the position of the elements of the group commutant.

Proof. Recall that the commutant is equal to the intersection of all one-dimensional characters.

Theorem. The character table determines all normal subgroups.

Proof. Denote by $N_i = \ker\chi_i$ a set of normal subgroups. Obviously, their finite intersections constitute the set of all normal subgroups.

Theorem. G is simple iff all kernels are trivial except for the kernel corresponding to the first row.

Proof. Trivial consequence of the previous theorem.

Theorem. Given a normal subgroup of given group G, the character table of G determines the character table of factor group (although not its isomorphism class).

Proof. Take all irreducible characters whose kernel contains given normal subgroup. These characters correspond to all irreducible characters of factor group. Now remove all duplicate columns.

Theorem. The character table determines the position of the center. It also determines whether the group is abelian and its isomorphism class is uniquely determined.

Proof. The center is the set of all elements whose conjugacy class size is 1. It is also the intersection of the centers of all characters.

Theorem. The character table determines the position of upper central series. Hence it determines whether the group is nilpotent.

Proof. Trivial consequence of definitions and previous results.

Theorem. The character tables determines whether the group is solvable.

Proof. A group is solvable iff there is a normal series with p-groups as factors.

Proposition. If H is a normal subgroup of G, then $|C_{\bar{G}}(\bar{g})| \leq |C_G(g)|$.

Second Burnside theorem. If $|G|$ is odd, then it is congruent to the number of conjugacy classes modulo 16.

Proof. The only real irreducible character is the trivial character. Also, the dimensions of irreducible characters are odd, since they divide the order of group.

Corollary. If $|G| \leq 19$ is odd, then it is odd.

Theorem. If $|G|$ is odd and every prime that divides $|G|$ is congruent to 1 modulo 4, then $|G|$ and the number of conjugacy classes are congruent modulo 32.

Chapter 3

Tensor Products and Invariants.

Multiplicative structure on character ring provides a good tool for computation. We can construct new irreducible representations from existing ones. For example, we can now determine the character table of A_5 given only its icosahedral representation.

Another kind of product (outer product): Let V be a kG_1-module and W be a kG_2-module. Then $V\#W = V \otimes_k W$ is a kG-module, where $G = G_1 \times G_2$. Here $(g_1, g_2)(v \otimes w) = g_1 v \otimes g_2 w$. This construction is a special case of tensor product if we regard V and W as kG-modules.

Proposition. Suppose that V and V' are irreducible kG_1-modules and W and W' are irreducible kG_2-modules. Then $V\#W$ is isomorphic to $V'\#W'$ iff V is isomorphic to V' and W is isomorphic to W'. If V and W are absolutely irreducible, then $V\#W$ is also absolutely irreducible.

Proof. The first part is trivial if we regard $V\#W$ as kG_1-module and decompose it into irreducible parts. This implies that V is isomorphic to V'. The same argument goes for W'. The second part is proven as follows. Recall that kG is isomorphic to $kG_1 \otimes kG_2$. By Burnside's theorem the maps $D_1 \colon kG_1 \to \mathrm{End}_k(V)$ and $D_2 \colon kG_2 \to \mathrm{End}_k(W)$ are epimorphisms. Hence $\mathrm{End}(V\#W)$ is isomorphic to $\mathrm{End}(V) \otimes \mathrm{End}(W)$. Therefore the map $kG \to \mathrm{End}(V\#W)$ is epimorphic, hence $V\#W$ is absolutely irreducible.

Theorem. Suppose that k is the splitting field for G_1 and G_2. Suppose that $\operatorname{char} k$ does not divide $|G|$. Tensor products of all pairs of irreducible representations of G_1 and G_2 form complete set of irreducible representations of $G_1 \times G_2$.

Proof. Obviously all of these representations are non-isomorphic and absolutely irreducible. The number of conjugacy classes of direct product of groups is equal to the product of the corresponding numbers of conjugacy classes.

Note that the character table of direct product of groups is equal to the tensor product of corresponding character tables.

In particular, if some representations have the same character tables, then their tensor products with anything also have the same character tables.

Let V be a simple $\mathbb{C}G$-module and let $\chi = \chi_V$.

Frobenius theorem. $\chi(1)$ divides $|G|$.

Schur's theorem. $\chi(1)$ divides $[G : Z(G)]$.

Generalized Schur's theorem. $\chi(1)$ divides $[G : Z(\chi)]$.

Proof. Recall that $H = Z(\chi) = \{h \in G \mid |\chi(h)| = \chi(1)\}$ is a normal subgroup of G. Note that $V^n = \#^n V$ is an irreducible $\mathbb{C}G^n$-module. Also H acts on V as complex numbers: $hv = \lambda(h)v$. Now we see that $K_n = \{(h_1, \ldots, h_n) \in H^n \mid \lambda(\prod_i h_i) = 1\}$ is a normal subgroup of G^n that acts trivially on V^n. Hence V^n is a simple G^n/K_n-module. Note that $|H|^{n-1}$ divides K_n because projection of K_n onto first $n-1$ arguments is surjective. Now $|G^n/K_n| = |H| \cdot [G : H]^n / x$. By Frobenius this is divisible by $\chi(1)^n$. Hence $\chi(1)$ divides $[G : H]$.

Burnside-Brauer theorem. Assume that char $k = 0$ and D is an arbitrary representation of G. Suppose that the character of D is faithful and takes m distrinct values. Then any irreducible character is contained in ϕ^i for $0 \le i < m$.

Proof. We just obtain Vandermonde system with m equations and m unknowns. All unknowns must be zero. Here jth unknown is the sum $\sum_{\chi(g)=a_j} \chi(g^{-1})$.

Generally, if field characterstic does not divide the order of the group, then the scalar product of any two characters is an integer number.

Definition. Poincaré series P of a character is $\sum_{n\ge 0}[\phi^n, \chi]t^n$ for a given character ϕ.

Theorem. Poincaré series is a rational function in t.

Proof. Expand $[\phi^n, \chi]$ and change the order of summation. We obtain $|G|^{-1} \sum_g \chi(g^{-1})(1 - \phi(g)t)^{-1}$.

We can use this theorem to prove Burnside-Brauer theorem. Instead of working with ϕ^n we can work with the symmetric power of ϕ, which is denoted by $\phi^{(n)}$. Using $\phi^{(n)}$ we can form a new Poincaré series Q.

Theorem. In characteristic zero we have $\sum_{n\ge 0} \phi^{(n)}(g)t^n = \det(I - gt)^{-1}$. Hence,

$$Q_{\phi,\chi}(t) = |G|^{-1} \sum_g \chi(g^{-1}) \det(I - gt)^{-1}.$$

Proof. We can assume that our field is algebraically closed. The action of an element of the group on vector space is diagonalizable. Now recall that S^nV has monomial basis.

We are interested in $T(V)^G$ and $S(V)^G$, which are the fixed points of $T(V)$ and $S(V)$. Note that $\dim(T^nV)^G$ is the number of times 1_G appears in $T^n(V)$, i.e., $[\phi^n, 1_G]$.

Using theorems that we proved above we see that $P_{T(V)^G}(t) = |G|^{-1} \sum_g (1-\phi(g)t)^{-1}$. Also $P_{S(V)^G}(t) = |G|^{-1} \sum_g \det(I - gt)^{-1}$.

Possible use of formulae: Suppose we locate a graded subspace of invariants $I_0 \subset I$. If $P_{I_0} = P_I$, then $I_0 = I$!

What can we say about $I = S(V)^G$. An affine k-algebra is a commutative finitely-generated k-algebra. By Hilbert basis theorem, this algebra is a noetherian ring. Let us study more general situation: R is an affine k-algebra, and let G is a finite group acting on R as a group of k-algebra automorphisms. Let $I = R^G$.

Noether theorem. R over I is an integral ring extension. R is a finitely-generated I-module. $I = R^G$ is an affine k-algebra.

Proof. Pick an $r \in R$. Then $f_r(t) = \prod_g (t - gr) \in R[t]$ is a G-invariant polynomial. Now take some generating set r_i for R. Take all coefficients of polynomials f_r for all r_i. Let S be a subalgebra generated by these elements. This subalgebra is G-invariant. Each r_i is integral over S, hence R is generated by r_i as S-module. Now note that I is a finitely-generated S-module. This implies that I is an affine k-algebra.

Chapter 4

Some applications.

Definition. We say that an irreducible character is prime to g if $(\chi(1), |g|) = 1$.

Recall three facts about irreducible characters: Frobenius integrality theorem: $|g|\chi(1)^{-1}\chi(g)$ is an algebraic integer. The center of a character contains the group center. Here $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$. Moreover, $Z(\chi)/\ker(\chi) = Z(G/\ker\chi)$.

Burnside theorem 3. If χ is prime to g, then $\chi(g) = 0$ or $g \in Z(\chi)$.

Proof. Suppose that $1 = m\chi(1) + n|g|$. Consider $\alpha = \chi(g)/\chi(1) = m\chi(g) + n|g|\chi(g)/\chi(1)$. Assume $\chi(g) \ne 0$. Let α_i be the conjugates of α. Each α_i is an average of some roots of 1. Hence $|\prod_i \alpha_i| = 1$ and $|\alpha| = 1$. Hence $|\chi(g)| = \chi(1)$.

Corollary. If G is a nonabelian simple group and $\chi \ne 1$ is prime to $g \ne 1$, then $\chi(g) = 0$.

Burnside theorem 4. Suppose there is an element g such that $|g| = p^e$, p is a prime. Then G is not a simple group.

Proof. Assume G is simple. Apply second orthogonality relation. All $\chi_i(1)$ must be divisible by p.

Burside theorem 5. If $G = p^a q^b$ for prime p and q, then G is solvable.

Proof. Induction by the size of the group. Assume that $p \neq q$ and a and b are positive. Take a q-Sylow subgroup Q. Fix a nontrivial $g \in Z(G)$. Then $C_G(g)$ contains Q. If $e = 0$, then g generates a nontrivial normal subgroup. Otherwise, we apply Burnside theorem 4.

Corollary. If G is a simple group with a p-Sylow subgroup P that is abelian and χ is an irreducible character with $\chi(1) = p^r$ for $r > 0$, then $|P| = p^r$.

Proof. G is nonabelian. Take $g \in P$ such that $g \neq 1$. Then χ is prime to g. Hence $\chi(g) = 0$. Therefore, $|g|$ is a p-prime number. Now $\langle \chi_P, 1_P \rangle = |P|^{-1}\chi(1)$. Hence $|P| = \chi(1) = p^r$.

Proposition. If G is a simple group, χ is an irreducible character, $\chi(1) = p$ where p is a prime number, then the p-part of $|G|$ is p and $p \neq 2$.

Proof. It suffices to show that p-Sylow subgroup P is abelian. Since p divides $|G|$, we have $P \neq \{1\}$. Recall that G acts faithfully on V and $Z(\chi) = Z(G) = \{1\}$.

If V_P is a simple $\mathbb{C}P$-module, then $1 \neq Z(P) \subset Z(\chi_P) = \{g \in P \mid |\chi_P(g)| = p\} \subset Z(\chi) = \{1\}$. Hence V_P is not simple, hence any 2 elements of P thus commute.

Classification of finite simple groups. Series of simples groups: cyclic groups of prime order, A_n for $n \geq 5$, linear groups (Jordan, Dickson) and other groups of Lie type by Chevalley, sporadic simple groups by Mathiue and others ending at Monster, 26 in total.

Burnside was a pioneer in this area. Brauer was a visionary. His idea was to study the centralizer of an involution (element of order 2). Feit and Thompson (young hotshots) proved that odd order group are solvable. Gorenstein was the field marshall (20 year war).

Relationship between simple groups and the prime 2: $\chi(1) \neq 2$, the number of distinct primes divisors of $|G|$ is not 2, and 2 divides $|G|$.

Brauer'sprogram of classifying finite simple groups: prove theorems like "if G is a finite simple group that has centralizer of involution isomorphic to a given group T then G is isomorphic to one of the finite number of given groups".

We will illustrate this philosophy in the simplest case:

Theorem. Suppose we have an involution u whose centralizer has degree 2. Then $[G : [G, G]] = 2$. In particular, if G is simple, then it has order 2.

Proof. We know that $\chi_i(u)$ is the sum of eigenvalues. Every eigenvalue for u is 1 or -1. On the other hand we have SOR $\sum_i \chi_i(u)\chi_i(u^{-1}) = |C_G(u)| = 2$. Hence $2 = \sum_i \chi_i(u)^2$. Say, $\chi_2(u) = \pm 1 = \epsilon$ and all other $\chi_i(u) = 0$. We have another SOR: $0 = 1 \cdot 1 + \epsilon\chi_2(1)$. Hence $\chi_2(1) = 1$ and $\epsilon = -1$. A linear character cannot have 0 as a value. Hence we have 2 linear characters and $[G : [G, G]] = 2$.

Important tool of Frobenius-Schur indicator.

Theorem. For arbitrary representation V of finite group G with character χ we have $\chi_{S^2V}(g) = (\chi(g)^2 + \chi(g^2))/2$ and $\chi_{\Lambda^2V}(g) = (\chi(g)^2 - \chi(g^2))/2$.

Definition. (Frobenius-Schur Indicator.) Let k be an algebraically closed field of characteristic 0. Then $s(\chi) = |G|^{-1}\sum_g \chi(g^2) = |G|^{-1}\sum_g(\chi_{S^2V} - \chi_{\Lambda^2V}(g)) = \langle \chi_S, 1\rangle - \langle \chi_A, 1\rangle$ is an integer number.

Note that $\langle \chi_S, 1\rangle + \langle \chi_A, 1\rangle = \langle \chi^2, 1\rangle = |G|^{-1}\sum_g \chi(g)\chi(g) = \langle \chi, \bar{\chi}\rangle = [\chi \text{ is real}]$.

Corollary. For $\chi \in \mathrm{Irr}(G)$ there are exactly 3 possibilities:

$\langle \chi_S, 1\rangle$	$\langle \chi_A, 1\rangle$	$s(\chi)$
1	0	1
0	1	-1
0	0	0

Hence we have 3 possible types of irreducible characters. We can call them type 1, type 2, and type 3 charcters.

An example of dihedral group of order 8 and quaternion group shows that $s(\chi)$ cannot be determined from the character table. Abelian group does not have type 2 characters. A nontrivial character of odd group is always indefinite.

Frobenius-Schur theorem. An irreducible character has type 1 iff V is defined over reals.

The number of square root of a group element is a class function, which is a virtual character: $\theta = \sum_\chi s(\chi)\chi$. The number of involutions in G is equal to $t = \sum_{\chi\neq 1} s(\chi)\chi(1)$. Moreover, $t^2 \leq (s-1)(n-1)$, where $n = |G|$ and s is the number of conjugacy classes.

Proof. First note that $\langle\theta,\chi\rangle = |G|^{-1}\sum_g \theta(g)\chi(g^{-1}) = |G|^{-1}\sum_h \chi(h^{-2}) = s(\chi)$. Next note that $1+t = \theta(1) = \sum_\chi s(\chi)\chi(1) = 1 + \sum_{1\neq\chi\in\mathrm{Irr}(G)} s(\chi)\chi(1)$. And now $t^2 \leq (\sum_{\chi\neq 1}\chi(1))^2 = (\sum_{\chi\neq 1} 1\cdot\chi(1))^2 \leq (s-1)\sum_{\chi\neq 1}\chi(1) = (s-1)(n-1)$.

Corollary. If the group is even, then there is a non-trivial conjugacy class of size not exceeding $((n-1)/t)^2$.

Brauer-Fowler theorem. Let m be an integer number and G be a simple group. Let u be an involution such that $|C_G(u)| \leq m$. Then $|G| < (m^2)!$. In particular there exists only finitely many such groups G.

This result motivated Brauer's program.

Involution count formula. The number of involutions equals $\sum_{\chi\neq 1} s(\chi)\chi(1)$.

Proof of Brauer-Fowler theorem. Let $n = |G|$. Then $t \geq |u| = |G|\cdot|C_G(u)|^{-1} \geq n/m$. Existence of small class theorem implies that there is $g \neq 1$ such that $|g| < (n/t)^2 \leq m^2$. We have a small conjugacy class. The group is simple, hence it must act faithfully on the cosets of this class, therefore we can embed it into symmetric group of required size. More precisely, if $H = G$, then $1 \neq g \in Z(G)$, therefore $G = Z(G)$ and $|G| = 2$. If $H \neq G$, then $r = [G : H] > 1$. And G maps to the symmetric group of the set G/H, which has more than 1 element. This action is faithful.

Brauer's vision was to study finite simple groups by looking at and controlling the centralizer of an involution.

Chapter 5

Induced representations.

Two ways to get induced representations: restriction and tensor product (over noncommutative rings). If U is a kG-module and $H \subset G$, then U_H is kH-module obtained by restriction. If V is a kH-modules, then V^G is a kG-modules obtained by tensor product. Now $(kG)_{kH}$ is free of finite rank $[G : H]$. In particular, $\dim(V^G) = [G : H]\dim V$.

Suppose V is 1-dimensional (over k) kH-module. Then V^G is called monomial kG-module. If we use matricial representation, we see that D^G is represented by block monomial (generalized permutation) matrices.

We can easily see that $\chi^G(g) = \sum_i \dot\chi(g_i^{-1}gg_i)$. Here $\dot\chi$ is χ extended by zeros. Hence $\chi^G(g) = 0$ whenever g does not belong to any conjugate of H. In particular, if H is a normal subgroup of G, then $\chi^G(G \setminus H) = 0$. Moreover, if $H \subset Z(G)$, then $\chi^G = [G : H]\dot\chi$. Note that in general $\dot\chi$ is not a class function on G. We can only say that $\dot\chi$ does not depend on H-conjugacy class.

Corollary. If $\operatorname{char} k$ does not divide $|H|$, then $\chi^G(g) = |H|^{-1}\sum_{t\in G}\dot\chi(t^{-1}gt)$.

Philosophy. Many irreducible representations of G are monomial.

Lemma. If $f\colon S \to R$ is a ring homomorphism. If V is an S-module, then $V^R = R\otimes_S V$ is an R-module. There is a natural abelian group isomorphism between $\mathrm{Hom}_R(V^R, U) = \mathrm{Hom}_S(V, U_S)$.

Proof. Obvious.

Frobenius reciprocity. Suppose that $\operatorname{char} k$ does not divide $|G|$. (Semisimple situation.) If $H \subset G$, U is a simple kG-module, V is a simple kH-module, m is the number of copies of V in U_H, n is the number of copies of U in V^G. Then $m > 0$ iff $n > 0$. Also $m = n$ iff $\dim_k \mathrm{End}_{kG} U = \dim_k \mathrm{End}_{kH} V$. The last assumption holds when U and V are absolutely irreducible (the dimension is 1 of endomorphism ring is 1 for such modules).

Proof. The first statement follows from the previous lemma. The second one follows from decomposition into irreducible modules. We have $m\,\mathrm{End}_{kH} V = n\,\mathrm{End}_{kG} U$.

Corollary. If $\operatorname{char} k$ does not divide $|G|$, $H \subset G$, k is a splitting field for G and H, and U_i and V_j are all simple kG and kH modulesw, then $(U_i)_H = \oplus_j a_{i,j} V_j$ iff $V_j^G = \oplus_i a_{i,j} U_i$.

Theorem. In semisimple case without splittig field assumption we have $i(W, U) = [\chi_W, \chi_U] \in k$, where W and U are finite-dimensional kG-modules. Here $i(W, U) = \dim \operatorname{Hom}_{kG}(W, U)$ is the intertwining number. Moreover, if U is a kG-module and V is a kH-module, then $[\chi_U, \chi_V^G] = [(\chi_U)_H, \chi_V]$.

Proof. Using additivity reduce everything to simple modules. Apply first orthogonality relation.

Theorem. (No splitting field assumption.) If U is a kG-module and V is a kH-module, then $V^G \otimes_k U = (V \otimes (U_H))^G$ as kG-modules. Taking characters we obtain $(\nu^G)\mu = (\nu \mu_H)^G$.

A couple of quick applications. Suppose that $\operatorname{char} k = 0$ and G acts transitively on E. Identify E with G/H where H is the stabilizer of a group element. Then kE is the induced representation of the trivial representation of H. Let π be the fixed point counter: $\pi = 1_H^G$. Now $[\pi, 1]_G = [1_H^G, 1]_G = [1, 1]_H = 1$ (Burnside lemma), $[\pi, \pi] = [1_H^G, \pi]_G = [1, \pi_H]_H$ is the number of H-orbits on E. This is another old result.

If H is a subgroup of abelian group G. Then any linear character ν of H extends to a character of G. Proof: $\mathbb{C}^*$ is a divisible, hence injective $\mathbb{Z}$-module. Character proof: $\nu^G = \oplus \mu_i$, hence $\nu = (\mu_i)_H$.

Restriction and induction for class functions. We assume that $\operatorname{char} k$ does not divide $|G|$. Let $F_k(G)$ be the k-algebra of class functions on G. Restriction: $F_k(G) \to F_k(H)$ is a ring homomorphism. Induction: if $\nu \in F_k(H)$, then $\nu^G : G \to k$ is defined by $\nu^G(g) = |H|^{-1} \sum_t \dot{\nu}(t^{-1} g t) = |H|^{-1} \sum_t \dot{\nu}(g^t)$. Here $\dot{\nu}$ is ν extended by zero. Note that if $g \notin \cup H^t$ then $\nu^G(g) = 0$. Here $\dot{\nu} = 0$ is an extension of ν. If ν is a character, then ν^G is the induced character.

Proposition. If $\nu \in F_k(H)$, then $\nu^G \in F_k(G)$. $(\nu^E)^G = \nu^G$. $[\nu^G, \mu]_G = [\nu, \mu_H]_H$. $(\nu \mu_H)^G = \nu^G \mu$.

Proof. Trivial.

Definition. A group G is called monomial group if every irreducible complex representation of G is monomial.

Examples. Q_8, D_n, A_4, S_4, S_3, special group of order 21.

Non-monomial groups. BT_{24} has 2-dimensional irreducible character, which is not monomial because the group does not have index-2 subgroup. A_5 is not monomial because three of its characters that have order 3 and 4 are not monomial because A_4 has no subgroup of index 3 and 4. Similarly, S_5 is not monomial.

Two facts about monomial groups. If $|G| < 24$, then it is monomial. Nilpotent groups are monomial. Monomial groups are solvable.

A group is nilpotent iff it is a direct product of Sylow p-groups. Monomial groups are closed under direct products. p-groups are monomial.

Definition. A Hall subgroup is a subgroup H such that $|H|$ and $[G : H]$ are relatively prime. This is a generalization of Sylow subgroup.

Third application. If H is a Hall subgroup and $h \in H \cap Z(G)$. Then $h \in G'$ iff $h \in H'$.

Proof. Assume that $h \in G'$ and $h \notin H'$. Fix a 1-dimensional $\mathbb{C}H$-module V on which h acts nontrivially. Fix a simple $\mathbb{C}G$-module $U \subset V^G$ whose dimension is relatively prime to p. This is possible only in Sylow case. From Frobenius reciprocity it follows that V is a simple submodule of U_H. Now h acts on U by scalar multiplication by some $\lambda \in \mathbb{C}^*$. Also the dimension of U and $|H|$ are relatively prime. If $h \in G'$ then $\det(D(h)) = \lambda^{\dim U} = 1$. On the other hand $\lambda^{|H|} = 1$.

Goal for the rest of the course. To prove the following application of induced representations: If G acts transitively on set E such that any non-indentity element has at most one fixed point, then for any two fixed-point free elements their product is also fixed-point free unless it is equal to 1.

Mackey theorems. Suppose H and K are subgroups of G, V is a kH-module. (1) What can we say about $(V^G)_K$? (2) When can we say V^G is simple.

If $G = \cup_i g_i H$, then $V^G = \oplus_i g_i \otimes V$. We have $V = 1 \otimes V \subset V^G$. In fact $gV = g(1 \otimes V) = g \otimes V$. Note: G permutes the gV's transitively. (If $g \in g_i H$, then $gV = g_i V$.) Also H is the isotropy subgroup of V.

Recognition criterion for an induced module. (H is not given a priori.) Suppose U is a kG-module such that $U = \oplus_i V_i$, where V_i are k-vector subspaces. Suppose G acts on U in such a way that U_i are permuted transitively. Let H be an isotropy subgroup of V. Then V is a kH-module and $U = V^G$.

Proof. It is sufficient to prove that $V^G = kG \otimes_{kH} V \to U$ is surjective and two modules have the same dimension.

Theorem. Suppose H and K are subgroups of G, V is a kH-module. Given K and H write the double-coset decomposition $G = \cup_{s\in S} KsH$. For any $s \in G$ define $K_s = sHs^{-1} \cap K$. Then $sV \subset V^G$ is a kK_s-submodule. We have $(V^G)_K = \oplus_{s\in S}(sV)^K$.

Proof. Define $V(s) = \sum_{g\in KsH} gV \subset V^G$. Note that $V(s)$ is a kK-submodule. Now we check that $V(s) = (sV)^K$ as kK-modules. To see this, apply induced module criterion: K acts transitively on $\{gV \mid g \in KsH\}$. Now the isotropy subgroup of sV is: $xsV = sV$ iff $x \in sHs^{-1} \cap K = K_s$.

Theorem. Mackey's irreducibility criterion. We assume that $\operatorname{char} k = 0$. V^G is absolutely irreducible iff V is absolutely irreducible and for any s in $S \setminus \{1\}$ two H_s-modules sV and V_{H_s} have no common composition factors.

Proof. We write $[U, W]$ for $[\chi_U, \chi_W]$. $[V^G, V^G] = [V, (V^G)_H]_H = [V, \oplus_{s\in S}(sV)^H]_H = \sum_{s\in S}[V, (sV)^H]_H = \sum_{s\in S}[V_{H_s}, sV]_{H_s} = [V, V]_H + \sum_{s\in S\setminus\{1\}}[V_{H_s}, sV]_{H_s} \in k$. Now V^G is absolutely irreducible iff $[V^G, V^G] = 1$ iff $[V, V]_H = 1$ and for all $s \in S \setminus \{1\}$ we have $[V_{H_s}, sV]_{H_s} = 0$.

Corollary. If $\operatorname{char} k = 0$ and H is a normal subgroup of G, then Write $G = \cup_i g_i H$. Let V be an absolutely irreducible kH-module. Then V^G is absolutely irreducible iff for any $i \geq 2$ we have $V \neq g_i V$ as kH-modules.

Corollary. If $\operatorname{char} k = 0$. Let $\nu\colon H \to k^*$ be a linear character. Then the monomial character ν^G is absolutely irreducible iff for any $s \in S$ such that $s \neq 1$ there is an $h \in sHs^{-1} \cap H$ such that $\neq (h) \neq \nu(s^{-1}hs) = \nu(h^s)$, i.e., iff $\nu(h^{-1}s^{-1}hs) \neq 1$.

Chapter 6

Frobenius groups.

If a finite group G acts on a finite nonempty set E. For $e \in E$ we denote by G_e the isotropy subgroup of e. We can easily see that $G_{ge} = g^{-1}G_e g$. If G is transitive on E, then all isotropy subgroups are conjugate. Recall that in transitive case $E = G/G_e$ as G-sets.

Definition. We say that G-action on E is semiregular or regular iff for any $e \in E$ we have $G_e = \{1\}$ (also called free) or the previous condition holds and it is transitive.

Definition. If E is a transitive G-set, then let $K = \{1\} \cup \{g \in G \mid \pi(g) = 0\} = G \setminus \cup_e G_e$, where π is fixed point counter. K is called the Frobenius kernel of the action. It is closed under inversion and conjugation. If K is closed under multiplication, then it is a normal subgroup.

Jordan's Inequality. If G acts transitively on E, then $|K| \geq |E|$.

Proof. Burnside: $|G| = \sum_g \pi(g) = \sum_{g\in K} \pi(g) + \sum_{g\in G\setminus K} \pi(g) \geq |E| + |G| - |K|$.

Jordan's theorem. If H is a subgroup of G, then $|G \setminus \cup H^g| \geq [G : H] - 1$.

Proposition. Consider statements: (1) $\pi(g) \leq 1$ for any $g \in G \setminus \{1\}$. (2) $|K| = |E|$. (3) K is a subgroup of G (hence K is a normal subgroup of G). We have (1) iff (2). If (3) holds, then there exists an H-set isomorphism $K = E$ where H acts on K by conjugation. In particular, (3) implies (1) and (2).

Proof. (1) iff (2) is clear from proof of Jordan inequality. Assume (3) holds. Define $\theta\colon K \to E$ such that $\theta(g) = ge$ where $g \in K$. Now we check that this is an H-set morphism. If $g \in K$ and $h \in H$, then $\theta({}^h g) = \theta(hgh^{-1}) = (hgh^{-1})(e) = h(ge) = h\theta(g)$. Now we prove that θ is injective.

Definition. (Frobenius.) If (1) holds, we say that G-action on E is Frobenius provided that $1 < |E| < |G|$. In this case G is a Frobenius group.

Big Frobenius theorem. Let E be a Frobenius G-set. Then K is a subgroup. This also means that (1), (2), and (3) are equivalent.

Corollary. Let $n = |E|$, where E is a Frobenius G-set. Then K acts regularly on E, G is a semidirect product of K and H, H acts semiregularly on $E \setminus \{e\}$ and $K^* = K \setminus \{1\}$, $|H|$ divides $n-1$, K and H are Hall subgroups of G, $K = \{x \in G \mid x^n = 1\}$.

Corollary. p-groups, abelian groups, and simple groups cannot be Frobenius.

K is called the Frobenius kernel, $H = G_e$ is called Frobenius complement, and we have $|G| = |K| \cdot |H|$. Big Frobenius states that K is a subgroup of G (hence a normal subgroup).

Given a subgroup H of G, when is G/H a Frobenius G-set?

Definition. A nontrivial subgroup H of G is said to have trivial intersection property if for any $g \in G \setminus H$ we have $H \cap H^g = \{1\}$.

Theorem. If H is a subgroup of G, then G acts Frobeniusly on G/H iff H has trivial intersection property.

Proof. Assume G/H is Frobenius. Then $1 \neq H \neq G$. Consider $g \notin H$. Now $H^g \cap H$ fixes e and $g^{-1}e$, which are different points, hence $H^g \cap H = \{1\}$. Now assume that H has trivial intersection property. Suppose $g \neq 1$ fixes cosets xH and yH. We have $xH = gxH$, hence $g \in H^{x^{-1}}$. Also $g \in H^{y^{-1}}$. Honce $H^{x^{-1}} \cap H^{y^{-1}} \neq \{1\}$. Conjugate this by y. We have $\{1\} \neq H^{x^{-1}y} \cap H$. By trivial intersection property $x^{-1}y \in H$, hence $xH = yH$.

Example. If G is nonabelian group of order pq, where $p < q$ are prime, then it is Frobenius.

Proof. Fix Sylow's subgroups P and Q. Note that Q is a normal subgroup because $[G : Q]$ is the smallest prime dividing $|G|$. It remains to check that H has trivial intersection property. Observe that $N_G(P) = P$ by cardinality consideration. Take $g \in G \setminus H$. We know that $H^g \neq H$. Both are of order p, hence $H^g \cap H = \{1\}$. Extra: Q^* is disjoint from $\cup_g H^g$. This implies that $Q^* \subset K^*$, hence $Q = K$.

Proposition. If H is a subgroup of G, the set K is a subset of G, $H \cap K = \{1\}$. Then K is a normal subgroup of G iff every $\phi \in \operatorname{Irr}(H)$ extends to some $\chi \in \operatorname{Irr}(G)$.

Proof. Consider extensions χ_i of all $\phi_i \in \operatorname{Irr}(H)$. Let $\tilde{K} = \cap_i \ker \chi_i \supset K$. It is easy to see that $\tilde{K} = K$ is a normal subgroup of G.

Key tool for proving big Frobenius. Suppose G is a group, H and K are its subgroups, $H \cap K = \{1\}$, and $|H| \cdot |K| = |G|$. To show that K is a normal subgroup of G it is sufficient (and necessary) to check that every $\phi \in \operatorname{Irr}(H)$ extends to some $\chi \in \operatorname{Irr}(G)$ with $\ker(\chi) \supset K$. Of course we may assume that $\phi \neq 1_H$.

Proof of the Big Frobenius. Suppose $\operatorname{Ch}(G)$ is the character ring of G (a ring of class functions on G). Let $\operatorname{Ch}_0(G) = \{\mu \in \operatorname{Ch}(G) \mid \mu(1) = 0\}$ be an ideal of $\operatorname{Ch}(G)$. Step 1: $\operatorname{res}\colon \operatorname{Ch}_0(G) \to \operatorname{Ch}_0(H)$ is surjective and split by ind. Step 2: $\operatorname{ind}\colon \operatorname{Ch}_0(H) \to \operatorname{Ch}_0(G)$ is an isometry (preserves inner product). Step 3: Let $\phi \in \operatorname{Irr}(H)$ be a nontrivial character. Define $\nu = d \cdot 1_H - \phi \in \operatorname{Ch}_0(H)$. Now χ is what goes to ν^G by going-in process. We must check that $\chi \in \operatorname{Irr}(G)$ and $K \subset \ker \chi$.

Step 1: If $\nu \in \operatorname{Ch}_0(H)$, then $\nu^G(g) = |H|^{-1} \sum_{t \in G} \dot{\nu}(g^t)$. We have $\nu^G(1) = 0$. If $h \neq 1$, then $(\nu^G)_H(h) = |H|^{-1} \sum_{t \in G} \dot{\nu}(h^t) = |H|^{-1} \sum_{t \in H} \nu(h) = \nu(h)$. Step 2: If $\mu, \nu \in \operatorname{Ch}_0(H)$, then $\langle \nu^G, \mu^G \rangle$. Step 3: Have $\nu = d \cdot 1_H - \phi$. Define χ by equation $d \cdot 1_G - \chi = \nu^G$. Since ν^G restricts to ν we clearly have $\chi_H = \phi$. In particular, $\chi(1) = \phi(1) = d$. Now $\langle \nu, 1_H \rangle = d$ and $\langle \nu, \nu \rangle = d^2 + 1$. On the other hand $\langle \nu, 1_H \rangle = \langle \nu^G, 1_G \rangle = \langle d \cdot 1_G - \chi, 1_G \rangle = d - \langle \chi, 1_G \rangle$ and $\langle \nu, \nu \rangle = \langle \nu^G, \nu^G \rangle = d^2 + \langle \chi, \chi \rangle$. Therefore, $\langle \chi, \chi \rangle = 1$. Hence χ or $-\chi$ is an irreducible charcter. Finally need to know that $K \subset \ker \chi$. This means that $g \in K$ implies $\chi(g) = \chi(1) = d$. We may assume that $g \neq 1$.

We want to find all irreducible representations of Frobenius group G with kernel K and complement H provided that we know all representations of K and H. (This covers affine groups in dimension 1, pq-groups, etc.)

Proposition. $|\operatorname{Irr}(G)| = |\operatorname{Irr}(H)| + (|\operatorname{Irr}(K)| - 1)/|H|$.

Proof. First count the number of conjugacy classes that are in $G \setminus K^*$. Two elements of H are conjugate in G iff they are conjugate in H (take a projection from G to H with kernel K). Since $G \setminus K^* = \cup_g H^g$, this gives us $|\operatorname{Irr}(H)|$ classes. Now we should count G-classes within K^*. Let H act by conjugation on K-classes of K^*. This action is semiregular. Hence we have $(|\operatorname{Irr}(K) - 1)/|H|$ classes in K^*. To prove the semiregularity, suppose that $h \in H$ acts trivially on K-class of x. This means that $hxh^{-1} = yxy^{-1}$, hence $h \in K \cap H$.

Theorem. Simple $\mathbb{C}G$-modules consist of those that come from lifting $\mathbb{C}H$-modules to G and $(|\operatorname{Irr}(K)| - 1)/|H|$ distinct simple $\mathbb{C}G$-modules induced by tensor product from nontrivial simple $\mathbb{C}K$-modules.

Proof. Look at the set of all nontrivial simple $\mathbb{C}K$-modules. Let H act on this set by twisting: $V \mapsto hV = h \otimes V \subset V^G$. Recall that $(gV)^G = V^G$. Hence any orbit of this action is mapped to the same $\mathbb{C}G$-module. Pick a simple submodule of this $\mathbb{C}G$-module. If we restrict it to H, we get a direct sum of some the modules of the orbit. This would product m (where m is the number of H-orbitsi of the twisting action) distinct (because they are distinct as $\mathbb{C}H$-modules) $\mathbb{C}G$-simple modules (not yet counted). But then we would have $n|H| = |\operatorname{Irr}(K)| - 1 \leq |H|m$, hence $n \leq m$, therefore $n = m$ and the size of each orbit is $|H|$. Hence H acts semiregularly on nontrivial simple $\mathbb{C}K$-modules. Frobenius Reciprocity implies that the restriction contains all the elements of the oribt, comparison of dimensions implies that it must be isomorphic to V^G where V is an element of the orbit. Hence if we induce a $\mathbb{C}G$-module from a $\mathbb{C}K$-module by tensor product we obtain a simple module.

Definition. A group H acting as a group of automorphisms of K is called fixed point free if for every $\sigma \in H$ and $x \in K$ if $\sigma(x) = x$, then $x = 1$.

Question. What kind of groups can be K and H?

Theorem. (Without proof.) K is nilpotent.